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Generalized Quasi Variational-Type Inequalities

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ABSTRACT

In this paper, we define the concepts of (η,h)-quasi pseudo-monotone operators on compact set in locally convex Hausdorff topological vector spaces and prove the existence results of solutions for a class of *generalized quasi variational type inequalities* in locally convex Hausdorff topological vector spaces.

Keywords: Generalized Quasi Variational Type Inequalities (GQVTI); (η,h)-Quasi Pseudo-Monotone Operator; Locally Convex Hausdorff Topological Vector Spaces; Compact Sets; Bilinear Functional; Lower Semicontinuous; Upper Semicontinuous

1. Introduction

Variational inequality theory has appeared as an effective and powerful tool to study and investigate a wide class of problems arising in pure and applied sciences including elasticity, optimization, economics, transportation, and structural analysis, see for instance [1,2]. In 1966, Browdev [3] first formulated and proved the basic existence theorems of solutions to a class of nonlinear variational inequalities. In 1980, Giannessi [1] introduced the vector variational inequality in a finite dimensional Euclidean space. Since then Chen *et al*. [4] have intensively studied vector variational inequalities in abstract spaces and have obtained existence theorems for their inequalities.

The pseudo-monotone type operators was first introduced in [5] with a slight variation in the name of this operator. Later these operators were renamed as pseudo-monotone operators in [6]. The pseudomonotone operators are set-valued generalization of the classical pseudo-monotone operator with slight variations. The classical definition of a single-valued pseudo-monotone operator was introduced by Brezis, Nirenberg and Stampacchia [7].

In this paper we obtained some general theorems on solutions for a new class of *generalized quasi variational type inequalities* for (η,h)-quasi-pseudo-monotone operators defined as compact sets in topological vector spaces. We have used the generalized version of Ky Fan's minimax inequality [8] due to Chowdhury and Tan [9].

Let X and Y be the topological spaces, $T: X \to 2^Y$ be the mapping and the graph of T is the set $G(T) = \{(x,y) \in X \times Y : y \in T(x)\}$. In this paper, Φ denotes either the real field $\mathbb{R}$ or the complex field $\mathbb{C}$. Let E be a topological vector space over Φ, F be a vector space over Φ and $\langle \cdot,\cdot \rangle : F \times E \to \Phi$ be a bilinear functional.

For each nonempty subset A of E and $\varepsilon > 0$, let $W(x_0;\varepsilon) = \{y \in F : |\langle y, x_0 \rangle| < \varepsilon\}$ and $U(A;\varepsilon) = \left\{y \in F : \sup_{x\in A} |\langle y, x \rangle| < \varepsilon\right\}$ for $x_0 \in E$. Let $\sigma(F,E)$ be the (weak) topology on F generated by the family $\{W(x;\varepsilon) : x \in E \text{ and } \varepsilon > 0\}$ as a subbase for the neighbourhood system at 0 and $\delta\langle F,E\rangle$ be the (strong) topology on F generated by the family $\{U(A;\varepsilon)$: A is a nonempty bounded subset of E and $\varepsilon > 0\}$ as a base for the neighbourhood system at 0. The bilinear functional $\langle \cdot,\cdot \rangle : F \times E \to \Phi$ separates points in F, *i.e.*, for each $0 \neq y \in F$, there exists $x \in E$ such that $\langle y, x \rangle \neq 0$, then F also becomes Hausdorff. Furthermore, for a net $\{y_\alpha\}_{\alpha\in\Gamma}$ in F and for $y \in F$,

1) $y_\alpha \to y$ in $\sigma\langle F,E\rangle$ if and only if $\langle y_\alpha, x\rangle \to \langle y, x\rangle$ for each $x \in E$ and

2) $y_\alpha \to y$ in $\sigma\langle F,E\rangle$ if and only if $\langle y_\alpha, x\rangle \to \langle y, x\rangle$ uniformaly for $x \in A$ for each nonempty bounded subset A of E.

Given a set-valued map $S: X \to 2^X$ and two set valued maps $M, T: X \to 2^F$, the *generalized quasi variational type inequality* (GQVTI) problem is to find $\hat{y} \in X$ and $\hat{w} \in T(\hat{y})$ such that $\hat{y} \in S(\hat{y})$ and

$$\operatorname{Re}\langle f - \hat{w}, \eta(\hat{y}, x)\rangle \le 0,$$
$$\text{for all } x \in S(\hat{y}) \text{ and } f \in M(\hat{y}),$$

where $\eta: X \times X \to E$.

If $\eta(\hat{y}, x) = \hat{y} - x$, then *generalized quasi variational type inequality* (*GQVTI*) is equivalent to generalized quasi variational inequality (GQVI).

Find $\hat{y} \in X$ and $\hat{w} \in T(\hat{y})$ such that $\hat{y} \in S(\hat{y})$ and

$$\operatorname{Re}\langle f - \hat{w}, \hat{y} - x\rangle \leq 0 \text{ for all } x \in S(y)$$

and $f \in M(\hat{y})$ was introduced by Shih and Tan [10] in 1989 and later was stated by Chowdhury and Tan in [11].

Definition 1. Let X be a nonempty subset of a topological vector space E over Φ and F be a topological vector space over Φ, which is equipped with the $\sigma(F,E)$-topology. Let $\langle\cdot,\cdot\rangle : F \times E \to \Phi$ be a bilinear functional. Suppose we have the following four maps.

1) $h : X \times X \to \mathbb{R}$
2) $\eta : X \times X \to E$
3) $M : X \to 2^F$
4) $T : X \to 2^F$.

1) Then T is said to be an (η,h)-quasi pseudo-monotone type operator if for each $y \in X$ and every net $\{y_\alpha\}_{\alpha\in\Gamma}$ in X converging to y (or weakly to y) with

$$\limsup_{\alpha}\left[\inf_{f\in M(y)}\inf_{u\in T(y_\alpha)} \operatorname{Re}\langle f-u, \eta(y_\alpha, y)\rangle + h(y_\alpha, y)\right] \leq 0.$$

We have

$$\limsup_{\alpha}\left[\inf_{f\in M(x)}\inf_{u\in T(y_\alpha)} \operatorname{Re}\langle f-u, \eta(y_\alpha, x)\rangle + h(y_\alpha, x)\right]$$
$$\geq \inf_{f\in M(x)}\inf_{w\in T(y)} \operatorname{Re}\langle f-w, \eta(y,x)\rangle + h(y,x),$$

for all $x \in X$;

2) T is said to be h-quasi-pseudomonotone operator if T is (η,h)-quasi-pseudomonotone operator with $\eta(x,y) = x - y$ and for some $h' : X \to \mathbb{R}$,

$$h(x,y) = h'(x) - h'(y) \text{ for all } x, y \in X.$$

3) a quasi-pseudo monotone operator if T is an h-quasi pseudo-monotone operator with $h \equiv 0$.

Remark 1. If $M \equiv 0$ and T is replaced by $-T$, then h-quasi-pseudo monotone operator reduces to the h-pseudo monotone operator, see for example [5]. The h-pseudo monotone operator defined in [5] is slightly more general than the definition of h-pseudo monotone operator given in [12]. Also we can find the generalization of quasi-pseudo monotone operator in [11] and for more detail see [13].

Theorem 1. [8] Let E be a topological vector space, X be a nonempty convex subset of E and $f : X \times X \to \mathbb{R} \cup \{-\infty, +\infty\}$ be such that

1) For each $A \in F(X)$ and each fixed $x \in co(A)$, $y \to f(x,y)$ is lower semicontinuous on $co(A)$;

2) For each $A \in F(X)$ and each $y \in co(A)$, $\min_{x\in A} f(x,y) \leq 0$;

3) For each $A \in F(X)$ and each $x, y \in co(A)$, every net $\{y_\alpha\}_{\alpha\in\Gamma}$ in X converging to y with $f(tx + (1-t)y, y_\alpha) \leq 0$ for all $\alpha \in \Gamma$ and all $t \in [0,1]$ we have $f(x,y) \leq 0$;

4) There exist a nonempty closed compact subset K of X and $x_0 \in K$ such that

$$f(x_0, y) > 0 \text{ for all } y \in X \setminus K.$$

Then there exists $\hat{y} \in K$ such that

$$f(x, \hat{y}) \leq 0 \text{ for all } x \in X.$$

2. Preliminaries

In this section, we shall mainly state some earlier work which will be needed in proving our main results.

Lemma 1. [14] Let X be a nonempty subset of a Hausdorff topological vector space E and $S : X \to 2^E$ be an upper semicontinuous map such that $S(x)$ is a bounded subset of E for each $x \in X$. Then for each continuous linear functional p on E, the map $f_p : X \to \mathbb{R}$ defined by

$$f_p(y) = \sup_{x\in S(y)} \operatorname{Re}\langle p, x\rangle \text{ is upper semicontinuous } i.e.,$$

for each $\lambda \in R$,

the set $\left\{ y \in X : f_p(y) = \sup_{x\in S(y)} \operatorname{Re}\langle p, x\rangle < \lambda \right\}$ is open in X.

Lemma 2. [15] Let X, Y be topological spaces, $f : X \to \mathbb{R}$ be non-negative and continuous and $g : Y \to \mathbb{R}$ be lower semicontinuous. Then the map $F : X \times Y \to \mathbb{R}$, defined by $F(x,y) = f(x)g(y)$ for all $(x,y) \in X \times Y$, is lower semicontinuous.

Lemma 3. [11] Let E be a topological vector space over Φ, X be a nonempty compact subset of E and F be a Hausdorff topological vector space over Φ. Let $\langle\cdot,\cdot\rangle : F \times E \to \Phi$ be a bilinear functional and $T : X \to 2^F$ be an upper semicontinuous map such that each $T(x)$ is compact. Let M be a nonempty compact subset of F, $x_0 \in X$ and $h : X \to \mathbb{R}$ be continuous. Define $g : X \to \mathbb{R}$ by

$$g(y) = \left[\inf_{f\in M(x)}\inf_{w\in T(y)} \operatorname{Re}\langle f - w, y - x_0\rangle\right] + h(y)$$

for each $y \in X$.

Suppose that $\langle\cdot,\cdot\rangle$ is continuous on the (compact) subset $\left[M - \bigcup_{y\in X} T(y)\right] \times X$ of $F \times E$. Then g is lower semicontinuous on X.

Lemma 4. [11] Let E be a topological vector space over Φ, F be a vector space over Φ and X be a nonempty convex subset of E. Let $\langle\cdot,\cdot\rangle : F \times E \to \Phi$ be a bilinear functional, equip F with the $\sigma\langle F, E\rangle$-

topology. Let $h: X \times X \to \mathbb{R}$ be convex with second argument and $h(x,x)=0$ for all $x \in X$. Let $M: X \to F$ be lower semicontinuous along line segments in X to the $\sigma\langle F,E\rangle$-topology on F. Let $S: X \to 2^X$ and $T: X \to 2^F$ be two maps. Let the continuous map $\eta: X \times X \to E$ be convex with second argument, $\eta(x,x)=0$ for every $x \in X$. Suppose that there exists $\hat{y} \in X$ such that $\hat{y} \in S(\hat{y})$, $S(\hat{y})$ is convex and

$$\inf_{f \in M(x)} \inf_{w \in T(y)} \operatorname{Re}\langle f-w, \eta(\hat{y},x)\rangle + h(x,\hat{y}) \le 0$$

for all $x \in S(\hat{y})$.

Then

$$\inf_{f \in M(\hat{y})} \inf_{w \in T(\hat{y})} \operatorname{Re}\langle f-w, \eta(\hat{y},x)\rangle + h(x,\hat{y}) \le 0$$

for all $x \in S(\hat{y})$.

Theorem 2. [16] Let X be a nonempty convex subset of a vector space and Y be a nonempty compact convex subset of a Hausdorff topological vector space. Suppose that f is a real-valued function on $X \times Y$ such that for each fixed $x \in X$, the map $y \to f(x,y)$, *i.e.*, $f(x,\cdot)$ is lower semicontinuous and convex on Y and for each fixed $y \in Y$, the map $x \to f(x,y)$, *i.e.*, $f(\cdot,y)$ is concave on X. Then

$$\min_{y\in Y} \sup_{x\in X} f(x,y) = \sup_{x\in X} \min_{y\in Y} f(x,y).$$

3. Existence Result

In this section, we prove the existence theorem for the solutions to the *generalized quasi variational type inequalities* for (η,h)-quasi-pseudo monotone operator with compact domain in locally convex Hausdorff topological vector spaces.

Theorem 3. Let E be a locally convex Hausdorff topological vector space over Φ, X be a nonempty compact convex subset of E and F a Hausdorff topological vector space over Φ. Let $\langle\cdot,\cdot\rangle: F \times E \to \Phi$ be a bilinear continuous functional on compact subset of $F \times X$. Suppose that

1) $S: X \to 2^X$ is upper semicontinuous such that each $S(x)$ is closed and convex;

2) $h: X \times X \to \mathbb{R}$ is convex with second argument, $h(\cdot,x)$ is lower semicontinuous and $h(x,x)=0$ for $x \in X$;

3) $\eta: X \times X \to E$ is convex with second argument, $\eta(\cdot,y)$ is continuous and $\eta(x,x)=0$ for all $x \in X$;

4) $T: X \to 2^F$ is an (η,h)-quasi-pseudo-monotone operator and is upper semicontinuous such that each $T(x)$ is compact, convex and $T(X)$ is strongly bounded;

5) $M: X \to F$ is a linear and upper semicontinuous map in X such that each $M(x)$ is (weakly) compact convex;

6) the set

$$\Sigma = \left\{ y \in X : \sup_{x\in S(y)} \inf_{w\in T(y)} \operatorname{Re}\langle M(x)-w, \eta(y,x)\rangle + h(y,x) - h(x,x) > 0 \right\}$$

is open in X.

Then there exists $\hat{y} \in X$ such that

a) $\hat{y} \in S(\hat{y})$ and

b) there exists $\hat{w} \in T(\hat{y})$ with

$$\operatorname{Re}\langle M(\hat{y}) - \hat{w}, \eta(\hat{y},x)\rangle + h(\hat{y},x) - h(x,x) \le 0$$

for all $x \in S(\hat{y})$.

Moreover if $S(x)=X$ for all $x \in X$, E is not required to be locally convex and if $T \equiv 0$, the continuity assumption on $\langle\cdot,\cdot\rangle$ can be weakened to the assumption that for each $f \in F$, the map $x \to \langle f,x\rangle$ is continuous on X.

Proof. We divide the proof into three steps.

Step 1. There exists $\hat{y} \in X$ such that $\hat{y} \in S(\hat{y})$ and

$$\sup_{x\in S(\hat{y})} \left[\inf_{w\in T(\hat{y})} \operatorname{Re}\langle M(x)-w, \eta(\hat{y},x)\rangle + h(\hat{y},x) - h(x,x) \right] \le 0.$$

Contrary suppose that for each $y \in X$, either $y \notin S(y)$ or there exists $x \in S(y)$ such that

$$\inf_{w\in T(y)} \operatorname{Re}\langle M(x)-w, \eta(y,x)\rangle + h(y,x) - h(x,x) > 0,$$

that is for each $y \in X$ either $y \notin S(y)$ or $y \in \Sigma$. If $y \notin S(y)$, then by a Hahn-Banach separation theorem for convex sets is locally convex Hausdorff topological vector spaces, there exists $p \in E^*$ such that

$$\operatorname{Re}\langle p,y\rangle - \sup_{x\in S(y)} \operatorname{Re}\langle p,x\rangle > 0.$$

For each $p \in E^*$, set

$$V_p = \left\{ y \in X : \operatorname{Re}\langle p,y\rangle - \sup_{x\in S(y)} \operatorname{Re}\langle p,x\rangle > 0 \right\}.$$

Then V_p is open in X by Lemma 1 and Σ is open in X by hypothesis. Now $X = \Sigma \cup \bigcup_{p\in E^*} V_p$ and $\left\{\Sigma, V_p : p \in E^*\right\}$ is an open covering for X. Since X is compact subset of E, there exists $p_1, p_2, \cdots, p_n \in E^*$ such that $X = \Sigma \cup \bigcup_{i=1}^{n} V_{p_i}$ for $i = 1,2,\cdots,n$. Let $V_i = V_{p_i}$ for $i = 1,2,\cdots,n$ and $\{\beta_0, \beta_1, \cdots, \beta_n\}$ be a continuous partition of unity on X subordinated to the covering $\{V_0, V_1, \cdots, V_n\}$. Then $\beta_0, \beta_1, \cdots, \beta_n$ are continuous non-negative real valued functions on X such that β_i vanishes on $X \setminus V_i$ for each $i = 0,1,\cdots,n$ and $\sum_{i=0}^{n} \beta_i(x) = 1$ for all $x \in X$ (see [17] p. 83).

Define $\varphi : X \times X \to \mathbb{R}$ by

$$\varphi(x,y) = \beta_0(y)\left[\inf_{w\in T(y)} \operatorname{Re}\langle M(x) - w, \eta(y,x)\rangle + h(y,x) - h(x,x)\right] + \sum_{i=1}^{n} \beta_i(y) \operatorname{Re}\langle p_i, \eta(y,x)\rangle$$

for each $x, y \in X$. Then we have

1) E is Hausdorff for each $A \in F(X)$ and each fixed $x \in co(A)$ the map

$$y \to \inf_{w\in T(y)} \operatorname{Re}\langle M(x) - w, \eta(y,x)\rangle + h(y,x) - h(x,x)$$

is lower semicontinuous on $co(A)$ by Lemma 3 and the fact that h is continuous on $co(A)$, therefore the map

$$y \to \beta_0(y)\left[\inf_{w\in T(y)} \operatorname{Re}\langle M(x) - w, \eta(y,x)\rangle + h(y,x) - h(x,x)\right]$$

is lower semicontinuous on $co(A)$ by Lemma 2. Also for each fixed $x \in X$,

$$y \to \sum_{i=1}^{n} \beta_i(y) \operatorname{Re}\langle p_i, \eta(y,x)\rangle$$

is continuous on X. Hence for each $A \in F(X)$ and each fixed $x \in co(A)$, the map $y \to \varphi(x,y)$ is lower semicontinuous on $co(A)$.

2) for each $A \in F(X)$ and each $y \in co(A)$, $\min_{x\in A} \varphi(x,y) \le 0$. Indeed, if these were false then for some $A = \{x_1, x_2, \cdots, x_n\} \in F(X)$ and some $y \in co(A)$ (say $y = \sum_{i=1}^{n} \lambda_i x_i$, where $\lambda_1, \lambda_2, \cdots, \lambda_n \ge 0$ with $\sum_{i=1}^{n} \lambda_i = 1$), we have $\min_{1\le i\le n} \varphi(x_i, y) > 0$. Then for each $i = 1,2,\cdots,n$,

$$\beta_0(y)\left[\inf_{w\in T(y)} \operatorname{Re}\langle M(x_i) - w, \eta(y,x_i)\rangle + h(y,x_i) - h(x_i,x_i)\right] + \sum_{i=1}^{n} \beta_i(y)\langle p_i, \eta(y,x_i)\rangle > 0.$$

So that

$$\begin{aligned}
0 = \varphi(y,y) &= \beta_0(y)\left[\inf_{w\in T(y)} \operatorname{Re}\left\langle M\left(\sum_{i=1}^{n}\lambda_i x_i\right) - w, \eta\left(y, \sum_{i=1}^{n}\lambda_i x_i\right)\right\rangle + h\left(y, \sum_{i=1}^{n}\lambda_i x_i\right) - h\left(\sum_{i=1}^{n}\lambda_i x_i, \sum_{i=1}^{n}\lambda_i x_i\right)\right] \\
&\quad + \sum_{i=1}^{n}\beta_i(y)\operatorname{Re}\left\langle p_i, \eta\left(y, \sum_{i=1}^{n}\lambda_i x_i\right)\right\rangle \\
&= \beta_0(y)\left[\inf_{w\in T(y)} \operatorname{Re}\left\langle \sum_{i=1}^{n}\lambda_i M(x_i) - w, \eta\left(y, \sum_{i=1}^{n}\lambda_i x_i\right)\right\rangle + h\left(y, \sum_{i=1}^{n}\lambda_i x_i\right) - h\left(\sum_{i=1}^{n}\lambda_i x_i, \sum_{i=1}^{n}\lambda_i x_i\right)\right] \\
&\quad + \sum_{i=1}^{n}\beta_i(y)\operatorname{Re}\left\langle p_i, \eta\left(y, \sum_{i=1}^{n}\lambda_i x_i\right)\right\rangle \\
&\ge \sum_{i=1}^{n}\lambda_i\left(\beta_0(y)\left[\inf_{w\in T(y)} \operatorname{Re}\langle M(x_i) - w, \eta(y,x_i)\rangle + h(y,x_i) - h\left(\sum_{i=1}^{n}\lambda_i x_i, \sum_{i=1}^{n}\lambda_i x_i\right)\right] + \sum_{i=1}^{n}\beta_i(y)\operatorname{Re}\langle p_i, \eta(y,x_i)\rangle\right) > 0
\end{aligned}$$

which is a contradiction.

Thus we have $\min_{x\in A} \varphi(x,y) \le 0$ for each $A \in F(x)$ and each $y \in co(A)$.

3) Suppose that $A \in F(X)$, $x, y \in co(A)$ and $\{y_\alpha\}_{\alpha\in\Gamma}$ is a net in X converging to y with $\varphi(tx + (1-t)y, y_\alpha) \le 0$ for all $\alpha \in \Gamma$, $t \in [0,1]$.

Case 1. $\beta_0(y) = 0$.

Note that $\beta_0(y_\alpha) \ge 0$ for each $\alpha \in \Gamma$ and $\beta_0(y_\alpha) \to 0$. Since $T(X)$ is strongly bounded and $\{y_\alpha\}_{\alpha\in\Gamma}$ is a bounded net, therefore

$$\limsup_{\alpha}\left[\beta_0(y_\alpha)\left(\min_{w\in T(y_\alpha)} \operatorname{Re}\langle M(x) - w, \eta(y_\alpha, x)\rangle + h(y_\alpha, x) - h(x,x)\right)\right] = 0. \tag{1}$$

Also

$$\beta_0(y)\left[\min_{w\in T(y)} \operatorname{Re}\langle M(x) - w, \eta(y,x)\rangle + h(y,x) - h(x,x)\right] = 0.$$

Thus

$$\limsup_{\alpha}\left[\beta_0(y_\alpha)\left(\min_{w\in T(y_\alpha)}\operatorname{Re}\langle M(x)-w,\eta(y_\alpha,x)\rangle+h(y_\alpha,x)-h(x,x)\right)\right]+\sum_{i=1}^{n}\beta_i(y)\operatorname{Re}\langle p_i,\eta(y,x)\rangle$$

$$=\sum_{i=1}^{n}\beta_i(y)\operatorname{Re}\langle p_i,\eta(y,x)\rangle \quad \text{by (1)} \tag{2}$$

$$=\beta_0(y)\left[\min_{w\in T(y)}\operatorname{Re}\langle M(x)-w,\eta(y,x)\rangle+h(y,x)-h(x,x)\right]+\sum_{i=1}^{n}\beta_i(y)\operatorname{Re}\langle p_i,\eta(y,x)\rangle.$$

When $t=1$, we have $\varphi(x,y_\alpha)\le 0$ for all $\alpha\in\Gamma$ *i.e.*,

$$\beta_0(y_\alpha)\left[\min_{w\in T(y_\alpha)}\operatorname{Re}\langle M(x)-w,\eta(y_\alpha,x)\rangle+h(y_\alpha,x)-h(x,x)\right]+\sum_{i=1}^{n}\beta_i(y_\alpha)\operatorname{Re}\langle p_i,\eta(y_\alpha,x)\rangle\le 0 \tag{3}$$

for all $\alpha\in\Gamma$.

Therefore by (3), we have

$$\limsup_{\alpha}\left[\beta_0(y_\alpha)\min_{w\in T(y_\alpha)}\operatorname{Re}\langle M(x)-w,\eta(y_\alpha,x)\rangle+h(y_\alpha,x)-h(x,x)\right]+\liminf_{\alpha}\left[\sum_{i=1}^{n}\beta_i(y_\alpha)\operatorname{Re}\langle p_i,\eta(y_\alpha,x)\rangle\right]$$

$$\le\limsup_{\alpha}\left[\beta_0(y_\alpha)\min_{w\in T(y_\alpha)}\operatorname{Re}\langle M(x)-w,\eta(y_\alpha,x)\rangle+h(y_\alpha,x)-h(x,x)+\sum_{i=1}^{n}\beta_i(y_\alpha)\operatorname{Re}\langle p_i,\eta(y_\alpha,x)\rangle\right]\le 0.$$

Thus

$$\limsup_{\alpha}\left[\beta_0(y_\alpha)\min_{w\in T(y_\alpha)}\operatorname{Re}\langle M(x)-w,\eta(y_\alpha,x)\rangle+h(y_\alpha,x)-h(x,x)\right]+\sum_{i=1}^{n}\beta_i(y)\operatorname{Re}\langle p_i,\eta(y,x)\rangle\le 0. \tag{4}$$

Hence by (2) and (4), we have $\varphi(x,y)\le 0$.

Case 2. $\beta_0(y)>0$.

Since $\beta_0(y_\alpha)\to\beta_0(y)$, there exists $\lambda\in\Gamma$ such that $\beta_0(y_\alpha)>0$ for all $\alpha\ge\lambda$. When $t=0$, we have $\varphi(y,y_\alpha)\le 0$ for all $\alpha\in\Gamma$, *i.e.*,

$$\beta_0(y_\alpha)\left[\inf_{w\in T(y_\alpha)}\operatorname{Re}\langle M(y)-w,\eta(y_\alpha,y)\rangle+h(y_\alpha,y)-h(y,y)\right]+\sum_{i=1}^{n}\beta_i(y_\alpha)\operatorname{Re}\langle p_i,\eta(y_\alpha,y)\rangle\le 0$$

for all $\alpha\in\Gamma$.

Thus

$$\limsup_{\alpha}\left[\beta_0(y_\alpha)\left(\inf_{w\in T(y_\alpha)}\operatorname{Re}\langle M(y)-w,\eta(y_\alpha,y)\rangle+h(y_\alpha,y)-h(y,y)\right)+\sum_{i=1}^{n}\beta_i(y_\alpha)\operatorname{Re}\langle p_i,\eta(y_\alpha,y)\rangle\right]\le 0. \tag{5}$$

Hence

$$\limsup_{\alpha}\left[\beta_0(y_\alpha)\left(\inf_{w\in T(y_\alpha)}\operatorname{Re}\langle M(y)-w,\eta(y_\alpha,y)\rangle+h(y_\alpha,y)-h(y,y)\right)\right]+\liminf_{\alpha}\left[\sum_{i=1}^{n}\beta_i(y_\alpha)\operatorname{Re}\langle p_i,\eta(y_\alpha,y)\rangle\right]$$

$$\le\limsup_{\alpha}\left[\beta_0(y_\alpha)\left(\inf_{w\in T(y_\alpha)}\operatorname{Re}\langle M(y)-w,\eta(y_\alpha,y)\rangle+h(y_\alpha,y)-h(y,y)\right)+\sum_{i=1}^{n}\beta_i(y_\alpha)\operatorname{Re}\langle p_i,\eta(y_\alpha,y)\rangle\right]\le 0 \ \text{(by (5))}.$$

Since

$$\liminf_{\alpha}\left[\sum_{i=1}^{n}\beta_i(y_\alpha)\operatorname{Re}\langle p_i,\eta(y_\alpha,y)\rangle\right]=0,$$

we have

$$\limsup_{\alpha}\left[\beta_0(y_\alpha)\left(\min_{w\in T(y_\alpha)}\operatorname{Re}\langle M(y)-w,\eta(y_\alpha,y)\rangle+h(y_\alpha,y)-h(y,y)\right)\right]\le 0. \tag{6}$$

Since $\beta_0(y_\alpha) > 0$ for all $\alpha > \lambda$. It follows that

$$\begin{aligned}&\beta_0(y_\alpha)\limsup_\alpha\left[\min_{w\in T(y_\alpha)}\operatorname{Re}\langle M(y)-w,\eta(y_\alpha,y)\rangle + h(y_\alpha,y)-h(y,y)\right]\\&=\limsup_\alpha\left[\beta_0(y_\alpha)\left(\min_{w\in T(y_\alpha)}\operatorname{Re}\langle M(y)-w,\eta(y_\alpha,y)\rangle + h(y_\alpha,y)-h(y,y)\right)\right].\end{aligned} \tag{7}$$

Since $\beta_0(y) > 0$ by (6) and (7), we have

$$\limsup_\alpha\left[\min_{w\in T(y_\alpha)}\operatorname{Re}\langle M(y)-w,\eta(y_\alpha,y)\rangle + h(y_\alpha,y)-h(y,y)\right]\le 0.$$

Since T is (η,h)-quasi pseudomonotone operator, we have

$$\begin{aligned}&\limsup_\alpha\left[\min_{w\in T(y_\alpha)}\operatorname{Re}\langle M(x)-w,\eta(y_\alpha,x)\rangle + h(y_\alpha,x)-h(x,x)\right]\\&\ge \min_{w\in T(y_\alpha)}\operatorname{Re}\langle M(x)-w,\eta(y,x)\rangle + h(y,x)-h(x,x) \text{ for all } x\in X.\end{aligned}$$

Since $\beta_0(y) > 0$, we have

$$\begin{aligned}&\beta_0(y)\left[\limsup_\alpha\left(\min_{w\in T(y_\alpha)}\operatorname{Re}\langle M(x)-w,\eta(y_\alpha,x)\rangle + h(y_\alpha,x)-h(x,x)\right)\right]\\&\ge \beta_0(y)\left[\min_{w\in T(y)}\operatorname{Re}\langle M(x)-w,\eta(y,x)\rangle + h(y,x)-h(x,x)\right].\end{aligned}$$

Thus

$$\begin{aligned}&\beta_0(y)\left[\limsup_\alpha\left(\min_{w\in T(y_\alpha)}\operatorname{Re}\langle M(x)-w,\eta(y_\alpha,x)\rangle + h(y_\alpha,x)-h(x,x)\right)\right]+\sum_{i=1}^n\beta_i(y)\operatorname{Re}\langle p_i,\eta(y,x)\rangle\\&\ge \beta_0(y)\left[\min_{w\in T(y)}\operatorname{Re}\langle M(x)-w,\eta(y,x)\rangle + h(y,x)-h(x,x)\right]+\sum_{i=1}^n\beta_i(y)\operatorname{Re}\langle p_i,\eta(y,x)\rangle.\end{aligned} \tag{8}$$

When $t=1$, we have $\varphi(x,y_\alpha)\le 0$ for all $\alpha\in\Gamma$, *i.e.*,

$$\beta_0(y_\alpha)\left[\min_{w\in T(y_\alpha)}\operatorname{Re}\langle M(x)-w,\eta(y_\alpha,x)\rangle + h(y_\alpha,x)-h(x,x)\right]+\sum_{i=1}^n\beta_i(y_\alpha)\operatorname{Re}\langle p_i,\eta(y_\alpha,x)\rangle\le 0$$

for all $\alpha\in\Gamma$.

Thus

$$\begin{aligned}0&\ge\limsup_\alpha\left[\beta_0(y_\alpha)\min_{w\in T(y_\alpha)}\operatorname{Re}\langle M(x)-w,\eta(y_\alpha,x)\rangle + h(y_\alpha,x)-h(x,x)+\sum_{i=1}^n\beta_i(y_\alpha)\operatorname{Re}\langle p_i,\eta(y_\alpha,x)\rangle\right]\\&\ge\limsup_\alpha[\beta_0(y_\alpha)\min_{w\in T(y_\alpha)}\operatorname{Re}\langle M(x)-w,\eta(y_\alpha,x)\rangle + h(y_\alpha,x)-h(x,x)]+\liminf_\alpha\left[\sum_{i=1}^n\beta_i(y_\alpha)\operatorname{Re}\langle p_i,\eta(y_\alpha,x)\rangle\right]\\&=\beta_0(y)\left[\limsup_\alpha\left\{\min_{w\in T(y_\alpha)}\operatorname{Re}\langle M(x)-w,\eta(y_\alpha,x)\rangle + h(y_\alpha,x)-h(x,x)\right\}\right]+\sum_{i=1}^n\beta_i(y)\operatorname{Re}\langle p_i,\eta(y,x)\rangle\\&\ge\beta_0(y)\left[\min_{w\in T(y)}\operatorname{Re}\langle M(x)-w,\eta(y,x)\rangle + h(y,x)-h(x,x)\right]+\sum_{i=1}^n\beta_i(y)\operatorname{Re}\langle p_i,\eta(y,x)\rangle \text{ (by (8))}.\end{aligned} \tag{9}$$

Hence, we have $\varphi(x,y)\le 0$.

Since X is a compact subset of the Hausdorff topological vector space E, it is also closed. Now if we take $K=X$, then for any $x_0\in K=X$, we have

$$\varphi(x_0,y)>0 \text{ for all } y\in X\setminus K\ (=X\setminus X=\varnothing).$$

Thus φ satisfies all the hypothesis of Theorem 1. Hence by Theorem 1, there exists $\hat{y}\in K$ such that

$$\varphi(x,\hat{y}) \le 0 \text{ for all } x \in X,$$

$$\beta_0(\hat{y})\left[\inf_{w\in T(\hat{y})} \operatorname{Re}\langle M(x)-w,\eta(\hat{y},x)\rangle + h(\hat{y},x)-h(x,x)\right] + \sum_{i=1}^{n}\beta_i(\hat{y})\operatorname{Re}\langle p_i,\eta(\hat{y},x)\rangle \le 0 \text{ for all } x \in X. \tag{10}$$

Now the rest of the proof of Step 1 is similar to the proof in Step 1 of Theorem 1 in [11]. Hence Step 1 is proved.

Step 2.

$$\inf_{w\in T(\hat{y})} \operatorname{Re}\langle M(\hat{y})-w,\eta(\hat{y},x)\rangle + h(\hat{y},x)-h(x,x) \le 0$$

for all $x \in S(y)$.

From Step 1, we have $\hat{y} \in S(\hat{y})$ and

$$\inf_{w\in T(\hat{y})} \operatorname{Re}\langle M(x)-w,\eta(\hat{y},x)\rangle + h(\hat{y},x)-h(x,x) \le 0$$

for all $x \in S(y)$.

Since $S(\hat{y})$ is a convex subset of X and M is linear, continuous along line segments in X, by Lemma 4 we have

$$\inf_{w\in T(\hat{y})} \operatorname{Re}\langle M(\hat{y})-w,\eta(\hat{y},x)\rangle + h(\hat{y},x)-h(x,x)) \le 0$$

for all $x \in S(y)$.

Step 3. There exists $\hat{w} \in T(\hat{y})$ with

$$\operatorname{Re}\langle M(\hat{y})-\hat{w},\eta(\hat{y},x)\rangle + h(\hat{y},x)-h(x,x) \le 0$$

for all $x \in S(y)$.

By Step 2 and applying Theorem 2 as proved in Step 3 of Theorem 1 in [11], we can show that there exists $\hat{w} \in T(\hat{y})$ such that

$$\operatorname{Re}\langle M(\hat{y})-\hat{w},\eta(\hat{y},x)\rangle + h(\hat{y},x)-h(x,x) \le 0$$

for all $x \in S(y)$.

We observe from the above proof that the requirement that E be locally convex is needed when and only when the separation theorem is applied to the case $y \notin S(y)$. Thus if $S: X \to 2^X$ is the constant map $S(x) = X$ for all $x \in X$, E is not required to be locally convex.

Finally, if $T \equiv 0$, in order to show that for each $x \in X$, $y \to \varphi(x,y)$ is lower semicontinuous, Lemma 3 is no longer needed and the weaker continuity assumption as $\langle\cdot,\cdot\rangle$ that for each $f \in E$, the map $x \to \langle f,x\rangle$ is continuous on X is sufficient. This completes the proof.

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