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On Solutions of Generalized Bacterial Chemotaxis Model in a Semi-Solid Medium

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Abstract

In this paper, the Adomian's decomposition method has been developed to yield approximate solution of bacterial chemotaxis model of fractional order in a semi-solid medium. The fractional derivatives are described in the Caputo sense. The method introduces a promising tool for solving many linear and nonlinear fractional differential equations.

Keywords: Decomposition Method, Bacterial Chemotaxis, Semi-Solid Medium, Fractional Calculus

1. Introduction and Preliminaries

This paper deals with numerical solutions of bacterial chemotaxis model of fractional order in a semi-solid medium. We are primarily interested in describing the behaviour of the generalized biological mechanisms that govern the bacterial pattern formation processes in the experiments of Budrene and Berg [1] for populations of E. coli. The model for the semi-solid medium experiment with E. coli is considered. The key players in this paper seem to be the bacteria, the chemoattractant (aspartate) and the stimulant (succinate) so the three variables is considered: the cell density u, the chemoattractant concentration v, and the stimulant concentration w. The bacteria diffuse, move chemotactically up gradients of the chemoattractant, proliferate and become non-motile. The non-motile cells can be thought of as dead, for the purpose of the model. The chemoattractant diffuses, and is produced and ingested by the bacteria while the stimulant diffuses and is consumed by the bacteria. The model consisting of three conservation equations is:

Rate of change of cell density, u	=	Diffusion of u	+	Chemotaxis of u to v	+	Proliferation (growth and death) of u
Rate of change of chemoattratant concentration, v	=	Diffusion of v	+	Production of v by u	−	Uptake of v by u
Rate of change of stimulant concentration, w	=	Diffusion of w	−	Uptake of w by u		

In recent years, fractional calculus starts to attract much more attention of physicists and mathematicians. It was found that various; especially interdisciplinary applications can be elegantly modeled with the help of the fractional derivatives. Other authors have demonstrated applications of fractional derivatives in the areas of electrochemical processes [2,3], dielectric polarization [4], colored noise [5], viscoelastic materials [6-9] and chaos [10]. Mainardi [11] and Rossikhin and Shitikova [12] presented survey of the application of fractional derivatives, in general to solid mechanics, and in particular to modeling of viscoelastic damping. Magin [13-15] presented a three part critical review of applications of fractional calculus in bioengineering. Applications of fractional derivatives in other fields and related mathematical tools and techniques could be found in [16-18]. Rida *et al.* presented a new solutions of some bio-mathematical models of fractional order [19-24].

In this paper, we implemented the Adomian's decomposition method (ADM) [25,26] to the generalized bacterial pattern formation models in a semi-solid medium:

$$\begin{aligned}
\frac{\partial^\alpha u}{\partial t^\alpha} &= D_u\nabla^2 u - \nabla\cdot\left(\frac{k_1 u}{(k_2+v)^2}\nabla v\right) + k_3 u\left(\frac{k_4 w^2}{k_9+w^2}-u\right),\\
\frac{\partial^\alpha v}{\partial t^\alpha} &= D_v\nabla^2 v + k_5 w\frac{u^2}{k_6+u^2} - k_7 uv \qquad (1.1)\\
\frac{\partial^\alpha w}{\partial t^\alpha} &= D_w\nabla^2 w + k_8 u\frac{w^2}{k_9+w^2}
\end{aligned}$$

where $0<\alpha\leq 1$, u,v and w are the cell density, the concentration of the chemoattractant and of the stimulant respectively. There are three diffusion coefficients, three initial values $(u,v,w\ at\ t=0)$ and nine parameters k in the model.

Subject to initial conditions:

$$u(\mathrm{X},0)=u_0(\mathrm{X}),\ v(\mathrm{X},0)=v_0(\mathrm{X}) \text{ and } w(\mathrm{X},0)=w_0(\mathrm{X})$$

where u,v and w are the cell density, the concentration of the chemoattractant and of the stimulant respectively. There are three diffusion coefficients, three initial values $(u,v,w \text{ at } t=0)$ and nine parameters k in the model.

The fractional systems of Equations (1.1) are obtained by replacing the first time derivative term by a fractional derivative of order $\alpha>0$. The Derivatives are understood in the Caputo sense. The general response expression contains a parameter describing the order of the fractional derivative that can be varied to obtain various responses.

In the case of $\alpha\to 1$, the fractional system equations reduce to the standard system of partial differential equations. The Adomian's decomposition method will be applied for computing solutions to the systems of fractional partial differential equations considered in this paper. This method has been used to obtain approximate solutions of a large class of linear or nonlinear differential equations. It is also quite straightforward to write computer codes in any symbolic languages. The method provides solutions in the form of power series with easily computed terms. It has many advantages over the classical techniques mainly; it provides efficient numerical solutions with high accuracy, minimal calculations.

The reason of using fractional order differential equations (FOD) is that FOD are naturally related to systems with memory which exists in most biological systems. Also they are closely related to fractals which are abundant in biological systems. The results derived of the fractional system (1.1) are of a more general nature. Respectively, solutions to the fractional reaction-diffusion equations spread at a faster rate than the classical diffusion equation, and may exhibit asymmetry. However, the fundamental solutions of these equations still exhibit useful scaling properties that make them attractive for applications.

Cherruault [27] proposed a new definition of the method and he then insisted that it would become possible to prove the convergence of the decomposition method. Cherruault and Adomian [28] proposed a new convergence series. A new approach of the decomposition method was obtained in a more natural way than was given in the classical presentation [29]. Recently, the application of the method is extended for fractional differential equations [30-33].

There are several approaches to the generalization of the notion of differentiation to fractional orders e.g. Riemann-Liouville, GruÖnwald-Letnikow, Caputo and Generalized Functions approach [34]. Riemann-Liouville fractional derivative is mostly used by mathematicians but this approach is not suitable for real world physical problems since it requires the definition of fractional order initial conditions, which have no physically meaningful explanation yet. Caputo introduced an alternative definition, which has the advantage of defining integer order initial conditions for fractional order differential equations [34]. Unlike the Riemann-Liouville approach, which derives its definition from repeated integration, the GruÖnwald-Letnikow formulation approaches the problem from the derivative side. This approach is mostly used in numerical algorithms.

2. Fractional Calculus

Here, we mention the basic definitions of the Caputo fractional-order integration and differentiation, which are used in the up coming paper and play the most important role in the theory of differential and integral equation of fractional order.

The main advantages of Caputo's approach are the initial conditions for fractional differential equations with Caputo derivatives take on the same form as for integer order differential equations.

Definition 2.1 The fractional derivative of $f(x)$ in the Caputo sense is defined as [34]:

$$\begin{aligned} D^\alpha f(x) &= I^{m-\alpha}D^m f(x) \\ &= \frac{1}{\Gamma(m-\alpha)}\int_0^x (x-t)^{m-\alpha+1} f^{(m)}(t)\mathrm{d}t \end{aligned}$$

for $m-1<\alpha\leq m,\ m\in N,\ x>0$.

For the Caputo derivative we have $D^\alpha C=0$, C is constant

$$D^\alpha t^n = \begin{cases} 0, & (n\leq\alpha-1) \\ \dfrac{\Gamma(n+1)}{\Gamma(n-\alpha+1)}t^{n-\alpha}, & (n>\alpha-1) \end{cases}$$

Definition 2.2 For m to be the smallest integer that exceeds α, the Caputo fractional derivatives of order $\alpha>0$ is defined as [34]:

$$D^\alpha u(x,t)=\frac{\partial^\alpha u(x,t)}{\partial t^\alpha} = \begin{cases} \dfrac{1}{\Gamma(m-\alpha)}\displaystyle\int_0^t (t-\tau)^{m-\alpha+1}\frac{\partial^m u(x,\tau)}{\partial\tau^m}\mathrm{d}\tau, & \text{for } m-1<\alpha<m \\ \dfrac{\partial^m u(x,t)}{\partial t^m}, & \text{for } \alpha=m\in N \end{cases}$$

3. Analysis of the Method

To give the approximate solution of nonlinear fractional-order differential equations by means of the ADM, we write the systems in the form

$$\begin{aligned} &D^{\alpha_1}u_1(\mathrm{X},t) = N_1(u_1,u_2,\cdots,u_m) + f_1(\mathrm{X},t) \\ &D^{\alpha_2}u_2(\mathrm{X},t) = N_2(u_1,u_2,\cdots,u_m) + f_2(\mathrm{X},t) \\ &\vdots \\ &D^{\alpha_m}u_m(\mathrm{X},t) = N_m(u_1,u_2,\cdots,u_m) + f_m(\mathrm{X},t) \end{aligned} \tag{3.1}$$

where $D^{\alpha_i}(i=1,2,\cdots,m)$ are the fractional operators, and $N_1,N_2,\cdots N_m$ are nonlinear operators.

Applying the inverse operators $I^{\alpha_1},I^{\alpha_2},\cdots,I^{\alpha_M}$ to the systems (3.1)

$$\begin{aligned} &u_1(\mathrm{X},t) = I^{\alpha_1}\left(N_1(u_1,u_2,\cdots,u_m) + f_1(\mathrm{X},t)\right) \\ &u_2(\mathrm{X},t) = I^{\alpha_2}\left(N_2(u_1,u_2,\cdots,u_m) + f_2(\mathrm{X},t)\right) \\ &\vdots \\ &u_m(\mathrm{X},t) = I^{\alpha_M}\left(N_m(u_1,u_2,\cdots,u_m) + f_m(\mathrm{X},t)\right) \end{aligned} \tag{3.2}$$

Subject to the initial conditions

$$u_i(\mathrm{X},0) = g_i(\mathrm{X}),\ (i=1,2,\cdots,m) \tag{3.3}$$

The Adomian decomposition method suggests that the linear terms $u_i(\mathrm{X},t)$ are decomposed by an infinite series of components

$$u_i(\mathrm{X},t) = \sum_{n=0}^{\infty} u_{i,n}(\mathrm{X},t),\ (i=1,2,\cdots,m) \tag{3.4}$$

and the nonlinear operators are defined by the infinite series of the so called Adomian polynomials

$$N_i = \sum_{n=0}^{\infty} A_{i,n} \tag{3.5}$$

where $u_{i,n}(\mathrm{X},t)$; $n \ge 0$ are the components of $u_i(\mathrm{X},t)$, that will be elegantly determined, and $A_{i,n}$; $n \ge 0$ are Adomian's polynomials that can be generated for all forms of non linearity [35]. Substituting (3.4) and (3.5) into (3.2) gives

$$\begin{aligned} &\sum_{n=0}^{\infty} u_{1,n}(\mathrm{X},t) = g_1(\mathrm{X}) + I^{\alpha_1}\left(\sum_{n=0}^{\infty} A_{1,n} + f_1(\mathrm{X},t)\right) \\ &\sum_{n=0}^{\infty} u_{2,n}(\mathrm{X},t) = g_2(\mathrm{X}) + I^{\alpha_2}\left(\sum_{n=0}^{\infty} A_{2,n} + f_2(\mathrm{X},t)\right) \\ &\vdots \\ &\sum_{n=0}^{\infty} u_{m,n}(\mathrm{X},t) = g_m(\mathrm{X}) + I^{\alpha_m}\left(\sum_{m=0}^{\infty} A_{m,n} + f_m(\mathrm{X},t)\right) \end{aligned} \tag{3.6}$$

Following adomian analysis, the nonlinear system (3.1) is transformed into a set of recursive relations given by

$$\begin{aligned} &u_{i,0}(\mathrm{X},t) = g_i(\mathrm{X}) \\ &u_{i,n+1}(\mathrm{X},t) = I^{\alpha}\left(A_{i,n} + f_i(\mathrm{X},t)\right) \quad n \ge 0, \end{aligned} \tag{3.7}$$

where $(i=1,2,\cdots,m)$.

It is an essential feature of the decomposition method that the zeroth components $u_{i,0}(\mathrm{X},t)$ are defined always by all terms that arise from initial data and from integrating the inhomogeneous terms. The remaining pairs $(u_{i,n}, n \ge 1)$ can be easily determined in a parallel manner. Additional pairs for the decomposition series normally account for higher accuracy. Have been determined the components of $u_i(\mathrm{X},t)$, the solutions of the system follow immediately in the form of a power series expansion upon using (3.4). The series obtained can be summed up in many cases to give a closed form solution for concrete problems, the n term approximants can be used for numerical purposes. Comparing the scheme presented above with existing techniques such as characteristics method and Riemann invariants, it is clear that the decomposition method introduces a fundamental qualitative difference in approach, because no assumptions are made. The approach is straightforward and the rapid convergence is guaranteed. To give a clear overview of the content of this work, several illustrative examples have been selected to demonstrate the efficiency of the method.

4. Applications and Numerical Results

In order to illustrate the advantages and the accuracy of the ADM, we consider time-fractional chemotaxis model of bacteria colonies in a semi-solid medium (1.1) in one dimensional in the form:

$$\begin{aligned} D^{\alpha}u &= D_u L_{xx} u \\ &\quad - L_x\left(\frac{k_1 u}{(k_2+v)^2} L_x v\right) + k_3 u\left(\frac{k_4 w^2}{k_9 + w^2} - u\right) \\ D^{\alpha}v &= D_v L_{xx} v + k_5 w \frac{u^2}{k_6 + u^2} \\ D^{\alpha}w &= D_w L_{xx} w + k_8 u \frac{w^2}{k_9 + w^2} \end{aligned} \tag{4.2}$$

Subject to the initial conditions

$$u(x,0) = n_0$$

$$v(x,0) = \lambda e^{-\mu x^2}$$

$$w(x,0) = s_0$$

where n_0, λ, μ, s_0 are constants.

Operating with I^{α} in both sides of system (4.2) we find

$$u(x,t) = u(x,0) + I^{\alpha}\left(D_u L_{xx} u - L_x\left(\frac{k_1 u}{(k_2+v)^2} L_x v\right) + k_3 u\left(\frac{k_4 w^2}{k_9 + w^2} - u\right)\right)$$

$$v(x,t) = v(x,0) + I^{\alpha}\left(D_v L_{xx} v + k_5 w \frac{u^2}{k_6 + u^2}\right)$$

$$w(x,t) = w(x,0) + I^{\alpha}\left(D_w L_{xx} w + k_8 u \frac{w^2}{k_9 + w^2}\right) \tag{4.3}$$

The ADM assumes a series solution for $u(x,t), v(x,t)$ and $w(x,t)$ given by:

$$u(x,t) = \sum_{n=0}^{\infty} u_n(x,t)$$
$$v(x,t) = \sum_{n=0}^{\infty} v_n(x,t) \tag{4.4}$$
$$w(x,t) = \sum_{n=0}^{\infty} w_n(x,t)$$

Substituting the decomposition series (4.4) into (4.3) yields

$$\sum_{n=0}^{\infty} u_n(x,t) = u(x,0) + I^{\alpha}\left(D_u L_{xx} \sum_{n=0}^{\infty} u_n(x,t) - k_1 L_x \sum_{n=0}^{\infty} A_n + k_3 k_4 \sum_{n=0}^{\infty} B_n - k_3 \sum_{n=0}^{\infty} C_n\right)$$

$$\sum_{n=0}^{\infty} v_n(x,t) = v(x,0) + I^{\alpha}\left(D_v L_{xx} \sum_{n=0}^{\infty} v_n(x,t) + k_5 \sum_{n=0}^{\infty} D_n\right)$$

$$\sum_{n=0}^{\infty} w_n(x,t) = w(x,0) + I^{\alpha}\left(D_w L_{xx} \sum_{n=0}^{\infty} w_n(x,t) + k_8 \sum_{n=0}^{\infty} B_n\right)$$

Identifying the zeros components, $u_0(x,t), v_0(x,t)$ and $w_0(x,t)$ by $u_0(x,0), v_0(x,0)$ and $w_0(x,0)$ the remaining components where $n \geq 0$ can be determined by using recurrence relation:

$$u_0(x,t) = u_0(x,0)$$
$$u_{n+1}(x,t) = I^{\alpha}\left(D_u L_{xx} u_n - k_1 L_x A_n + k_3 k_4 B_n - k_3 C_n\right), \quad n \geq 0 \tag{4.5}$$

$$v_0(x,t) = v_0(x,0)$$
$$v_{n+1}(x,t) = I^{\alpha}\left(D_v L_{xx} v_n + k_5 D_n\right), \quad n \geq 0 \tag{4.6}$$

$$w_0(x,t) = w_0(x,0)$$
$$w_{n+1}(x,t) = I^{\alpha}\left(D_w L_{xx} w_n + k_8 B_n\right), \ n \geq 0 \tag{4.7}$$

where A_n, B_n, C_n, and D_n are the Adomian's polynomials calculated for all forms of nonlinearity according to specific algorithms constructed by Adomian as:

$$A_0 = \frac{k_1 u_0 v_{0x}}{(k_2+v)^2} = u_0 \frac{\partial}{\partial x}\frac{v_0}{k_2+v_0},$$
$$A_1 = u_1 \frac{\partial}{\partial x}\frac{v_0}{k_2+v_0} + k_2 u_0 v_1 \frac{\partial}{\partial x}\frac{1}{(k_2+v_0)^2} \tag{4.8}$$

$$B_0 = \frac{u_0 w_0^2}{k_9 + w_0^2}$$
$$B_1 = \frac{u_1 w_0^2}{k_9 + w_0^2} + \frac{2k_9 u_0 w_0 w_1}{(k_9 + w_0^2)} \tag{4.9}$$

$$C_0 = u_0^2$$
$$C_1 = 2u_0 u_1 \tag{4.10}$$

and

$$D_0 = \frac{w_0 u_0^2}{k_6 + u_0^2}$$
$$D_1 = \frac{w_1 u_0^2}{k_6 + u_0^2} + \frac{2k_6 w_0 u_0 u_1}{(k_6 + u_0^2)} \tag{4.11}$$

Using Equations (4.5)-(4.11), we can calculate some of the terms of the decomposition series (4.4) as:

$$u_0(x,t) = f(x)$$
$$u_1(x,t) = f_1(x)\frac{t^{\alpha}}{\Gamma(\alpha+1)}$$
$$u_2(x,t) = f_2(x)\frac{t^{2\alpha}}{\Gamma(2\alpha+1)}$$
$$v_0(x,t) = g(x)$$
$$v_1(x,t) = g_1(x)\frac{t^{\alpha}}{\Gamma(\alpha+1)}$$
$$v_2(x,t) = g_2(x)\frac{t^{2\alpha}}{\Gamma(2\alpha+1)}$$

and

$$w_0(x,t) = h(x)$$
$$w_1(x,t) = h_1(x)\frac{t^{\alpha}}{\Gamma(\alpha+1)}$$
$$w_2(x,t) = h_2(x)\frac{t^{2\alpha}}{\Gamma(2\alpha+1)}$$

where:

$$f(x) = n_0$$
$$f_1(x) = D_u f^{(2)} - k_1 f \frac{\partial}{\partial x}\frac{g}{k_2+g} + k_3 k_4 \frac{f h^2}{k_9 + h^2} - k_3 f^2$$
$$f_2(x) = D_u f_1^{(2)} - k_1\left(f_1 \frac{\partial}{\partial x}\frac{g}{k_2+g} + k_2 f g_1 \frac{\partial}{\partial x}\frac{g}{(k_2+g)^2}\right) + k_3 k_4\left(\frac{f_1 h^2}{k_9+h^2} + \frac{2k_9 f h h_1}{(k_9+h^2)^2}\right) - 2k_3 f\, f_1,$$

$$g(x) = \lambda e^{-\mu x^2}$$

$$g_1(x) = D_v g^{(2)} + k_5 \frac{h f^2}{k_6 + f^2}$$

$$g_2(x) = D_v g_1^{(2)} + k_5 \left(\frac{h_1 f^2}{k_6 + f^2} + \frac{2k_6 h f f_1}{(k_6 + f^2)^2} \right)$$

and

$$h(x) = s_0$$

$$h_1(x) = D_w h^{(2)} + k_8 \frac{f h^2}{k_9 + h^2}$$

$$h_2(x) = D_w h_1^{(2)} + k_8 \left(\frac{f_1 h^2}{k_9 + h^2} + \frac{2k_9 f h h_1}{(k_9 + h^2)^2} \right)$$

and so on, substituting $u_0, u_1, u_2, \cdots, v_0, v_1, v_2, \cdots$ and $w_0, w_1, w_2, \cdots$ into (4.4) gives the solution $u(x,t)$, $v(x,t)$ and $w(x,t)$ in a series form by:

$$\begin{aligned} u(x,t) &= u_0 + u_1 + u_2 + \cdots \\ v(x,t) &= v_0 + v_1 + v_2 + \cdots \\ w(x,t) &= w_0 + w_1 + w_2 + \cdots \end{aligned} \tag{4.12}$$

See **Figure 1** and **Table 1**.

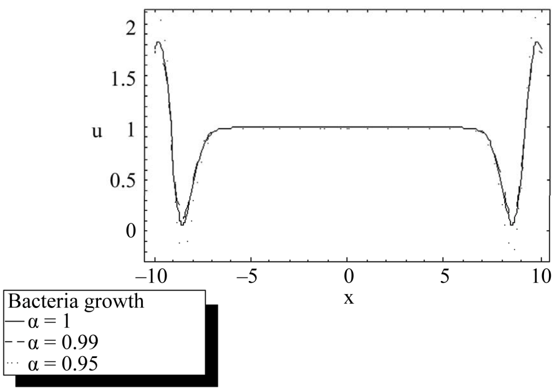

Figure 1. The Numerical results *u*(*x*,*t*).

Table 1. Density of the active bacteria in a semi-solid medium.

x	$\alpha = 1$	$\alpha = 0.99$	$\alpha = 0.95$
–10	1.7620E+00	1.7078E+00	1.9998E+00
–8	4.3328E–01	4.7356E–01	2.5642E–01
–6	9.9630E–01	9.9656E–01	9.9515E–01
–4	9.9997E–01	9.9997E–01	9.9995E–01
–2	1.0000E+00	1.0000E+00	1.0000E+00
0	1.0000E+00	1.0000E+00	1.0000E+00
2	1.0000E+00	1.0000E+00	1.0000E+00
4	9.9997E–01	9.9997E–01	9.9995E–01
6	9.9630E–01	9.9656E–01	9.9515E–01
8	4.3328E–01	4.7356E–01	2.5642E–01
10	1.7620E+00	1.7078E+00	1.9998E+00

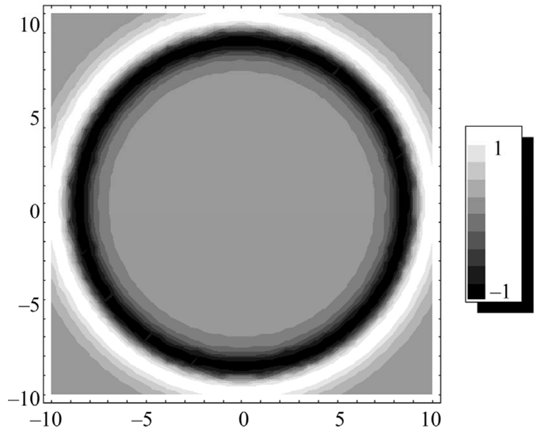

Figure 2. The movement of bacteria cells *u*(*x*,y,*t*) at *α* = 1, white color corresponds to high cell density.

5. Conclusions

In this paper, the decomposition method was implemented to describe the evolution of the bacterial chemotaxis model in a semi-solid medium. The results shows that the solution continuously depends on the time-fractional derivative see **Figure 1**. In other words, we solve the model in two dimensional forms. **Figure 2** is time evolution of the bacteria density $u(x, y, t)$ at, $\alpha \to 1$ which the light regions represent high density of bacteria. In **Figure 2**, Bacteria patterns obtained in semi-solid medium begin with a very low density bacterial lawn spreading out from the initial inoculums. But high density ring of bacteria appears at some radius less than the radius of the lawn, which is very similar to the experimental results (Budrene and Berg (1991)). Finally, it may be concluded that the decomposition method does not change the problem into a convenient one for use of linear theory. It therefore provides more realistic solutions. It provides series solutions which generally converge very rapidly in real physical problems. Respectively, the recent appearance of fractional differential equations as models in some fields of applied mathematics makes it necessary to investigate methods of solution for such equations (analytical and numerical) and we hope that this work is a step in this direction.

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Note: Parameters Taken as:

$n_0 = 1,\ \lambda = 0.01,\ \mu = 0.1,$

$k_1 = 3.9(10)^{-9},\ k_2 = 5(10)^{-6},\ k_3 = 1.62(10)^{-8},$

$k_5 = k_6 = k_7 = k_8 = 1,\ k_9 = 4(10)^{-6},\ s_0 = 1 - 3(10)^{-3},$

$D_u = 2 - 4(10)^{-6},\ D_v = 8.9(10)^{-6},\ D_w = 9(10)^{-6},\ t = 100$

