Applied Mathematics, 2011, 2, 1546-1550
doi:10.4236/am.2011.212220 Published Online December 2011 (http://www.SciRP.org/journal/am)

Scientific Research

Test of Generating Function and Estimation of Equivalent Radius in Some Weapon Systems and Its Stochastic Simulation

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Received November 16, 2011; revised December 6, 2011; accepted December 14, 2011

Abstract

We discuss three-dimensional uniform distribution and its property in a sphere; give a method of assessing the tactical and technical indices of cartridge ejection uniformity in some type of weapon systems. Meanwhile we obtain the test of generating function and the estimation of equivalent radius. The uniformity of distribution is tested and verified with ω^2 test method on the basis of stochastic simulation example.

Keywords: Uniform Distribution in a Sphere, Weapon Systems, Generating Function, Equivalent Radius, Stochastic Simulation

1. Introduction

Uniform distribution is very important in the probability statistics, many scholars pay attention to it. The following questions have been explored: the estimate of interval length about uniform distribution in $[a,b]$ [1,2], the estimate of regional area about two dimension uniform distribution in a rectangle [3], the estimate of cuboid volume about three dimension uniform distribution [4], the estimate of regional area about two-dimensional uniform distribution in a circle [5,6], estimate of radius on three-dimensional uniform distribution in a sphere [7]. In addition, many scholars get useful test statistics and limit theorems [8-12]. In this paper, basing on some articles [13-18], according to t the indices of cartridge ejection uniformity in some type of weapon systems, we give the test of generating function and the estimation of equivalent radius by simulation example.

Definition 1 [7]. If (X,Y,Z) is three-dimensional continuous random variable, its probability density function is

$$f(x,y,z)=\begin{cases}\dfrac{3}{4\pi R_0^3}, & (x,y,z)\in G,\\ 0 & (x,y,z)\notin G.\end{cases} \tag{1.1}$$

where $G=\left\{(x,y,z)\middle|x^2+y^2+z^2<R_0^2\right\}, R_0>0$, then we call that (X,Y,Z) obeys uniform distribution in $G=\left\{(x,y,z)\middle|x^2+y^2+z^2<R_0^2\right\}$, recorded as $(X,Y,Z)\sim U(G)$.

Give a transformation

$$\begin{cases}x=r\sin\varphi\cos\theta\\ y=r\sin\varphi\sin\theta\\ z=r\cos\varphi\end{cases} (0<r<R_0, 0<\varphi<\pi, 0<\theta<2\pi) \tag{1.2}$$

The probability density function of three-dimensional $r.v.$ (R,Φ,Θ) is

$$h(r,\varphi,\theta)=f(r\sin\varphi\cos\theta, r\sin\varphi\sin\theta, r\cos\varphi)\left|\frac{\partial(x,y,z)}{\partial(r,\varphi,\theta)}\right| \tag{1.3}$$

in which $0<r<R_0, 0<\varphi<\pi, 0<\theta<2\pi$, $\dfrac{\partial(x,y,z)}{\partial(r,\varphi,\theta)}$ is Jacobi determinant of the transformation (1.2), and

$$\frac{\partial(x,y,z)}{\partial(r,\varphi,\theta)}=\begin{vmatrix}\dfrac{\partial x}{\partial r} & \dfrac{\partial x}{\partial\varphi} & \dfrac{\partial x}{\partial\theta}\\ \dfrac{\partial y}{\partial r} & \dfrac{\partial y}{\partial\varphi} & \dfrac{\partial y}{\partial\theta}\\ \dfrac{\partial z}{\partial r} & \dfrac{\partial z}{\partial\varphi} & \dfrac{\partial z}{\partial\theta}\end{vmatrix}$$

$$=\begin{vmatrix}\sin\varphi\cos\theta & r\cos\varphi\cos\theta & -r\sin\varphi\sin\theta\\ \sin\varphi\sin\theta & r\cos\varphi\sin\theta & r\sin\varphi\cos\theta\\ \cos\varphi & -r\sin\varphi & 0\end{vmatrix} \tag{1.4}$$

$$=r^2\sin\varphi$$

Therefore the probability density function of (R,Φ,Θ) is

$$h(r,\varphi,\theta)=\begin{cases}\dfrac{3r^2\sin\varphi}{4\pi R_0^{\ 3}}, & 0<r<R_0, 0<\varphi<\pi, 0<\theta<2\pi \\ 0, & \text{otherwise}\end{cases} \tag{1.5}$$

Theorem 1. If the marginal density functions of $r.v.$ (R,Φ,Θ) about R,Φ,Θ are $h_1(r)$, $h_2(\varphi)$, $h_3(\theta)$, then

1) $h_1(r)=\begin{cases}\dfrac{3r^2}{R_0^3}, & 0<r<R_0, \\ 0, & \text{otherwise.}\end{cases}$

2) $h_2(\varphi)=\begin{cases}\dfrac{1}{2}\sin\varphi, 0<\varphi<\pi, \\ 0, \quad \text{otherwise.}\end{cases}$

3) $h_3(\theta)=\begin{cases}\dfrac{1}{2\pi}, & 0<\theta<2\pi, \\ 0, & \text{otherwise.}\end{cases}$

Proof. According to (1.5) and the definition of marginal density function, we have

$$\begin{aligned}h_1(r)&=\int_0^{\pi}\int_0^{2\pi}h(r,\varphi,\theta)\mathrm{d}\varphi\mathrm{d}\theta\\&=\int_0^{\pi}\int_0^{2\pi}\frac{3r^2\sin\varphi}{4\pi R_0^3}\mathrm{d}\varphi\mathrm{d}\theta=2\pi\times\frac{3r^2}{4\pi R_0^{\ 3}}\int_0^{\pi}\sin\varphi\mathrm{d}\varphi\\&=2\pi\times\frac{3r^2}{4\pi R_0^{\ 3}}\times 2=\frac{3r^2}{R_0^{\ 3}},\end{aligned}$$

where $0<r<R_0$,

$$\begin{aligned}h_2(\varphi)&=\int_0^{R_0}\int_0^{2\pi}h(r,\varphi,\theta)\mathrm{d}r\mathrm{d}\theta=\int_0^{R_0}\int_0^{2\pi}\frac{3r^2\sin\varphi}{4\pi R_0^{\ 3}}\mathrm{d}r\mathrm{d}\theta\\&=2\pi\times\frac{3\sin\varphi}{4\pi R_0^3}\int_0^{R_0}r^2\mathrm{d}r=\frac{1}{2}\sin\varphi,\end{aligned}$$

where $0<\varphi<\pi$,

$$\begin{aligned}h_3(\theta)&=\int_0^{R_0}\int_0^{\pi}h(r,\varphi,\theta)\mathrm{d}r\mathrm{d}\varphi=\int_0^{R_0}\int_0^{\pi}\frac{3r^2\sin\varphi}{4\pi R_0^{\ 3}}\mathrm{d}r\mathrm{d}\varphi\\&=\frac{3}{4\pi R_0^3}\int_0^{R_0}r^2\mathrm{d}r\int_0^{\pi}\sin\varphi\mathrm{d}\varphi=\frac{3}{4\pi R_0^{\ 3}}\times\frac{1}{3}R_0^{\ 3}\times 2\\&=\frac{1}{2\pi},\end{aligned}$$

where $0<\theta<2\pi$.

Corollary 1 [7]. If $r.v.$ (R,Φ,Θ) is defined by (1.5), then three $r.v.$ R,Φ,Θ are independent each other.

Corollary 2. If $r.v.$ (R,Φ,Θ) is defined by (1.5), the marginal distribution function of $r.v.$ (R,Φ,Θ) about R,Φ,Θ are $H_1(r),H_2(\varphi),H_3(\theta)$, then

$$H_1(r)=\begin{cases}0, & r\le 0\\ \dfrac{r^3}{R_0^3}, & 0<r<R_0\\ 1, & r\ge R_0\end{cases},$$

$$H_2(\varphi)=\begin{cases}0, & \varphi\le 0\\ \dfrac{1}{2}(1-\cos\varphi), & 0<\varphi<\pi\\ 1, & \varphi\ge\pi\end{cases},$$

$$H_3(\theta)=\begin{cases}0, & \theta\le 0\\ \dfrac{\theta}{2\pi}, & 0<\theta<2\pi\\ 1, & \theta\ge 2\pi\end{cases}.$$

Proof. According to theorem 1, we can get it easily.

Corollary 3. If $E(R)=\mu$, $Var(R)=\sigma^2$, then the probability of cartridges falling into a ball with radius μ is about 42.2%, and the probability of cartridges falling into a sphere with radius $\mu+\sigma$ is about 84.0%.

Proof. By the definition of Mathematical expectation, we have

$$\mu=E(R)=\int_0^{R_0}rh_1(r)\mathrm{d}r=\frac{3}{4}R_0 \tag{1.6}$$

$$E(R^2)=\int_0^{R_0}r^2h_1(r)\mathrm{d}r=\int_0^{R_0}\frac{3r^4}{R_0^3}\mathrm{d}r=\frac{3}{5}R_0^2 \tag{1.7}$$

then

$$\sigma^2=D(R)=E(R^2)-E^2(R)=\frac{3}{5}R_0^2-\frac{9}{16}R_0^2=\frac{3}{80}R_0^2$$

and $\sigma=\frac{\sqrt{15}}{20}R_0$, then the probability of cartridges falling into a sphere with radius μ is about

$$H_1(\mu)=H_1\left[E(R)\right]=H_1\left[\frac{3}{4}R_0\right]\approx 42.2\% \tag{1.8}$$

then the probability of cartridges falling into a sphere with radius $\mu+\sigma$ is about

$$H_1\left(\mu+\sigma\right)=H_1\left(\frac{3}{4}R_0+\frac{\sqrt{15}}{20}R_0\right)\approx 84.0\% \tag{1.9}$$

2. Test of Generating Distribution Function

Usually there are χ^2 test method, ω^2 test method and Cole Moge Rove test method (K test method) [17] to test distribution function. Here, we use ω^2 test method, we want to know the sub-sample is uniform distribution or not. Because the locations of any cartridges are ascertained by three-dimensional $r.v.$ (R,Φ,Θ), so we should test them one by one, test $R \sim H_1(r)$, $\Phi \sim H_2(\varphi)$, $\Theta \sim H_3(\theta)$. We give testing hypotheses H_0

$$H_0: F(x) = \Phi_0(x)$$

where $F(x)$ is generating distribution function, $\Phi_0(x)$ is known distribution function, and $\varphi_0(x)$ is the derivative of $\Phi_0(x)$.

Tests for generating function should be independent, $y_{(1)}, y_{(2)}, \cdots, y_{(n)}$ is the sequent sub-sample of the test, under hypotheses H_0 is correct, the statistic

$$n\omega^2 = \frac{1}{12n} + \sum_{i=1}^{n}\left[\Phi_0(y_{(i)}) - \frac{2i-1}{2n}\right]^2 \qquad (2.1)$$

is Smirnov distribution. For the given confidence level α, according to the Table 10 in [17], we obtain the boundary value z_α of $n\omega^2$, in which $P(n\omega^2 > z_\alpha) = \alpha$. Then (z_α, ∞) is rejection region of the hypotheses H_0, when $n\omega^2 > z_\alpha$, we reject H_0, if $n\omega^2 \le z_\alpha$, we should accept H_0.

3. Estimation of Equivalent Radius

On the supposition that N is the number of cartridges from a shrapnel, n is the actual observed number of cartridges within a certain region near the centre of dispersion. When calculating equivalent radius, we presume all the cartridges are found. The distances from any cartridges to the dispersion centre point A are recorded as $r_i (i = 1, 2, \cdots, n)$, let $\overline{r} = \frac{1}{n}\sum_{i=1}^{n} r_i$. According to the properties of density function $h_1(r)$, we know that R obeys uniform distribution in a ball with radius

$r_{n0} (0 < r_{n0} < R_0)$, $E(R) = \frac{3}{4} r_{n0}$ (by 1.6), owing to

$$E(\overline{r}) = E\left(\frac{1}{n}\sum_{i=1}^{n} r_i\right) = \frac{1}{n}\sum_{i=1}^{n}\int_0^{r_{n0}} r_i \frac{3r_i^2}{r_{n0}^3} dr_i = \frac{3}{4} r_{n0} \qquad (3.1)$$

so $\hat{r}_{n0}$ is a unbiased estimate of r_{n0} [7]. on the basis of the properties of distribution function, let $\varphi = \pi, \theta = 2\pi$, we have $H(r) = H(r, \varphi, \theta)\Big|_{\theta = 2\pi}^{\varphi = \pi}$, let $N \to \infty$, (N is amount of test cartridges), then

$$\lim_{N \to \infty} H(r, \varphi, \theta)\Big|_{\theta = 2\pi}^{\varphi = \pi} = \lim_{N \to \infty} \frac{n}{N} = \frac{r_{n0}^3}{R_0^3} \qquad (3.2)$$

Therefore let $\hat{R}_0 = \sqrt{\frac{N}{n}} \hat{r}_{n0}$, that is $\hat{R}_0 = \frac{4}{3n}\sqrt{\frac{N}{n}}\sum_{i=1}^{n} r_i$, and

$$D(\hat{R}_0) = \frac{16N}{9n^2} D(r_i) \qquad (3.3)$$

As well as by (1.7), $D(r_i) = \frac{3}{80} r_{n0}^2$, substitute into (3.3), we obtain

$$\begin{aligned} D(\hat{R}_0) &= \frac{16N}{9n^2} D(r_i) = \frac{16N}{9n^2} \times \frac{3}{80} r_{n0}^2 = \frac{N r_{n0}^2}{15n^2}, \\ \sigma(\hat{R}_0) &= \frac{\sqrt{15N} r_{n0}}{15n} \end{aligned} \qquad (3.4)$$

Based on Formula (3.4), if N is large enough, when $n \to N$, $\sigma(\hat{R}_0)$ becomes small enough. In order to decrease estimating error of equivalent radius R_0, sample size is large enough.

4. Stochastic Simulation

In order to verify the correctness of above methods, we give a simulation example. For some type of weapon systems, assuming the tactical technical requirements, the cartridges from single shrapnel should be more evenly scattered in a ball with equivalent radius (120 ± 20) m, launching a shrapnel, the number of cartridges is N = 400, measuring the coordinates of one hundred cartridges near the dispersion centre (n = 100), they were produced by computer simulation basing on uniform requirements in a sphere, *i.e.* (r, φ, θ) were produced by stochastic simulation according to the following formulas

$$\begin{aligned} r = \sqrt[3]{H_1(r)} r_{n0}, \varphi &= \arccos\left[1 - 2H_2(\varphi)\right], \\ \theta &= 2\pi H_3(\theta) \end{aligned} \qquad (4.1)$$

where, $r_{n0} = 60$, $H_1(r)$, $H_2(\varphi)$, $H_3(\theta)$ are random number produced by stochastic simulation in (0,1), coordinates for cartridges as below **Table 1**, and the MATLAB program as below,

```
>> clear
for k=1:100
    r=rand(1,3);
        x=60*sqrt(r(1));
        y=acos(-2*r(2)+1);
        z=2*pi*r(3);
[x, y, z]
  end
>>clear
for k=1:100
a=(x^2-((2*k-1)/200))^2;
b=(y^2/3600-((2*k-1)/200))^2;
c=(z^2/3600-((2*k-1)/200))^2;
[a, b, c]
end

>> clear
for k=1:100
s=1/1200+sum(a);
u=1/1200+sum(b);
v=1/1200+sum(c);
[s u v]
End
```

Table 1. Polar coordinates points of fall for cartridges.

$(r/m, \varphi/rad, \theta/rad)$			$(r/m, \varphi/rad, \theta/rad)$			$(r/m, \varphi/rad, \theta/rad)$			$(r/m, \varphi/rad, \theta/rad)$		
57.8262	1.1283	0.0936	56.9007	1.7096	0.0811	55.0439	0.4464	5.2025	47.7960	0.6985	5.5996
58.8251	1.3574	1.8108	33.8961	1.4923	1.9503	43.4513	2.1067	5.7655	33.1923	2.1410	3.0668
41.6750	1.5768	5.1315	19.1279	1.4612	4.8952	45.6907	2.4810	0.7106	48.3936	1.3183	6.2367
45.5454	2.0309	6.1921	54.1207	0.6013	1.9308	47.0793	1.1287	5.1026	51.7387	2.2742	2.3455
46.6895	1.1728	0.1093	48.7537	1.4576	5.8226	50.8327	1.0500	5.7070	15.1818	0.4352	3.3389
31.8301	0.6831	5.1484	39.0650	1.3001	4.2644	33.6245	2.6167	0.9827	56.5426	1.7679	1.1391
30.8478	1.4572	3.9025	43.0160	1.1647	0.4668	7.5595	0.7407	0.7672	55.7802	2.6869	3.1535
48.6323	1.5043	3.5198	14.0717	2.3512	0.4442	55.4707	2.6500	4.7922	53.2184	1.1347	2.6528
53.9209	0.2431	1.5331	57.6967	2.1165	0.0748	48.0499	1.9865	4.5352	46.3803	2.4616	4.1494
44.1604	1.9052	5.1648	57.1905	2.6896	1.4275	35.8438	2.3398	4.0941	26.1303	0.6488	4.2330
42.6194	2.0355	1.6537	38.0081	1.6869	3.2440	28.1616	0.9503	4.7375	56.1754	0.5168	6.0149
39.4989	1.1188	4.7350	49.7335	0.2389	2.8790	32.3887	1.4809	4.1670	34.6740	1.0105	1.2057
57.2477	1.0742	4.1444	32.5254	1.7644	4.4183	44.4830	0.5776	5.5512	45.8214	2.6183	0.6987
51.3376	2.0009	1.3452	50.4480	2.2554	3.6600	44.4939	2.3493	1.7103	14.1080	0.5078	3.5506
37.3488	2.1746	3.7831	41.5086	2.8378	3.1994	22.4951	1.6951	2.6352	40.5506	1.0797	6.0897
59.5588	2.9061	3.8007	52.2026	0.9810	0.4668	58.8127	1.2010	1.3383	57.3967	3.0969	0.1489
51.7199	1.5174	4.1438	57.1376	1.9904	1.2139	31.8798	1.3179	0.2237	56.5065	0.9570	5.4676
36.7509	2.5075	1.1523	49.6781	1.6150	2.3851	43.6260	2.3973	0.5102	41.5920	1.5676	0.1690
52.7956	1.4728	3.9992	59.6074	2.6175	1.7367	40.6554	1.3123	5.3445	57.5123	1.1385	3.2641
52.3921	2.2256	1.0700	55.5081	2.0117	4.8437	33.1399	0.5499	2.1375	46.9686	1.9237	1.2083
30.7568	2.2887	3.3904	32.0629	0.9956	1.9723	57.8563	0.9268	2.9292	49.4790	2.7288	4.4969
16.9386	0.8401	3.9169	56.4546	1.4698	4.0099	41.1545	0.4487	5.7416	51.0491	2.1332	1.5752
38.4027	1.3570	4.3096	34.6080	0.8558	6.1990	54.1232	1.7046	1.4363	52.2896	1.9094	5.8679
29.3040	1.6124	4.2556	51.6795	2.7865	3.1598	44.6063	0.7133	5.4161	51.0684	0.7404	0.8621
24.6450	2.0222	5.5091	52.4759	1.2780	5.9546	44.2010	1.6150	4.1255	52.8935	0.6280	3.2773

According to the data in **Table 1** and methods in this paper, using above MATLAB program, we obtain

$$n\omega_r^2 = 0.0888, n\omega_\varphi^2 = 0.0425, n\omega_\theta^2 = 0.0749$$

$$\hat{R}_0 = 117.4909, \sigma(\hat{R}_0) = 3.0984$$

Take conspicuous level $\alpha = 10\%$, seeing the Table 10 in [17], we obtain the boundary value $z_\alpha = 0.3472$ for $n\omega^2$. Because $n\omega_r^2$, $n\omega_\varphi^2$, $n\omega_\theta^2$ are less than 0.3472, so we consider that cartridges obeys uniform distribution in a sphere.

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