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# Uniqueness of Common Fixed Points for a Family of Mappings with $\phi$-Contractive Condition in 2-Metric Spaces*

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## ABSTRACT

In this paper, we will introduce a class of 5-dimensional functions $\Phi$ and prove that a family of self-mappings $\{T_{i,j}\}_{i,j\in\mathbb{N}}$ in 2-metric space have an unique common fixed point if 1) $\{T_{i,j}\}_{i\in\mathbb{N}}$ satisfies $\phi_j$-contractive condition, where $\phi_j \in \Phi$, for each $j\in\mathbb{N}$; 2) $T_{m,\mu}\cdot T_{n,\nu} = T_{n,\nu}\cdot T_{m,\mu}$ for all $m,n,\mu,\nu\in\mathbb{N}$ with $\mu\neq\nu$. Our main result generalizes and unifies many known unique common fixed point theorems in 2-metric spaces.

**Keywords:** 2-Metric Space; 5-Dimensional Functions $\Phi$; $\phi$-Contractive Condition; Cauchy Sequence; Common Fixed Point

## 1. Introduction and Preliminaries

There have appeared many unique common fixed point theorems for self-maps $\{f_i\}_{i\in\mathbb{N}}$ with some contractive condition on 2-metric spaces. But most of them held under subsidiary conditions [1-4], for examples: commutativity of $\{f_i\}_{i\in\mathbb{N}}$ or uniform boudedness of $\{f_i\}_{i\in\mathbb{N}}$ at some point, and so on. In [5], the author obtained similar results under removing the above subsidiary conditions. The result generalized and improved many same type unique common fixed point theorems. Recently, the author discussed unique common fixed point theorems for a family of contractive or quasi-contractive type mappings on 2-metric spaces, see [6-8], these results improve the above known common fixed point theorems.

In this paper, in order to generalize and unify further these results, we will prove that a family of self-maps $\{T_{i,j}\}_{i,j\in\mathbb{N}}$ satisfying $\phi_j$-contractive condition on 2-metric spaces have an unique common fixed point if $\{T_{i,j}\}_{i,j\in\mathbb{N}}$ satisfy the condition 2.

The following definitions are well known results.

**Definition 1.1.** [4] 2-metric space $(X,d)$ consists of a nonempty set $X$ and a function $d: X\times X\times X\to[0,+\infty)$ such that

1) for distant elements $x,y\in X$, there exists an $u\in X$ such that $d(x,y,u)\neq 0$;

2) $d(x,y,z)=0$ if and only if at least two elements in $\{x,y,z\}$ are equal;

3) $d(x,y,z)=d(u,v,w)$, where $\{u,v,w\}$ is any permutation of $\{x,y,z\}$;

4) $d(x,y,z)\le d(x,y,u)+d(x,u,z)+d(u,y,z)$ for all $x,y,z,u\in X$.

**Definition 1.2.** [4] A sequence $\{x_n\}_{n\in\mathbb{N}}$ in 2-metric space $(X,d)$ is said to be cauchy sequence, if for each $\varepsilon>0$ there exists a positive integer $N\in\mathbb{N}$ such that $d(x_n,x_m,a)<\varepsilon$ for all $a\in X$ and $n,m>N$.

**Definition 1.3.** [4,5] A sequence $\{x_n\}_{n\in\mathbb{N}}$ is said to be convergent to $x\in X$, if for each $a\in X$, $\lim_{n\to+\infty} d(x_n,x,a)=0$. And write $x_n\to x$ and call $x$ the limit of $\{x_n\}_{n\in\mathbb{N}}$.

**Definition 1.4.** [4,5] 2-metric space $(X,d)$ is said to be complete, if every cauchy sequence in $X$ is convergent.

Let $\Phi$ denotes a family of mappings such that each $\phi\in\Phi$, $\phi:(\mathbb{R}^+)^5\to\mathbb{R}^+$ is continuous and increasing in each coordinate variable, and $\lambda(t)=\phi(t,t,t,2t,t)<t$ for all $t>0$.

There are many functions $\phi$ which belongs to $\Phi$:

**Example 1.5.** Let $\phi:(\mathbb{R}^+)^5\to\mathbb{R}^+$ be defined by

$$\phi(t_1,t_2,t_3,t_4,t_5)=\frac{1}{7}(t_1+t_2+t_3+t_4+t_5).$$

Then obviously, $\phi\in\Phi$

**Example 1.6.** Let $\phi:(\mathbb{R}^+)^5\to\mathbb{R}^+$ be defined by

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$$\phi(t_1,t_2,t_3,t_4,t_5) = \frac{1}{7}(\arctan t_1 + \arctan t_2 + \arctan t_3 + \arctan t_4 + \arctan t_5).$$

Then obviously, $\phi$ is continuous and increasing in each coordinate variable, and

$$\lambda(t) = \phi(t,t,t,2t,t) = \frac{1}{7}(\arctan t + \arctan t + \arctan t + \arctan 2t + \arctan t) = \frac{1}{7}(4\arctan t + \arctan 2t) < \frac{1}{7}(4t+2t) < t.$$

Hence $\phi \in \Phi$.

The following two lemmas are known.

**Lemma 1.7.** [1-4] Let $(X,d)$ be a 2-metric space and $\{x_n\}_{n\in\mathbb{N}}$ a sequence. If there exists $h \in [0,1)$ such that $d(x_{n+2},x_{n+1},a) \le hd(x_{n+1},x_n,a)$ for all $a \in X$ and $n \in \mathbb{N}$, then $d(x_n,x_m,x_l) = 0$ for all $n,m,l \in \mathbb{N}$, and $\{x_n\}_{n\in\mathbb{N}}$ is a cauchy sequence

**Lemma 1.8.** [1-4] If $(X,d)$ is a 2-metric space and sequence $\{x_n\}_{n\in\mathbb{N}} \to x \in X$, then $\lim_{n\to+\infty} d(x_n,b,c) = d(x,b,c)$ for each $b,c \in X$.

## 2. Main Result

The following theorem is the main result in this present paper.

**Theorem 2.1.** Let $(X,d)$ be a complete 2-metric space, $\{T_{i,j}\}_{i,j\in\mathbb{N}}$ a family of maps from $X$ into itself, $\{m_{i,j}\}_{i,j\in\mathbb{N}}$ a family of positive integers, and $0 < q_j < 1$ and $\phi_j \in \Phi$ for each $j \in \mathbb{N}$. If the following $\phi_j$-contractive conditions hold

$$d\left(T_{\alpha,j}^{m_{\alpha,j}}x, T_{\beta,j}^{m_{\beta,j}}y, a\right) \le q_j\phi_j\left(d\left(x,T_{\alpha,j}^{m_{\alpha,j}}x,a\right), d\left(y,T_{\beta,j}^{m_{\beta,j}}y,a\right), d\left(x,T_{\beta,j}^{m_{\beta,j}}y,a\right), d\left(T_{\alpha,j}^{m_{\alpha,j}}x,y,a\right), d(x,y,a)\right),$$
$$\forall\ j\in\mathbb{N};\ x,y,a\in X;\ \alpha,\beta\in\mathbb{N};\ \alpha\ne\beta \tag{1}$$

and $T_{\alpha,\mu}\cdot T_{\beta,\nu} = T_{\beta,\nu}\cdot T_{\alpha,\mu}$ for all $\alpha,\beta,\mu,\nu \in \mathbb{N}$ with $\mu \ne \nu$. Then $\{T_{i,j}\}_{i,j\in\mathbb{N}}$ have an unique common fixed point in *X*.

**Proof** Fix $j \in \mathbb{N}$ and let $S_{n,j} = T_{n,j}^{m_{n,j}}$ for each $n \in \mathbb{N}$, then (1) becomes the following

$$d\left(S_{\alpha,j}(x),S_{\beta,j}(y),a\right) \le q_j\phi_j\left(d\left(x,S_{\alpha,j}(x),a\right), d\left(y,S_{\beta,j}(y),a\right), d\left(x,S_{\beta,j}(y),a\right), d\left(S_{\alpha,j}(x),y,a\right), d(x,y,a)\right),$$
$$\forall\ j\in\mathbb{N};\ x,y,a\in X;\ \alpha,\beta\in\mathbb{N};\ \alpha\ne\beta \tag{2}$$

Take an $x_{0,j} \in X$ and define a sequence as follows

$$x_{n,j} = S_{n,j}(x_{n-1,j}),\ n = 1,2,3,\cdots,$$

Then

$$\begin{aligned}
&d(x_{n+1,j},x_{n,j},a) = d\left(S_{n+1}(x_{n,j}),S_n(x_{n-1,j}),a\right)\\
&\le q_j\phi_j\left(d\left(x_{n,j},S_{n+1}(x_{n,j}),a\right), d(x_{n-1,j},S_n(x_{n-1,j}),a), d\left(x_{n,j},S_n(x_{n-1,j}),a\right), d\left(S_{n+1}(x_{n,j}),x_{n-1,j},a\right), d\left(x_{n,j},x_{n-1,j},a\right)\right)\\
&= q_j\phi_j\left(d(x_{n,j},x_{n+1,j},a), d(x_{n-1,j},x_{n,j},a), 0, d(x_{n+1,j},x_{n-1,j},a), d(x_{n,j},x_{n-1,j},a)\right)\\
&\le q_j\phi_j\left(d(x_{n,j},x_{n+1,j},a), d(x_{n-1,j},x_{n,j},a), 0, \left[d(x_{n-1,j},x_{n,j},a) + d(x_{n+1,j},x_{n,j},a) + d(x_{n-1,j},x_{n,j},x_{n+1,j})\right], d(x_{n,j},x_{n-1,j},a)\right).
\end{aligned} \tag{3}$$

If $d(x_{n-1,j},x_{n,j},x_{n+1,j}) > 0$, then

$$\begin{aligned}
&d(x_{n-1,j},x_{n,j},x_{n+1,j}) = d\left(S_{n+1,j}(x_{n,j}),S_{n,j}(x_{n-1,j}),x_{n-1,j}\right)\\
&\le q_j\phi_j\Big(d\left(x_{n,j},S_{n+1,j}(x_{n,j}),x_{n-1,j}\right), d\left(x_{n-1,j},S_{n,j}(x_{n-1,j}),x_{n-1,j}\right), d\left(x_{n,j},S_{n,j}(x_{n-1,j}),x_{n-1,j}\right),\\
&\quad d\left(S_{n+1,j}(x_{n,j}),x_{n-1,j},x_{n-1,j}\right), d\left(x_{n,j},x_{n-1,j},x_{n-1,j}\right)\Big)\\
&= q_j\phi_j\left(d(x_{n,j},x_{n+1,j},x_{n-1,j}),0,0,0,0\right)\\
&\le q_j\phi_j\left(d(x_{n,j},x_{n+1,j},x_{n-1,j}), d(x_{n,j},x_{n+1,j},x_{n-1,j}), d(x_{n,j},x_{n+1,j},x_{n-1,j}), 2d(x_{n,j},x_{n+1,j},x_{n-1,j}), d(x_{n,j},x_{n+1,j},x_{n-1,j})\right)\\
&< q_j d(x_{n,j},x_{n+1,j},x_{n-1,j})
\end{aligned} \tag{4}$$

 

which is a contradiction since $0 < q_j < 1$, hence $d\left(x_{n-1,j}, x_{n,j}, x_{n+1,j}\right) = 0$. And therefore, (3) becomes

$$\begin{aligned} & d\left(x_{n+1,j}, x_{n,j}, a\right) \\ & \leq q_j \phi_j \left(d\left(x_{n,j}, x_{n+1,j}, a\right), d\left(x_{n-1,j}, x_{n,j}, a\right), 0, [d\left(x_{n-1,j}, x_{n,j}, a\right) + d\left(x_{n+1,j}, x_{n,j}, a\right)], d\left(x_{n,j}, x_{n-1,j}, a\right)\right). \end{aligned} \tag{5}$$

If there exists an $a \in X$ such that $d\left(x_{n-1,j}, x_{n,j}, a\right) < d\left(x_{n+1,j}, x_{n,j}, a\right)$, then (5) becomes

$$\begin{aligned} d\left(x_{n+1,j}, x_{n,j}, a\right) & \leq q_j \phi_j \left(d\left(x_{n,j}, x_{n+1,j}, a\right), d\left(x_{n,j}, x_{n+1,j}, a\right), d\left(x_{n,j}, x_{n+1,j}, a\right), 2d\left(x_{n+1,j}, x_{n,j}, a\right), d\left(x_{n,j}, x_{n+1,j}, a\right)\right) \\ & < q_j d\left(x_{n+1,j}, x_{n,j}, a\right), \end{aligned}$$

which is a contradiction since $0 < q_j < 1$ and $d\left(x_{n+1,j}, x_{n,j}, a\right) > 0$, hence he have that

$d\left(x_{n+1,j}, x_{n,j}, a\right) \leq d\left(x_{n-1,j}, x_{n,j}, a\right)$ for all $a \in X$. In this case, (5) becomes

$$\begin{aligned} d\left(x_{n+1,j}, x_{n,j}, a\right) & \leq q_j \phi_j \left(d\left(x_{n-1,j}, x_{n,j}, a\right), d\left(x_{n-1,j}, x_{n,j}, a\right), d\left(x_{n-1,j}, x_{n,j}, a\right), 2d\left(x_{n-1,j}, x_{n,j}, a\right), d\left(x_{n,j}, x_{n-1,j}, a\right)\right) \\ & \leq q_j d\left(x_{n-1,j}, x_{n,j}, a\right). \end{aligned} \tag{6}$$

(6) implies that $\left\{x_{n,j}\right\}_{n\in\mathbb{N}}$ is a cauchy sequence by Lemma 1, hence by the completeness of $X$, $\left\{x_{n,j}\right\}_{n\in\mathbb{N}}$ converges to some element $x_j \in X$. (7)

Now, we prove that $x_j$ is the unique common fixed point of $\left\{S_{n,j}\right\}_{n\in\mathbb{N}}$. In fact, for any fixed $n \in \mathbb{N}$ and any $x_{m+1,j}$ with $m+1 > n$ and any $a \in X$,

$$\begin{aligned} & d\left(x_j, S_{n,j}\left(x_j\right), a\right) \leq d\left(x_{m+1,j}, S_{n,j}\left(x_j\right), a\right) + d\left(x_j, x_{m+1,j}, a\right) + d\left(x_j, S_{n,j}\left(x_j\right), x_{m+1,j}\right) \\ & = d\left(S_{m+1,j}\left(x_{m,j}\right), S_{n,j}\left(x_j\right), a\right) + d\left(x_j, x_{m+1,j}, a\right) + d\left(x_j, S_{n,j}\left(x_j\right), x_{m+1,j}\right) \\ & \leq q_j \phi_j \left(\left(d(x_{m,j}, S_{m+1,j}\left(x_{m,j}\right), a\right), d\left(x_j, S_{n,j}\left(x_j\right), a\right), d\left(x_{m,j}, S_{n,j}\left(x_j\right), a\right), d\left(S_{m+1,j}\left(x_{m,j}\right), x_j, a\right), d\left(x_{m,j}, x_j, a\right)\right) \\ & \quad + d\left(x_j, x_{m+1,j}, a\right) + d\left(x_j, S_{n,j}\left(x_j\right), x_{m+1,j}\right) \\ & = q_j \phi_j \left(d\left(x_{m,j}, x_{m+1,j}, a\right), d\left(x_j, S_{n,j}\left(x_j\right), a\right), d\left(x_{m,j}, S_{n,j}\left(x_j\right), a\right), d\left(x_{m+1,j}, x_j, a\right), d\left(x_{m,j}, x_j, a\right)\right) \\ & \quad + d\left(x_j, x_{m+1,j}, a\right) + d\left(x_j, S_{n,j}\left(x_j\right), x_{m+1,j}\right) \end{aligned}$$

Let $m \to +\infty$, then by Lemma 2, the continuity of $\phi_j$ and (7), the above becomes

$$\begin{aligned} & d\left(x_j, S_{n,j}\left(x_j\right), a\right) \\ & \leq q_j \phi_j \left(0, d\left(x_j, S_{n,j}\left(x_j\right), a\right), d\left(x_j, S_{n,j}\left(x_j\right), a\right), 0, 0\right). \\ & \leq q_j d\left(x_j, S_{n,j}\left(x_j\right), a\right) \end{aligned}$$

But $0 < q_j < 1$, hence $d\left(x_j, S_{n,j}\left(x_j\right), a\right) = 0$ for all $a \in X$, and therefore, $S_{n,j}\left(x_j\right) = x_j$ for all $n \in \mathbb{N}$. This completes that $x_j$ is a common fixed point of $\left\{S_{n,j}\right\}_{n\in\mathbb{N}}$.

Let $y_j$ be a common fixed point of $\left\{S_{n,j}\right\}_{n\in\mathbb{N}}$, If there exists an $a \in X$ such that $d\left(x_j, y_j, a\right) > 0$, then

$$\begin{aligned} & d\left(x_j, y_j, a\right) = d\left(S_{n+1,j}\left(x_j\right), S_{n,j}\left(y_j\right), a\right) \\ & \leq q_j \phi_j \left(d\left(x_j, S_{n+1,j}\left(x_j\right), a\right), d\left(y_j, S_{n,j}\left(y_j\right), a\right), d\left(x_j, S_{n,j}\left(y_j\right), a\right), d\left(S_{n+1,j}\left(x_j\right), y_j, a\right), d\left(x_j, y_j, a\right)\right) \\ & = q_j \phi_j \left(0, 0,, d\left(x_j, y_j, a\right), d\left(x_j, y_j, a\right), d\left(x_j, y_j, a\right)\right) < q_j d\left(x_j, y_j, a\right), \end{aligned}$$

which is a contradiction since $0 < q_j < 1$, hence $d\left(x_j, y_j, a\right) = 0$ for all $a \in X$, and therefore $x_j = y_j$. This completes that $\left\{S_{n,j}\right\}_{n\in\mathbb{N}}$ has an unique common fixed point $x_j$ for all $j \in \mathbb{N}$.

Next, we will prove that $x_j$ is the unique common fixed point of $\left\{T_{n,j}\right\}_{n\in\mathbb{N}}$ for each fixed $j \in \mathbb{N}$. Indeed, for fixed $j \in \mathbb{N}$, Since $x_j = S_{n,j}\left(x_j\right) = T_{n,j}^{m_{n,j}}\left(x_j\right)$ for each $n \in \mathbb{N}$, hence

 

$$T_{n,j}(x_j) = T_{n,j}\left(T_{n,j}^{m_{n,j}}(x_j)\right) = T_{n,j}^{m_{n,j}}\left(T_{n,j}(x_j)\right) = S_{n,j}\left(T_{n,j}(x_j)\right)$$

for each $n \in \mathbb{N}$, which means that $T_{n,j}(x_j)$ is a fixed point of $S_{n,j}$ for each $n \in \mathbb{N}$. Now, fix $n \in \mathbb{N}$ and let $i \in \mathbb{N}$ with $i \neq n$, if there exists an $a \in X$ such that $d\left(T_{n,j}(x_j), S_{i,j}\left(T_{n,j}(x_j)\right), a\right) > 0$, then

$$\begin{aligned}
&d\left(T_{n,j}(x_j), S_{i,j}\left(T_{n,j}(x_j)\right), a\right) = d\left(S_{n,j}\left(T_{n,j}(x_j)\right), S_{i,j}\left(T_{n,j}(x_j)\right), a\right) \\
&\leq q_j\phi_j(d\left(T_{n,j}(x_j), S_{n,j}\left(T_{n,j}(x_j)\right), a\right), d\left(T_{n,j}(x_j), S_{i,j}(T_{n,j}(x_j)), a\right), \\
&\quad d\left(T_{n,j}(x_j), S_{i,j}\left(T_{n,j}(x_j)\right), a\right), d\left(S_{n,j}\left(T_{n,j}(x_j)\right), T_{n,j}(x_j), a\right), d\left(T_{n,j}(x_j), T_{n,j}(x_j), a\right) \\
&= q_j\phi_j\left(0, d\left(T_{n,j}(x_j), S_{i,j}\left(T_{n,j}(x_j)\right), a\right), d\left(T_{n,j}(x_j), S_{i,j}\left(T_{n,j}(x_j)\right), a\right), 0, 0\right) \\
&< q_j d\left(T_{n,j}(x_j), S_{i,j}(T_{n,j}(x_j)), a\right),
\end{aligned}$$

which is a contradiction since $0 < q_j < 1$, hence $d\left(T_{n,j}(x_j), S_{i,j}\left(T_{n,j}(x_j)\right), a\right) = 0$ for all $a \in X$, and therefore $T_{n,j}(x_j) = S_{i,j}\left(T_{n,j}(x_j)\right)$. This means that $T_{n,j}(x_j)$ is a common fixed point of $\{S_{i,j}\}_{i\in\mathbb{N}}$. But $x_j$ is the unique common fixed point of $\{S_{i,j}\}_{i\in\mathbb{N}}$, hence $T_{n,j}(x_j) = x_j$ for all $n \in \mathbb{N}$, which means that $x_j$ is a common fixed point of $\{T_{n,j}\}_{n\in\mathbb{N}}$ for all $j \in \mathbb{N}$.

If $y_j$ is a common fixed point of $\{T_{n,j}\}_{n\in\mathbb{N}}$, then $S_{n,j}(y_j) = T_{n,j}^{m_{n,j}}(y_j) = T_{n,j}(y_j) = y_j$ for all $n \in \mathbb{N}$, which means that $y_j$ is a common fixed point of $\{S_{i,j}\}_{i\in\mathbb{N}}$. But $x_j$ is the unique common fixed point of $\{S_{i,j}\}_{i\in\mathbb{N}}$, hence $x_j = y_j$. This completes that $\{T_{n,j}\}_{n\in\mathbb{N}}$ has the unique common fixed point $x_j$ for each $j \in \mathbb{N}$.

Finally, we will prove that $x_\mu = x_\nu$ for all $\mu, \nu \in \mathbb{N}$. In fact, for any fixed $m, n, \mu, \nu \in \mathbb{N}$ with $\mu \neq \nu$, since $T_{m,\mu}(x_\mu) = x_\mu$ and $T_{n,\nu}(x_\nu) = x_\nu$, hence $T_{n,\nu}(x_\mu) = T_{n,\nu}\left(T_{m,\mu}(x_\mu)\right) = T_{m,\mu}\left(T_{n,\nu}(x_\mu)\right)$ by condition 2). Which means that $T_{n,\nu}(x_\mu)$ is a common fixed point of $\{T_{m,\mu}\}_{m\in\mathbb{N}}$ for all $\mu \in \mathbb{N}$. But the unique com- mon fixed point of $\{T_{m,\mu}\}_{m\in\mathbb{N}}$ is $x_\mu$, hence $T_{n,\nu}(x_\mu) = x_\mu$ for all $\mu \in \mathbb{N}$, this means that $x_\mu$ is a common fixed point of $\{T_{n,\nu}\}_{n\in\mathbb{N}}$, and therefore $x_\mu = x_\nu$ since $x_\nu$ is the unique common fixed point of $\{T_{n,\nu}\}_{n\in\mathbb{N}}$. Let $x^* = x_j$, then $x^*$ is the unique common fixed point of $\{T_{i,j}\}_{i,j\in\mathbb{N}}$.

The following is a particular form of Theorem 2.1:

**Theorem 2.2.** Let $(X, d)$ be a complete 2-metric space, $\{f_i\}_{i\in\mathbb{N}}$ a family of maps from $X$ into itself and $0 < q < 1$ and $\phi \in \Phi$. If the following $\phi$-contractive condition holds

$$\begin{aligned}
&d(f_i x, f_j y, a) \leq q\phi\big(d(x, f_i x, a), d(y, f_j y, a), \\
&d(x, f_j y, a), d(f_i x, y, a), d(x, y, a)\big), \\
&\forall\, x, y, a \in X, i \neq j,
\end{aligned}$$

then $\{f_i\}_{i\in\mathbb{N}}$ has an unique common fixed point in $X$.

Next theorem is the main result in [5].

**Theorem 2.3.** Let $(X, d)$ be a complete 2-metric space, $\{T_i\}_{i=1}^{\infty}$ a family of maps from $X$ into itself. If there exist a family non-negative integers $\{m_i\}_{i=1}^{\infty}$ and non-negative real numbers $\alpha, \beta, \gamma$ with $\alpha + \beta + \gamma < 1$ such that for all $x, y, a \in X$ and all natural numbers $i, j$ with $i \neq j$, the following holds

$$\begin{aligned}
&d\left(T_i^{m_i} x, T_j^{m_j} y, a\right) \\
&\leq \alpha d\left(x, T_i^{m_i} x, a\right) + \beta d\left(y, T_j^{m_j} y, a\right) + \gamma d(x, y, a).
\end{aligned}$$

Then $\{T_i\}_{i=1}^{\infty}$ have an unique common fixed point in $X$.

**Remark.** Obviously, Theorem 2.3 is a very particular form of Theorem 2.1. In fact, Let $\phi(u_1, u_2, u_3, u_4, u_5) = \alpha u_1 + \beta u_2 + \gamma u_5$, and take $0 < q < 1$ satisfying $\frac{\alpha + \beta + \gamma}{q} < 1$, then $\phi$ and $q$ sa- tisfy all conditions of Theorem 2.1. Hence we sure that our main result generalized and improve many corres- ponding common fixed point theorems in 2-metric spaces.

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