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An Efficient Combinatorial-Probabilistic Dual-Fusion Modification of Bernstein's Polynomial Approximation Operator

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Abstract

The celebrated Weierstrass Approximation Theorem (1885) heralded intermittent interest in polynomial approximation, which continues unabated even as of today. The great Russian mathematician Bernstein, in 1912, not only provided an interesting proof of the Weierstrass' theorem, but also displayed a sequence of the polynomials which approximate the given function $f(x) \in C[0,1]$. An efficient "Combinatorial-Probabilistic Dual-Fusion" version of the modification of Bernstein's Polynomial Operator is proposed. The potential of the aforesaid improvement is tried to be brought forth and illustrated through an empirical study, for which the function is assumed to be known in the sense of simulation.

Keywords: Approximation, Bernstein Operator, Dual-Fusion, Simulated Empirical Study

1. Introduction

The problem of approximation arises in many contexts of "Numerical Analyses and Computing" [1-4]. Weirstrass [5] proved his celebrated approximation theorem: "…If, $f \in C[a,b]$, then for every $\delta > 0$, $\exists$ a polynomial "*p*" such that "$\|f - p\| < \delta$". In other words, result established the existence of an algebraic polynomial in concerned variable capable of approximating the unknown function in that variable, as closely as we please!

This result was a big beginning of the Mathematicians' interest in "Polynomial Approximation" [4,6-8] of an unknown function using its values generated, experimentally or otherwise, at certain equidistant knots in the impugned interval of the relevant variable. The Great Russian mathematician Bernstein proved the Weirstrass theorem in a style, which was very thought-provoking and curious in many ways. He first noted a simple though a very significant feature of this theorem, namely that if it holds for $C[0,1]$, it does hold for $C[a,b]$ also vice-versa. In fact, $C[0,1]$ and $C[a,b]$ are essentially identical, for all practical purposes, inasmuch as they are linearly "isometric" as normed spaces, order isomorphic as lattices, and isomorphic as algebras (rings) [9].

Also, the most important contribution in the Bernstein's proof of the Weirstrass' theorem consisted in the fact that Bernstein actually displayed a sequence of polynomials that approximate a given function $f \in C[0,1]$.

If, $f(x)$ is any bounded function on $C[0,1]$, the sequence of "Bernstein Polynomials" [6] for $f(x)$ is defined by:

$$\left(\mathrm{Bn}(f)\right)(x) = \sum_{k=0}^{k=n} \binom{n}{k} . x^k \ . \ (1-x)^{(n-k)} . f(k/n) \quad (1)$$

$x \in \mathrm{C}[0,1]$, Say $E[f(x)]$

The aim of this paper is to propose a more efficient polynomial approximation operator exploiting the "combinatorial structure" of the "Bernstein's Polynomial Approximation operator", and the fact that the unknown function might, without any loss of the generality, be assumed to be $f \in C\left[0, \frac{1}{2}\right]$, as in [10-14].

2. The Proposition of the Variant of the Bernstein Polynomial Approximator

In context of the aforementioned sequence of "Bernstein Polynomials" for $f(x)$, a significant observation which must be taken note of is that the use is made of the values of the unknown function "$f(x)$" at the equidistant-knots "$\frac{k}{n}; k = 0(1)n$", assumed to be knowable through the experiment (s) in the relevant scientific field of investigation or known otherwise.

In any approximating polynomial operator use is made of the "Knots" and of the corresponding "Weights".

In our proposition of a variant of the "Bernstein's Polynomial" we propose to systematically introduce new corresponding weights, without essentially changing the location of the "equi-distant" "knots", except for the fact that the impugned interval is $C\left[0, \frac{1}{2}\right]$, rather than $C[0,1]$, thanks to "isometric" spaces noted earlier. We propose a variant of the "Bernstein Polynomial" which is having a better combinatorial structure in favor of the interval for $f \in C\left[0, \frac{1}{2}\right]$, and that is the main strategy for making it better than the original/usual "Bernstein's Polynomial"! We consider the following PRIMAL variant of the Bernstein's Polynomial:

Say,

$$B^P(f;x)[n] = \sum_{k=0}^{k=n} \binom{n}{k} .(0.67+x)^k \cdot (0.33-x)^{(n-k)} \cdot f\left(k/(2^*n)\right)$$

3. The Combinatorial-Probability & Dual-Fusion Variant for Bernstein's Polynomial

The original interval $C[0,1]$ isometric to the impugnned interval $C\left[0, \frac{1}{2}\right]$ could be thought of in terms of its two parts, namely "$0.33-x$" and "$0.67+x$" for $x \in C[0,1]$ with "k" and "$n-k$" "knots" sitting in each, respectively. The expected number of points sitting these two parts. Respectively, would be $n^{(0.33-x)}$ and $n^{(0.67+x)}$. The "combinatorial probability" of description would be: binomial

$$\left(n^*(0.33-x), k\right) \text{binomial}\left(n^*(0.67+x), n-k\right) / \text{binomial}(n,n)$$

$$\equiv \text{binomialn}\left((0.33-x), n-k\right) \text{binomialn}\left((0.67+x), n-k\right)$$

Hence the PRIMAL-Variant of the "Bernstein's Polynomial" in (2.1) comes off to be as below.

Say,

$$BV^P(f;x)[n] = \sum_{k=0}^{k=n} \text{binomial } \left(n^*(0.33-x), k\right) * \text{ binomial } \left(n^*(0.67+x), n-k\right).f\left(k/(2^*n)\right)$$

The correspondingly DUAL (-Weights) variant of Bernstein Polynomial would be:

Say,

$$BV^P(f;x)[n] = \sum_{k=0}^{k=n} \text{binomial } \left(n^*(0.33+x), k\right) * \text{ binomial } \left(n^*(0.67-x), n-k\right) f\left(k/(2^*n)\right)$$

We define the "PRIMAL-DUAL-Fusion-Weights" variant of the Bernstein Polynomial as

Follows: Say,

$$PDFBV(f;x)[n] = \left[BV^P(f;x) + BV^D(f;x)\right]/2$$

To make comprehensive the combinatorial systematicness of PRIMAL-DUAL Fusion variant of Bernstein polynomial say $PDFBV(f;x)[n]$, we note that it will work for an approximation polynomial focusing interval $[(1/3-x)/2, (1/3+x)/2]$ around "0.33", which will $\sim [0,1/3]$ for $x = 1/3$.

Impugned interval will be wider, the greater the value of "$x < 1/3$"! For example, in the approximating polynomial in "x" for values of $x \in [0,1/3]$; interval will be symmetrically, centered on "0.165", e.g. $\sim [0.035, 0.295]$ for $x = 0.26$.

To balance the "Pull", systematically, the weights "(1/2)" and "(1/2)" are assigned to the relevant weights in $B^P(f;x)[n]$ & $B^D(f;x)[n]$. These weights are also, respectively, "DUAL" to each-other, again!

The aforesaid (PRIMAL-DUAL Fusion) variant of the Bernstein Polynomial, namely, $PDFBV(f;x)[n]$ will, apparently induce a "(Systematic)Bias" in the approximating "Polynomial", which is amenable more systematically than that in the original "Bernstein's Polynomial".

Similar to what was noted in Sahai (2004) [10], in the absence of any conclusive analytical study [The derivable "Upper" bounds on the error of approximation (as noted in the paper by Sahai (2004) [8]) are not of much use. In fact, a smaller/lower "Upper Bound" does not guarantee a better approximation and the extent of the resultant "GAIN" is unavailable, too! Hence, we go for an empirical simulation study to illustrate the potential "GAIN" through our PRIMAL-DUAL Fusion variant of the Bernstein Polynomial, namely, $PDFBV(f;x)[n]$.

4. The Empirical Simulation Study

To illustrate gain in efficiency by using our proposed "Dual-Fusion" variant of Bernstein Polynomial Approximation, we have carried an empirical study. We have taken example-cases of n = 3, 6, and 9 (*i.e.* n + 1 = 4, 7, and 10 knots) in the empirical study.

To numerically illustrate the relative gain in efficiency in using "Dual-Fusion" variant of Bernstein Polynomial proposed vis-à-vis the original (Primal) Bernstein's polynomial operator in each example case of *n*-value.

Essentially, the empirical study is a simulation one wherein we would assume that approximated function, namely "$f(x)$", is known to us.

We have confined to illustrations of relative gain in efficiency by Iterative Improvement for the following four illustrative-functions:

$$f(x) = \exp(x);\ \ln(2+x);\ \sin(2+x),\ \text{and } 10^x.$$

To illustrate the POTENTIAL of improvement with our proposed Dual-Fusion Operator $\mathrm{PDFBV}(f;x)[n]$', we have TWO numerical values of quantities ~ two percentage relative errors (PREs) corresponding to original (Primal) Bernstein's Operator $\mathrm{B}^{\mathrm{P}}(f;x)[n]$: Say; $\mathrm{PRE\_PFB}(f;x)[n]$ verses that of the proposed Dual-Fusion Operator *i.e.*; $\mathrm{PRE\_PDFBV}(f;x)[n]$. We calculated Percentage Relative Gains (PRGs) in using our "Dual-Fusion" variant of Bernstein Polynomial in place of Original "Primal" variant of Bernstein Polynomial $\mathrm{PRG\_UPDFB}(f;x)[n]$. These quantities are defined: $\mathrm{PRE\_PFB}(f;x)[n] = 100$.

$$\left[\left\{\int_0^{0.33} \text{abs.}\left(\mathrm{PFB}\ (f;x)[n] - f\ (x)\right)\mathrm{d}x\right\} \Big/ \int_0^{0.33} f(x)\ \mathrm{d}x\right]$$

$$\&\mathrm{PRE\_PDFBV}(f;x)[n] = 100.$$

$$\left[\left\{\int_0^{0.33} \text{abs.}\left(\mathrm{PDFBV}\ (f;x)[n] - f\ (x)\right)\mathrm{d}x\right\} \Big/ \int_0^{0.33} f(x)\ \mathrm{d}x\right]$$

.

Hence, $\mathrm{PRG\_\ PDFBV}(f;x)[n] = 100$.

$$\left[\left\{\mathrm{PRE\_PFB}(f;x)[n] - \mathrm{PRE\_PDFBV}(f;x)[n]\right\} \big/ \mathrm{PRE\_PFB}(f;x)[n]\right\}\right]$$

PREs for Original-Primal/Variant Primal-Dual Bernstein polynomial respectively for each of example # of approximation Knots/Intervals.

PRGs by using Proposed Dual-Fusion Polynomials with the *n* intervals in $[0,1/2]$ over using the Original Primal-Bernstein Polynomial for approximation of function, "$f(x)$" are tabulated in APPENDIX in **Tables 1-4**.

5. Conclusions

For all the FOUR illustrative functions, namely $f(x) = \exp 9x;\ \ln(2+x);\ \sin(2+x)$, and 10^x, the PRGs are above 99.9% for $n = 3, 6,$ and 9. It is very significant to note that the PRGs are (almost) 100% for $n = 6$ for all example-functions, *i.e.* for only SEVEN "Knots"!

6. References

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APPENDIX

Table 1. (Iterative) algorithmic (In %) relative (absolute) efficiency/gain for $f(x) = \exp(x)$.

Items ↓	$n \to 3$	6	9
PRE_PFB $(f; x)[n]$	7.7098	7.97506	8.0614
PRE_PDFBV $(f; x)[n]$	0.0004	0.00000	0.0000
PRG_PDFBV $(f; x)[n]$	99.994	100.000	99.999

Table 2. (Iterative) algorithmic (In %) relative (absolute) efficiency/gain for $f(x) = \ln(2 + x)$.

Items ↓	$n \to 3$	6	9
PRE_PFB $(f; x)[n]$	5.0970	5.0217	4.9961
PRE_PDFBV $(f; x)[n]$	0.0001	0.0000	0.0000
PRG_PDFBV $(f; x)[n]$	99.996	100.00	99.999

Table 3. (Iterative) algorithmic (In %) relative (absolute) efficiency/gain for $f(x) = \sin(2 + x)$.

Items ↓	$n \to 3$	6	9
PRE_PFB $(f; x)[n]$	5.0404	5.3105	5.4020
PRE_PDFBV $(f; x)[n]$	0.0004	0.0000	0.0000
PRG_PDFBV $(f; x)[n]$	99.991	99.999	99.999

Table 4. (Iterative) algorithmic (In %) relative (absolute) efficiency/gain for $f(x) = 10^x$.

Items ↓	$n \to 3$	6	9
PRE_PFB $(f; x)[n]$	16.112	17.498	17.935
PRE_PDFBV $(f; x)[n]$	0.0120	0.0000	0.0000
PRG_PDFBV $(f; x)[n]$	99.925	99.999	99.999

