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Controllability of Neutral Impulsive Differential Inclusions with Non-Local Conditions

Dimplekumar N. Chalishajar[1], **Falguni S. Acharya**[2*]
[1]*Department of Mathematics and Computer Science, Virginia Military Institute (VMI), Lexington, USA*
[2]*Department of Applied Sciences Humanities, Institute of Technology and Management Universe, Vadodara, India*
E-mail: *falguni 69@yahoo.co.in, dipu17370@yahoo.com, chalishajardn@vmi.edu*
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Abstract

In this short article, we have studied the controllability result for neutral impulsive differential inclusions with nonlocal conditions by using the fixed point theorem for condensing multi-valued map due to Martelli [1]. The system considered here follows the P.D.E involving spatial partial derivatives with α-norms.

Keywords: Controllability, Neutral Impulsive Differential Inclusions, Spatial Partial Derivative, Martelli Fixed Point Theorem

1. Introduction

In this paper we have discussed the controllability of nonlocal Cauchy problem for neutral impulsive differential inclusions of the form

$$\begin{cases} \dfrac{\mathrm{d}}{\mathrm{d}t}\left[x(t)-F\left(t,x\left(h_1(t)\right)\right)\right]\in Ax(t)+Bu(t)+G\left(t,x\left(h_2(t)\right)\right); \\ \qquad\qquad t\in J:=[0,b]; t\neq t_k \\ \Delta x\mid_{t=t_k}=I_k\left(x(t_k^-)\right); k=1,2,\cdots m;\ x(0)+g(x)=x_0\in X \end{cases} \tag{1}$$

where the linear operator $(-A)$ generates an analytic semigroup $\{T(t)\}_{t\geq 0}$; G is a multi-valued map and $\Delta x_{t=t_k}=x(t_k^+)-x(t_k^-)$, $x(t_k^+)=\lim_{h\to 0^+}x(t_k+h)$ and $x(t_k^-)=\lim_{h\to 0^+}x\left(t_k-h\right)$ represent the right and left limits of $x(t)$ at $t=t_k$ respectively, $x_0\in X$, $F:J\times X\to P(X)\setminus\phi$ is a multi-valued map [$P(X)$ is the family of all subsets of X] and $g\in C(J,X)$. Also the control function $u\in L^2(J,U)$, a Banach space of admissible control functions with U as a Banach space. B is a bounded linear operator from U to X and X is a separable Banach space with norm $\|\cdot\|$. $I_k:X\to \overline{D(A)}$; $k=1,2,\cdots,m$ and $h_1,h_2\in C(J,J)$.

As a model we consider the following system of heat equations;

$$\begin{cases} \dfrac{\partial}{\partial t}\left[z(t,x)-F(t,z(\cos t,x)),\dfrac{\partial z}{\partial x}(\cos t,x)\right] \\ =\dfrac{\partial^2 z(t,x)}{\partial x^2}+u(t,x)+G\left(t,z(\cos t,x),\dfrac{\partial z}{\partial x}(\cos t,x)\right), \\ z(t,0)=z(t,\pi)=0; \\ z(t_k^+)-z(t_k^-)=I_k(z(t_k^-)), \\ t\neq t_k; k=1,2,\cdots,m \\ z(0,x)+g(z(t,x))=z_0(x), 0\leq x\leq \pi, t\in[0,1] \end{cases} \tag{2}$$

Since F and G involve spatial partial derivative, the results obtained by other authors cannot be applied to our system even if $g(.)=0$. This is the main motivation of this paper.

The existence and controllability of the following system is studied by Benchohra and Ntouyas [2]

$$\begin{cases} \dfrac{\mathrm{d}}{\mathrm{d}t}\left[x(t)-g(t,x_t)\right]\in Ax(t)+Bu(t)+F(t,x_t); \\ \qquad\qquad t\in J:=[0,b]; t\neq t_k \\ \Delta x\mid_{t=t_k}=I_k(x(t_k^-)); k=1,2,\cdots,m; x(t)=\phi(t), t\in(-\infty,0). \end{cases} \tag{3}$$

Here authors have proved exact controllability by using fixed point theorem for condensing multi-valued maps due to Martelli. In this paper, we have discussed controllability results with α-norms as in [3] with deviating arguments in terms involving spatial partial derivatives.

As indicated in [4], and reference therein, the nonlocal Cauchy problem $x(0)+g(x)=x_0$ can be applied in different fields with better effect than the classical initial condition $x(0)=x_0$. For example in [5], the author described the diffusion phenomenon of a small amount of gas in a transparent tube by using the formula

$$g(x)=\sum_{i=0}^{p} c_i x(t_i),$$

where $c_i, i=0,1,\cdots,p$ are given constants and $0<t_0,t_1,\cdots,t_p<b.$ In this case the above equation allows the additional measurement at t_i, $i=0,1,\cdots,p$. In the past several years theorems about controllability of differential, integro-differential, fractional differential systems and inclusions with nonlocal conditions have been studied by Chalishajar and Acharya [6-9], Benchohra and Ntouyas [10,11], and Hernandez, Rabello and Henriquez [12] and the references therein. In [13], Chalishajar discussed exact controllability of third order nonlinear integro-differential dispersion system without compactness of semigroup.

Xianlong Fu and Yueju Cao [14], has discussed the existence of mild solution for neutral partial differential inclusions involving spatial partial derivative with α-norms in Banach space. However in their work authors impose some severe assumptions on the operator family generator by $(-A)$, *i.e.* $(-A): D(A)\subset X\to X$ is an infinitesimal generator of a compact analytic semigroup of a uniformly bounded linear operator $\{T(t)\}_{t\geq 0}$, which imply that underlying space X has finite dimension and so the example considered in [14], and subsequently in Section 4 is ordinary differential equation but not partial differential equation which shows lack of existence (exact controllability) in abstract (control) system (refer [15]). This fact and several other applications of neutral equation (inclusions) are the main motivation of this paper.

In Section 3 (followed by Preliminaries) of present paper we discuss the controllability of neutral impulsive differential inclusion with nonlocal condition with deviating arguments with α-norm, which is the generalization of [14], in a finite dimensional space. The example is given in Section 4 to support the theory. In Section 5 we study exact controllability of same system in infinite dimension space by dropping the compactness assumption of semigroup $\{T(t)\}_{t\geq 0}$. Here we generalized the result proved in Section 3.

2. Preliminaries

In this section, we shall introduce some basic definitions, notations and lemmas which are used throughout this paper.

Let $(X,\|\cdot\|)$ be a Banach space. $C(J,X)$ is the Banach space of continuous functions from J into X with the norm defined by

$$\|x\|_J := \sup\{\|x(t)\| : t\in J\}.$$

Let $B(X)$ be the Banach space of bounded linear operators from X into X with standard norm

$$\|N\|_{B(X)} := \sup\{\|N(x)\| : \|x\|=1\}.$$

A measurable function $x: J\to X$ is Bochner integrable if and only if $\|x\|$ is Lebesgue integrable. (For properties of the Bochner integral see [16]). Let $L^1(J,X)$ denotes the Banach space of Bochner integrable functions $x: J\to X$ with norm $\|x\|_{L^1}=\int_0^b \|x(t)\|\mathrm{d}t$ for all $x\in L^1(J,X)$.

We use the notations $P(X)=\{Y\in 2^X : Y\neq\phi\}$, $P_{cl}(X)=\{Y\in P(X): Yclosed\}$, $P_b(X)=\{Y\in P(X): Ybounded\}$, $P_c(X)=\{Y\in P(X): Yconvex\}$, and $P_{cp}(X)=\{Y\in P(X): Ycompact\}$.

A multi-valued map $G: X\to 2^X$ is convex (respectively closed) valued if $G(x)$ is convex (respectively closed) for all $x\in X$.

The map G is bounded on bounded sets if $G(B)=\cup_{x\in B}G(x)$ is bounded in X for any bounded set B of X. $\left(i.e.\ \sup_{x\in B}\{\sup\{\|x\| : x\in G(x)\}\}<\infty\right)$.

G is called upper semi-continuous (u.s.c.) on X if for each $x_0\in X$, the set $G(x_0)$ is a nonempty closed subset of X and if for each open set B of X containing $G(x_0)$, there exists an open neighborhood A of x_0 such that $G(A)\subseteq B.$

The map G is said to be completely continuous if $G(B)$ is relatively compact for every bounded subset $B\subseteq X.$

If the multi-valued map G is completely continuous with nonempty compact values, then G is u.s.c. if and only if G has a closed graph, That is, if $x_n\to x_0, y_n\to y_0$, where $y_n\in G(x_n)$ then $y_0\in G(x_0)$. G has a fixed point if there is $x\in X$ such that $x\in G(x)$.

A multi-valued map $G: J\to BCC(X)$ is said to be measurable, if for each $x\in X$, the distance function $Y: J\to R$ defined by

$$Y(t)=d(x,G(t))=\inf\{\|x-z\| : z\in G(t)\}$$

is measurable.

An upper semi-continuous map $G: X\to 2^X$ is said to be condensing, if for any bounded subset $B\subseteq X$, with $\alpha(B)\neq 0$, we have $\alpha(G(B))<\alpha(B)$, where α denotes the Kuratowski measure of non-compactness.

We remark that a completely continuous multi-valued map is the easiest example of a condensing map. For more details on multivalued maps see the books of Deimling [17].

Throughout this paper, $A : D(A) \subset X \to X$ will be the infinitesimal generator of a compact analytic semigroup of uniformly bounded linear operator $T(t)$. Let $0 \in \rho(A)$, then it is possible to define the fractional power A^{α}, *for* $0 \le \alpha \le 1$, as a closed linear operator on its domain $D(A^{\alpha})$. Furthermore, the subspace $D(A^{\alpha})$ is dense in X and the expression

$$\|x\|_{\alpha} = \|A^{\alpha} x\|;\ x \in D(A^{\alpha})$$

defines a norm on $D(A^{\alpha})$. Hereafter we denote by X_{α}, the Banach space $D(A^{\alpha})$ normed with $\|x\|_{\alpha}$. Then for each $0 < \alpha \le 1$, X_{α} is a Banach space, and $X_{\alpha} \rightarrowtail X_{\beta}$ *for* $0 < \beta < \alpha \le 1$ and the imbedding is compact whenever the resolvent operator of A is compact.

Semigroup $\{T(t)\}_{t \ge 0}$ satisfies the following properties:

a) there is a $M \ge 1$ such that

$$\|T(t)\| \le M \text{ for all } 0 \le t \le \alpha;$$

b) for any $0 < \alpha \le 1$, there exists a positive constant C_{α} such that

$$\|A^{\alpha} T(t)\| \le \frac{C_{\alpha}}{t^{\alpha}};\ 0 < t \le \alpha$$

For more details about the above preliminaries, we refer to ([18,19]).

In order to define the solution of the system (1) we shall consider the space

$$\Omega = \{x : [0,b] \to X_{\alpha};\ x_k \in C(J_k, X_{\alpha});\ k = 0,1,\cdots,m$$
$$\text{and there exist } x(t_k^-) \text{ and } x(t_k^+);\ k = 0,1,\cdots,m$$
$$\text{with } x(t_k^-) = x(t_k),\ x(0) + g(x) = x_0\},$$

which is a Banach space with the norm

$$\|x\|_{\Omega} = max\left\{\|x_k\|_{J_k};\ k = 0,1,\cdots,m\right\}$$

where x_k is the restriction of x to $J_k = (t_k, t_{k+1}],\ k = 0,1,\cdots,m$ and $\|x_k\|_{J_k} = sup_{s \in J_k} \|x_k(s)\|_{\alpha}$.

For the system (1) we assume that the following hypotheses are satisfied for some $\alpha \in (0,1)$:

(H1) Let $W : L^2(J,U) \to X_{\alpha}$ be the linear operator defined by

$$Wu = \int_0^b T(b-s) Bu(s) \mathrm{d}s$$

The $W : L^2(J,U)/kerW \to X_{\alpha}$ induces a bounded invertible operator $\tilde{W}^{-1}$ and there exists positive constant M_1 and M_2 such that and $\|B\| \le M_1$ and $\|\tilde{W}^{-1}\| \le M_2$.

(H2) i) there exists a constant $\beta \in (0,1)$ such that $F : [0,b] \times X_{\alpha} \to X_{\beta}$ is a continuous function, and $A^{\beta} F : [0,b] \times X_{\alpha} \to X_{\beta}$ satisfies the Lipschitz condition, that is, there exists a constant $L > 0$ such that

$$\left\|A^{\beta} F(t_1, x_1) - A^{\beta} F(t_2, x_2)\right\|_{\alpha} \le L\left(|t_1 - t_2| + \|x_1 - x_2\|_{\alpha}\right),$$

for any $0 \le t_1, t_2 \le b;\ x_1, x_2 \in X_{\alpha}$.

ii) Moreover, there exists a constant $L_1 > 0$ such that the inequality

$$\left\|A^{\beta} F(t,x)\right\|_{\alpha} \le L_1\left(\|x\|_{\alpha} + 1\right),$$

holds for any $x \in X_{\alpha}$.

(H3) The multi-valued map $G : J \times X_{\alpha} \to P_{c,cp}(X)$ satisfies the following conditions:

i) for each $t \in J$, the function $G(t,.) : X_{\alpha} \to P_{c,cp}(X)$ is u.s.c. and for each $x \in X_{\alpha}$, the function $G(.,x) : J \to P_{c,cp}(X)$ is measurable. Also for each fixed $y \in \Omega$ the set

$$S_{G,x} = \left\{v \in L^1(J,X) : v(t) \in G(t, x(h_2(t)))\ for a.e.\ t \in J\right\}$$

is nonempty.

ii) for each positive number $l \in N$, there exists a positive function $w(l)$ dependent on l such that

$$\sup_{\|x\| \le l} \|G(t,x)\| \le w(l)$$

and $\liminf_{l \to \infty} \frac{w(l)}{l} = \gamma < \infty$ where $\|G(t,x)\| = \sup\{\|v\| : v \in G(t,x)\}$, $\|x\|_{\alpha} = \sup_{0 \le s \le \alpha} \|x(s)\|_{\alpha}$.

(H4) $h_i \in C(J,J), i = 1,2.$ $g : \Omega \to X_{\alpha}$ is continuous and satisfies that

i) there exists positive constants L_2 and $L_{2'}$ such that

$$\|g(y)\|_{\alpha} \le L_2 \|y\|_{\Omega} + L_2' \text{ for all } y \in \Omega.$$

ii) $A^{\alpha} g$ is completely continuous map.

(H5) $I_k \in C(X_{\alpha}, X_{\alpha}), k = 1,2,\cdots,m,$ are all bounded, that is, there exist constants $d_k, k = 1,2,\cdots,m,$ such that

$$\|I_k(x)\|_{\alpha} \le d_k, \text{ for each } x \in X_{\alpha}.$$

Now we define the mild solution for the system (1).

DEFINITION 2.1 *The system* (1) *is said to be non-locally controllable on the interval* J *if for every* $x(0) + g(x) \in D(A)$ *and* $x_0, z_1 \in X$, *there exists a control* $u \in L^2(0, b : L^2(0, 2\pi)) = L^2(J,U)$ *such that the corresponding solution* $x(\cdot)$ *of* (1) *satisfies*

i) $x(b) + g(x) = z_1$ with $x(0) + g(x) = x_0$;

ii) $\Delta x|_{t=t_k} = I_k(x(t_k^-));\ k = 1,2,\cdots,m;$

iii) there exists a function $v \in L^1(J,X)$ such that

$v(t) \in G(t, x(h_2(t)))$ a.e. on J and

$$x(t) = T(t)\left[x_0 - g(x) - F(0, x(h_1(0)))\right] + F(t, x(h_1(t))) + \int_0^t AT(t-s)F(s, x(h_1(s)))\mathrm{d}s + \int_0^t T(t-s)v(s)\mathrm{d}s + \int_0^t T(t-s)(Bu)(s)\mathrm{d}s + \sum_{0<t_k<t} T(t-t_k)I_k(x(t_k^-)); t \in J, v \in S_{G,x}. \quad (2.1)$$

The following lemmas are crucial in the proof of our main theorem.

LEMMA 2.2 [20] *Let* X *be a Banach space. Let* $G: J \times X \to P_{b,cl,c}(X)$ *satisfies that*

i) For each $x \in X$, $(t,x) \mapsto G(t,x)$ is measurable with respect to t and for each $t \in J$, $(t,x) \to G(t,x)$ is u.s.c. with respect to x .

ii) For each fixed $x \in C(J, X)$, the set

$S_{G,x} = \left\{v \in L^1(J,X) : v(t) \in G(t, x(h_2(t))), \text{for } a.e.\ t \in J\right\}$

is nonempty.

Let Γ be a linear continuous mapping from $L^1(J,X)$ to $C(J,X)$ then the operator $\Gamma OS_G : C(J,X) \to P_{cp,c}(C(J,X))$,

$x \to (\Gamma OS_G)(x) := \Gamma(S_{G,x})$ is a closed graph operator in $C(J,X) \times C(J,X)$.

LEMMA 2.3 [17] *Let* Ω *be a bounded and convex set in Banach space X.* $F: \Omega \to 2^{\Omega} \setminus \phi$ *be an upper semi-continuous and condensing multi-valued map. If for every* $x \in \Omega$, $F(x)$ *is closed and convex set in* Ω, *then* F *has a fixed point in* Ω.

3. Controllability Result

We are now able to state and prove our main controllability result.

THEOREM 3.1 *Let* $x_0 \in X_\alpha$. *If the hypotheses* (*H*1)-(*H*5) *are satisfied, then the system* (1) *is controllable provided*

$$L_0 := L\left[(M+1)M_0 + \frac{1}{\beta}C_1 - \beta\alpha^\beta\right] < 1 \quad (3.1)$$

$$(M_0L_1 + L_2)M + M_0L_1 + \frac{1}{\beta}C_1 - \beta\alpha^\beta L_1 + \frac{1}{(1-\alpha)}C_\alpha \alpha^{1-\alpha}\gamma < 1 \quad (3.2)$$

and

$$M_3 = \|z_1\|_\alpha + M_1\|x_0\|_\alpha + (M+1)(L_2 l + L_2') + (M+1)M_0L_1(l+1) + \frac{C_{1-\beta}}{\beta}L_1(l+1)b^\beta + \frac{C_\alpha}{1-\alpha}w(l)b^{1-\alpha} + M\sum_{k=1}^m d_k, \quad (3.3)$$

where, $M_0 = \left\|A^{-\beta}\right\|$.

Proof. Let $C := C(J, \overline{D(A)})$ denote the Banach space of continuous functions from J to $\overline{D(A)}$ normed by

$$\|x\|_C = sup\left\{|x(t)| : t \in J\right\}$$

Using hypothesis (H5) for an arbitrary function $x(.)$ and $z_1 \in \overline{D(A)}$ define the control

$$u_x(t) = \tilde{W}^{-1}\Big[z_1 - g(x) - T(b)\left[x_0 - g(x) - F(0, x(h_1(0)))\right] - F(b, x(h_1(b))) + \int_0^b AT(b-s)F(s, x(h_1(s)))\mathrm{d}s + \int_0^b T(b-s)v(s)\mathrm{d}s + \sum_{0<t_k<t} T(t-t_k)I_k(x(t_k^-))\Big]$$

Using the above control, define a multi-valued map $N: \Omega \to 2^{\Omega}$ by

$$N(x) = \Big\{y \in \Omega : y(t) = T(t)\left[x_0 - g(x) - F(0, x(h_1(0)))\right] + F(t, x(h_1(t))) + \int_0^t AT(t-s)F(s, x(h_1(s)))\mathrm{d}s + \int_0^t T(t-s)v(s)\mathrm{d}s + \int_0^t T(t-s)(Bu)(s)\mathrm{d}s + \sum_{0<t_k<t} T(t-t_k)I_k(x(t_k^-)); t \in J, v \in S_{G,x}\Big\}.$$

By assumption on F, g, I_k and the fact that $x_0 \in X_\alpha$, it is obvious that $y(t) \in X_\alpha$.

Clearly the fixed points of N are mild solutions to (1). We shall show that N satisfies the hypotheses of Lemma 2.3. The proof will be given in several steps. **Step 1:** There exists a positive number $l \in N$ such that $N(H_l) \subset H_l$, where $H_l = \left\{x \in \Omega : \|x(t)\|_\alpha \le l, 0 \le t \le \alpha\right\}$.

For each positive number l, H_l is clearly a bounded closed convex set in Ω. We claim that there exists a positive integer l such that $N(H_l) \subset H_l$, where $N(H_l) = \cup_{x \in H_l} N(x)$. If it is not true, then for each positive integer l, there exist the functions $x_l(.) \in H_l$ and $y_l \in N(x_l)$, but $y_l(.) \notin H_l$, that is $\|y_l(t)\|_\alpha > l$ for some $t(l) \in [0,b]$, where $t(l)$ denotes t is dependent on l. However on the other hand we have,

$$l < \|y_l(t)\|_\alpha = \Big\|T(t)\left[x_0 - g(x_l) - F(0, x_l(h_1(0)))\right] + F(t, x_l(h_1(t))) + \int_0^t AT(t-s)F(s, x_l(h_1(s)))\mathrm{d}s + \int_0^t T(t-s)v_l(s)\mathrm{d}s + \int_0^t T(t-s)(Bu_l)(s)\mathrm{d}s + \sum_{0<t_k<t} T(t-t_k)I_k(x_l(t_k^-))\Big\|_\alpha, \text{ where } v_l \in S_{G,x_l}$$

Hence,

$$l < \left\| T(t)\left[x_0 - g(x_l) - A^{-\beta}A^{\beta}F(0, x_l(h_1(0)))\right]\right\|_\alpha + \left\|A^{-\beta}A^{\beta}F(t, x_l(h_1(t)))\right\|_\alpha + \left\|\int_0^t A^{1-\beta}T(t-s)A^{\beta}F(s, x_l(h_1(s)))\mathrm{d}s\right\|_\alpha + \left\|\int_0^t T(t-s)v_l(s)\mathrm{d}s\right\|_\alpha + \left\|\int_0^t T(t-s)(Bu_l)(s)\mathrm{d}s\right\|_\alpha + \sum_{0<t_k<t}\|T(t-t_k)\|_\alpha \left\|I_k(x_l(t_k^-))\right\|_\alpha$$

$$l \le M\left[\|x_0\|_\alpha + L_2 l + L_2' + M_0 L_1(l+1)\right] + M_0 L_1(l+1) + \int_0^t \frac{C_{1-\beta}}{(t-s)^{(1-\beta)}} L_1(l+1)\mathrm{d}s + \int_0^t \frac{C_\alpha}{(t-s)^\alpha} w_1(l)\mathrm{d}s + \int_0^t \frac{C_\alpha}{(t-s)^\alpha} M_1 M_2 M_3 \mathrm{d}s + M\sum_{k=1}^m d_k$$

Dividing on both sides by l and taking the lower limit as $l \to +\infty$ we get

$$(M_0 L_1 + L_2)M + M_0 L_1 + C_{1-\beta}\frac{b^\beta}{\beta}L_1 + C_\alpha \frac{b^{1-\alpha}}{1-\alpha}\gamma \ge 1.$$

This is a contradiction with Formula (2). Hence for some positive integer $N(H_l) \subseteq H_l$.

Step 2: $N(x)$ is convex for each $x \in \Omega$.

Indeed if $y_1, y_2 \in N(x)$ then there exists $v_1, v_2 \in S_{G,x}$ such that for each $t \in J$, we have

$$y_i(t) = T(t)\left[x_0 - g(x) - F(0, x(h_1(0)))\right] + F(t, x(h_1(t))) + \int_0^t AT(t-s)F(s, x(h_1(s)))\mathrm{d}s + \int_0^t T(t-s)v_i(s)\mathrm{d}s + \int_0^t T(t-s)(Bu_i)(s)\mathrm{d}s + \sum_{0<t_k<t} T(t-t_k)I_k(x(t_k^-)); i = 1,2.$$

Let $0 \le \lambda \le 1$. Then for each $t \in J$ we have

$$\left[\lambda y_1 + (1-\lambda)y_2\right](t) = T(t)\left[x_0 - g(x) - F(0, x(h_1(0)))\right] + F(t, x(h_1(t))) + \int_0^t AT(t-s)F(s, x(h_1(s)))\mathrm{d}s + \int_0^t T(t-s)\left[\lambda v_1(s) + (1-\lambda)v_2(s)\right]\mathrm{d}s + \int_0^t T(t-s)B\left[\lambda u_1(s) + (1-\lambda)u_2(s)\right]\mathrm{d}s + \sum_{0<t_k<t} T(t-t_k)I_k(x(t_k^-))$$

Since $S_{G,x}$ is convex because G has convex values, $\lambda y_1 + (1-\lambda)y_2 \in N(x)$.

Step 3: $N(x)$ is closed for each $x \in \Omega$.

Let $\{y_n\}_{n\ge 0} \in N(x)$ such that $y_n \to y$ in Ω. Then $y \in \Omega$ and there exists $v_n \in S_{G,x}$ such that for every $t \in J$,

$$y_n(t) = T(t)\left[x_0 - g(x) - F(0, x(h_1(0)))\right] + F(t, x(h_1(t))) + \int_0^t AT(t-s)F(s, x(h_1(s)))\mathrm{d}s + \int_0^t T(t-s)v_n(s)\mathrm{d}s + \int_0^t T(t-s)(Bu)(s)\mathrm{d}s + \sum_{0<t_k<t} T(t-t_k)I_k(x(t_k^-))$$

Using the fact that G has compact values, we may pass to a subsequence if necessary to get that v_n conver-ges to $v \in L^1(J, X)$ and hence $v \in S_{G,x}$. Then for each $t \in J$,

$$y_n(t) \to y(t) = T(t)\left[x_0 - g(x) - F(0, x(h_1(0)))\right] + F(t, x(h_1(t))) + \int_0^t AT(t-s) \times F(s, x(h_1(s)))\mathrm{d}s + \int_0^t T(t-s)v(s)\mathrm{d}s + \int_0^t T(t-s)(Bu)(s)\mathrm{d}s + \sum_{0<t_k<t} T(t-t_k)I_k(x(t_k^-)); t \in J.$$

Hence $y \in N(x)$.

Step 4: Next we show that the operator N is u.s.c and condensing.

For this purpose, we decompose N as $N = N_1 + N_2$, where the operators N_1, N_2 are defined on H_l respectively by

$$(N_1 x)(t) = F(t, x(h_1)(t)) - T(t)F(0, x(h_1(0)) + \int_0^t AT(t-s)F(s, x(h_1(s)))\mathrm{d}s$$

$$N_2 x = \Big\{ y \in \Omega : y(t) = T(t)\left[x_0 - g(x) - F(0, x(h_1(0)))\right] + F(t, x(h_1(t))) + \int_0^t AT(t-s)F(s, x(h_1(s)))\mathrm{d}s + \int_0^t T(t-s)v(s)\mathrm{d}s + \int_0^t T(t-s)(Bu)(s)\mathrm{d}s + \sum_{0<t_k<t} T(t-t_k)I_k(x(t_k^-)); v \in S_{G,x}\Big\}.$$

We will verify that N_1 is a contraction while N_2 is a completely continuous operator.

To prove that N_1 is a contraction, we take $x_1, x_2 \in H_l$ arbitrarily. Then for each $t \in J$ and by condition (H2), we have that

$$\begin{aligned}
&\|(N_1 x_1)(t) - (N_1 x_2)(t)\|_\alpha \\
&\le \|F(t, x_1(h_1(t))) - F(t, x_2(h_1(t)))\|_\alpha + \|T(t)[F(0, x_1(h_1(0))) - F(0, x_2(h_1(0)))]\|_\alpha \\
&\quad + \left\|\int_0^t AT(t-s)\left[F(s, x_1(h_1(s))) - F(s, x_2(h_1(s)))\right]\mathrm{d}s\right\|_\alpha \\
&= \left\|A^{-\beta}\left[A^\beta F(t, x_1(h_1(t))) - A^\beta F(t, x_2(h_1(t)))\right]\right\|_\alpha + \left\|T(t)A^{-\beta}\left[A^\beta F(0, x_1(h_1(0)) - A^\beta F(0, x_2(h_1(0)))\right]\right\|_\alpha \\
&\quad + \left\|\int_0^t A^{1-\beta}T(t-s) \cdot \left[A^\beta F(s, x_1(h_1(s))) - A^\beta F(s, x_2(h_1(s)))\right]\mathrm{d}s\right\|_\alpha \\
&\le \left[(M+1)M_0 L + \int_0^t \frac{C_{1-\beta}}{(t-s)^{1-\beta}} L\mathrm{d}s\right] . \sup_{0\le s\le\alpha}\|x_1(s) - x_2(s)\|_\alpha \\
&\le L\left[(M+1)M_0 + \frac{1}{\beta}C_{1-\beta}\alpha^\beta\right]\sup_{0\le s\le\alpha}\|x_1(s) - x_2(s)\|_\alpha \\
&= L_0 \sup_{0\le s\le\alpha}\|x_1(s) - x_2(s)\|_\alpha
\end{aligned}$$

Thus, $\|N_1x_1 - N_2x_2\|_\alpha \le L_0\|x_1 - x_2\|_\alpha$. Therefore by assumption $0 < L_0 < 1$, N_1 is a contraction.

Next we show that N_2 is u.s.c. and condensing.

i) $N_2(H_l)$ is clearly bounded.

ii) $N_2(H_l)$ is equi-continuous.

Let $\tau_1, \tau_2 \in J, \tau_1 < \tau_2$. Let $x \in H_l$ and $y \in N_2(x)$. Then there exists $v \in S_{G,x}$ such that for each $t \in J$, we have

$$y(t) = T(t)\left[x_0 - g(x)\right] + \int_0^t T(t-s)v(s)\mathrm{d}s + \int_0^t T(t-s)(Bu)(s)\mathrm{d}s + \sum_{0<t_k<t} T(t-t_k)I_k(x(t_k^-))$$

Then,

$$\begin{aligned}
&\|y(\tau_2) - y(\tau_1)\|_\alpha \\
&\le \|[T(\tau_2) - T(\tau_1)](x_0 - g(x))\|_\alpha \\
&+ \left\|\int_0^{\tau_1}[T(\tau_2 - s) - T(\tau_1 - s)]v(s)\mathrm{d}s\right\|_\alpha + \left\|\int_{\tau_1}^{\tau_2} T(\tau_2 - s)v(s)\mathrm{d}s\right\|_\alpha \\
&+ \left\|\int_0^{\tau_1}\left[T(\tau_2 - s) - T(\tau_1 - s)\right]B\tilde{W}^{-1}\right. \\
&\quad \cdot\left\{z_1 - g(x) - T(b)\left[x_0 - g(x) - F(0, x(h_1(0)))\right]\right. \\
&\quad - F(b, x(h_1(b))) - \int_0^b AT(b-\eta)F(\eta, x(h_1(\eta)))\mathrm{d}\eta \\
&\quad \left.\left. - \int_0^b T(b-\eta)v(\eta)\mathrm{d}\eta - \sum_{0\le t_k\le\tau_1} T(t-t_k)I_k(x(t_k^-))\right\}\mathrm{d}s\right\|_\alpha \\
&+ \left\|\int_{\tau_1}^{\tau_2} T(\tau_2 - s)B\tilde{W}^{-1}\right. \\
&\quad \cdot\left\{z_1 - g(x) - T(b)\left[x_0 - g(x) - F(0, x(h_1(0)))\right]\right. \\
&\quad - F(b, x(h_1(b))) - \int_0^b AT(b-\eta)F(\eta, x(h_1(\eta)))d\eta \\
&\quad \left.\left. - \int_0^b T(b-\eta)v(\eta)\mathrm{d}\eta + \sum_{\tau_1\le t_k\le\tau_2} T(t-t_k)I_k(x(t_k^-))\right\}\mathrm{d}s\right\|_\alpha
\end{aligned}$$

The right hand side tends to zero as $(\tau_2 - \tau_1) \to 0$, since $T(t)$ is strongly continuous and the compactness of $\{T(t)\}_{t\ge 0}$ implies the continuity in the uniform operator topology. Thus $N_2(.)$ is equi-continuous on H_l.

iii) $(N_2H_l)(t)$ is relatively compact for each $t \in J$, where $(N_2H_l)(t) = \{y(t) : y \in (N_2H_l)\}$.

Obviously, by condition $(H4)(ii), (N_2H_l)(t)$ is relatively compact in X_α for $t = 0$. Let $0 < t \le b$ be fixed and $0 < \varepsilon < t$. For $x \in H_l$ and $y \in N_2(x)$, there exists a function $v \in S_{G,x}$ such that

$$\begin{aligned}
y(t) &= T(t)[x_0 - g(x)] + \int_0^{t-\varepsilon} T(t-s)v(s)\mathrm{d}s \\
&+ \int_{t-\varepsilon}^t T(t-s)v(s)\mathrm{d}s + \int_0^{t-\varepsilon} T(t-s)(Bu)(s)\mathrm{d}s \\
&+ \int_{t-\varepsilon}^t T(t-s)(Bu)(s)\mathrm{d}s + \sum_{0<t_k<t} T(t-t_k)I_k(x(t_k^-))
\end{aligned}$$

Define,

$$\begin{aligned}
y_\varepsilon(t) &= T(t)\left[x_0 - g(x)\right] + \int_0^{t-\varepsilon} T(t-s)v(s)\mathrm{d}s \\
&+ \int_0^{t-\varepsilon} T(t-s)(Bu)(s)\mathrm{d}s + \sum_{0<t_k<t} T(t-t_k)I_k(x(t_k^-)) \\
y_\varepsilon(t) &= T(t)\left[x_0 - g(x)\right] + T(\varepsilon)\int_0^{t-\varepsilon} T(t-\varepsilon-s)v(s)\mathrm{d}s \\
&+ T(\varepsilon)\int_0^{t-\varepsilon} T(t-\varepsilon-s)(Bu)(s)\mathrm{d}s \\
&+ \sum_{0<t_k<t} T(t-t_k)I_k(x(t_k^-))
\end{aligned}$$

Since $T(t)$ is compact, the set $Y_\varepsilon(t) = \{y_\varepsilon(t) : y \in N_2(H_l)\}$ is relatively compact in X_α for every $\varepsilon, 0 < \varepsilon < t$.

Moreover, for every $y \in N_2(H_l)$,

$$\begin{aligned}
&\|y(t) - y_\varepsilon(t)\|_\alpha \\
&= \left\|\int_{t-\varepsilon}^t T(t-s)v(s)\mathrm{d}s\right\|_\alpha + \left\|\int_{t-\varepsilon}^t T(t-s)(Bu)(s)\mathrm{d}s\right\|_\alpha \\
&\le M\int_{t-\varepsilon}^t w(l)\mathrm{d}s \\
&+ MM_1M_2\int_{t-\varepsilon}^t \Bigg\{\|z_1\|_\alpha + \|g(x)\|_\alpha \\
&\qquad + \|T(b)\|\left[\|x_0\| + \|g(x)\| + \|F(0, x(h_1(0)))\|\right] \\
&\qquad + \|F(b, x(h_1(b)))\| \\
&\qquad + \left\|\int_0^b AT(b-\eta)F(\eta, x(h_1(\eta)))\mathrm{d}\eta\right\| \\
&\qquad + \left\|\int_0^b T(b-\eta)v(\eta)\mathrm{d}\eta\right\| \\
&\qquad + \left\|\sum_{0<t_k<t} T(t-t_k)I_k(x(t_k^-))\right\|\Bigg\}\mathrm{d}s \\
&\le Mw(l)\varepsilon \\
&+ MM_1M_2\int_{t-\varepsilon}^t \Bigg\{\|z_1\|_\alpha + (L_2l + L_2') \\
&\qquad + M\left[\|x_0\| + (L_2l + L_2') + M_0L_1(l+1)\right] \\
&\qquad + M_0L_1(l+1) + \int_0^b \frac{C_{1-\beta}}{(b-\eta)^{(1-\beta)}}L_1(l+1)\mathrm{d}\eta \\
&\qquad + \int_0^b \frac{C_\alpha}{(b-\eta)^\alpha}w(l)\mathrm{d}\eta + M\sum_{k=1}^m d_k\Bigg\}\mathrm{d}s \\
&\le Mw(l)\varepsilon + MM_1M_2M_3\varepsilon \\
&\le M\varepsilon\left[w(l) + M_1M_2M_3\right]
\end{aligned}$$

Therefore, letting $\varepsilon \to 0$, we see that there are relatively compact sets arbitrarily close to the set $\{y(t) : y \in N_2(H_l)\}$. Hence the set $\{y(t) : y \in N_2(H_l)\}$ is relatively compact in X_α.

As a consequence of (i),(ii), (iii) and together with the Arzela-Ascoli theorem we can conclude that $N_2 : H_l \to 2^{H_l}$ is a completely continuous multi-valued map and, therefore, a condensing multi- valued map.

iv) N_2 has a closed graph.

From the above steps we can see, for every $x \in H_l$, $N_2(x)$ is relatively compact and closed set, which can be testified as in Step 3. Hence $N_2(x)$ is a compact set.

Let $x_n \to x_*$, $x_n \in H_l, y_n \in N_2(x_n)$ and $y_n \to y_*$. We must show that $y_* \in N_2(x_*)$; $y_n \in N_2(x_n)$ means that there exists $v_n \in S_{G,x_n}$ such that , for each $t \in J$.

$$y_n(t) = T(t)\left[x_0 - g(x_n)\right] + \int_0^t T(t-s)v_n(s)\mathrm{d}s + \int_0^t T(t-s)(Bu_n)(s)\mathrm{d}s + \sum_{0<t_k<t} T(t-t_k)I_k(x_n(t_k^-))$$

where

$$u_{x_n}(t) = \tilde{W}^{-1}\Big[z_1 - g(x_n) - T(b)\left[x_0 - g(x_n) - F(0, x(h_1(0)))\right] - F(b, x(h_1(b))) + \int_0^b AT(b-s)F(s, x(h_1(s)))\mathrm{d}s + \int_0^b T(b-s)v_n(s)\mathrm{d}s + \sum_{0<t_k<t} T(t-t_k)I_k(x_n(t_k^-))\Big]$$

We must prove that there exists $v_* \in S_{G,x_*}$ such that

$$y_*(x) = T(t)\left[x_0 - g(x_*)\right] + \int_0^t T(t-s)v_*(s)\mathrm{d}s + \int_0^t T(t-s)(Bu_*)(s)\mathrm{d}s + \sum_{0<t_k<t} T(t-t_k)I_k(x_*(t_k^-))$$

where

$$u_{x_*}(t) = \tilde{W}^{-1}\Big[z_1 - g(x_*) - T(b)\left[x_0 - g(x_*) - F(0, x(h_1(0)))\right] - F(b, x(h_1(b))) + \int_0^b AT(b-s)F(s, x(h_1(s)))\mathrm{d}s + \int_0^b T(b-s)v_*(s)\mathrm{d}s + \sum_{0<t_k<t} T(t-t_k)I_k(x_*(t_k^-))\Big]$$

Clearly, since $I_k, k = 1, 2, \cdots, m$ and g are continuous we have that

$$\left\| \left(y_n(t) - T(t)\left[x_0 - g(x_n)\right] - \int_0^t T(t-s)(Bu_n)(s)\mathrm{d}s - \sum_{0<t_k<t} T(t-t_k)I_k(x_n(t_k^-)) \right) - \left(y_*(x) - T(t)\left[x_0 - g(x_*)\right] - \int_0^t T(t-s)(Bu_*)(s)\mathrm{d}s - \sum_{0<t_k<t} T(t-t_k)I_k(x_*(t_k^-)) \right) \right\|_\Omega \to 0, \text{ as } n \to \infty.$$

Consider the linear continuous operator

$$\Gamma : L^1(J, X) \to C(J, X), v \mapsto \Gamma(v)(t) = \int_0^t T(t-s)v(s)\mathrm{d}s.$$

From Lemma (H3) it follows that $\Gamma O S_G$ is a closed graph operator.

Moreover, we obtain that

$$\left(y_n(t) - T(t)\left[x_0 - g(x_n)\right] - \int_0^t T(t-s)(Bu_n)(s)\mathrm{d}s - \sum_{0<t_k<t} T(t-t_k)I_k(x_n(t_k^-)) \right) \in \Gamma(S_{G,x_n}).$$

Since $x_n \to x_*$, it follows from (H3) that

$$y_*(t) - T(t)\left[x_0 - g(x_*)\right] \int_0^t T(t-s)(Bu_*)(s)\mathrm{d}s - \sum_{0<t_k<t} T(t-t_k)I_k(x_*(t_k^-)) \in \Gamma(S_{G,x_*})$$

that is, there must exist a $v_*(t) \in S_{G,x_*}$ such that

$$y_*(t) - T(t)\left[x_0 - g(x_*)\right] - \int_0^t T(t-s)(Bu_*)(s)\mathrm{d}s - \sum_{0<t_k<t} T(t-t_k)I_k(x_*(t_k^-)) = \Gamma(v_*(t)) = \int_0^t T(t-s)v_*(s)\mathrm{d}s.$$

Therefore, N_2 has a closed graph. Since N_2 is a completely continuous multi-valued map with compact value, N_2 is u.s.c. On the other hand N_1 is a contraction. Hence $N = N_1 + N_2$ is u.s.c. and condensing.

By Lemma 2.3, there exists a fixed point $x(.)$ for N on H_l. Therefore, the nonlocal Cauchy problem with impulsive effect (1) is controllable on J.

Particularly, if $G(t, x(h_2(t)))$ is a single-valued map, then the system (1) will become

$$\begin{cases} \dfrac{\mathrm{d}}{\mathrm{d}t}\left[x(t) - F\left(t, x\left(h_1(t)\right)\right)\right] = Ax(t) + Bu(t) + G\left(t, x\left(h_2(t)\right)\right); \\ \qquad t \in J := [0, b]; \\ \Delta x|_{t=t_k} = I_k(x(t_k^-)); k = 1, 2, \cdots, m;\ t \neq t_k; \\ \qquad x(0) + g(x) = x_0 \end{cases} \tag{4}$$

by using Sadovskii's fixed-point theorem for condensing map, we can analogously study the controllability of the system (4).

(H3)' The function $G : J \times X_\alpha \to X$ satisfies the following conditions:

i) for each $t \in J$, the function $G(t,.) : X_\alpha \to X$ is continuous; and for each $x \in X_\alpha$, the function $G(.,x): J \to X$ is strongly measurable.

ii) for each positive number $l \in N$, there exists a positive function $w(l)$ dependent on l such that

$$sup\left\|G(t,x)\right\| \le w(l) \text{ and } \lim_{l\to\infty}\frac{w(l)}{l} = \gamma < \infty$$

where

$$\|x\|_\alpha = \sup_{0\le s\le\alpha}\|x(s)\|_\alpha .$$

THEOREM 3.2 *Let* $x_0 \in X_\alpha$. *If the hypotheses* $(H1),(H2),(H3)',(H4)$ and $(H5)$ *are satisfied, then the system* (4) *is controllable on* J *provided* (1), (2) *and* (1) *hold.*

Proof The mild solution of the system (4) is given by

$$\begin{aligned} x(t) &= T(t)\left[x_0 - g(x) - F(0,x(h_1(0)))\right] \\ &+ F(t,x(h_1(t))) + \int_0^t AT(t-s)F(s,x(h_1(s)))\mathrm{d}s \\ &+ \int_0^t T(t-s)G(s,x(h_2(s))\mathrm{d}s + \int_0^t T(t-s)(Bu)(s)\mathrm{d}s \\ &+ \sum_{0<t_k<t} T(t-t_k)I_k(x(t_k^-)); t \in J. \end{aligned}$$

We define the operator $N:\Omega\to\Omega$ by

$$\begin{aligned} (Nx)(t) &= T(t)\left[x_0 - g(x) - F(0,x(h_1(0)))\right] \\ &+ F(t,x(h_1(t))) + \int_0^t AT(t-s)F(s,x(h_1(s)))\mathrm{d}s \\ &+ \int_0^t T(t-s)G(s,x(h_2(s)))\mathrm{d}s + \int_0^t T(t-s)(Bu)(s)\mathrm{d}s \\ &+ \sum_{0<t_k<t} T(t-t_k)I_k(x(t_k^-)) \end{aligned}$$

Then we can decompose N as $N = N_1 + N_2$, where

$$\begin{aligned} (N_1x)(t) &= F(t,x(h_1)(t)) - T(t)F(0,x(h_1(0)) \\ &+ \int_0^t AT(t-s)F(s,x(h_1(s)))\mathrm{d}s \end{aligned}$$

and

$$\begin{aligned} (N_2x)(t) &= T(t)\left[x_0 - g(x)\right] + \int_0^t T(t-s)G(s,x(h_2(s)))\mathrm{d}s \\ &+ \int_0^t T(t-s)(Bu)(s)\mathrm{d}s \end{aligned}$$

and verify that N_1 is a contraction while N_2 is a compact operator, thus Sadovskii's fixed-point theorem can be applied to the operator N and hence N has atleast a fixed point on Ω, which shows that system (4) is controllable on the interval J. The desired proof is similar to Step 4 of Theorem 3.1.

4. Example

As an application of Theorem 3.2, we study the following impulsive partial function differential system with nonlocal condition

$$\begin{cases} \dfrac{\partial}{\partial t}\left[z(t,x) - \int_0^\pi b(t,y,x)\left[z(\sin t,y) + \dfrac{\partial z}{\partial y}(\sin t,y)\right]\mathrm{d}y\right] \\ = \dfrac{\partial^2 z(t,x)}{\partial x^2} + u(t,y) + h\left(t, z(\sin t,x), \dfrac{\partial z}{\partial x}(\sin t,x)\right), \\ \qquad 0 \le t \le 1, 0 \le x \le \pi, t \ne t_k, k = 1,2,\cdots,m; \\ z(t,0) = z(t,\pi) = 0;\ z(t_k^+) - z(t_k^-) = I_k(z(t_k)), \\ \qquad k = 1,2,\cdots,m \\ z(0,x) + \sum_{i=0}^{p}\int_0^\pi k_i(y,x)z(s_i,y)\mathrm{d}y = z_0(x), 0 \le x \le \pi \end{cases} \tag{4.1}$$

where p is a positive integer, $0 < s_0 < s_1 < \cdots < s_p < 1$, and $0 < t_1 < t_2 < \cdots t_m < \cdots < 1; z_0(x) \in X = L^2([0,\pi])$. A is defined by $A\omega = \omega''$ with the domain

$$\begin{aligned} D(A) = H_0^2([0,\pi]) = \{\omega(.) \in X : \omega, \\ \omega' \text{ are absolutely continuous}, \\ \omega'' \in X, \omega(0) = \omega(\pi) = 0\} \end{aligned}$$

Then A generates a strongly continuous semigroup $T(.)$ which is compact, analytic and self-adjoint.

a') Also A has a discrete spectrum representation

$$A\omega = \sum_{n=1}^{\infty}(-n^2) < \omega, \omega_n > \omega_n, \omega \in D(A), n \in N;$$

where $\omega_n(x) = \sqrt{\dfrac{2}{\pi}}sin(nx); n = 1,2,\cdots$ is the orthogonal set of eigenvector of A. The eigenvalues are $-n^2, n \in N$.

b') The operator $A^{\frac{1}{2}}$ is given by

$$A^{\frac{1}{2}}\omega = \sum_{n=0}^{\infty} n < \omega, \omega_n > \omega_n$$

on the space $D(A^{\frac{1}{2}}) = \left\{\omega(.) \in X : \sum_{n=1}^{\infty} n < \omega, \omega_n > \omega_n \in X\right\}$.

The control operator $B: L^2(J,X) \to X$ is defined by $(Bu)(t)(y) = u(t,y); y \in (0,\pi)$ which satisfies condition (H5). Here B is an identity operator and the control function $u(.)$ is given in $L^2([0,\pi],U)$.

We assume that the following conditions hold:

i) The function b is measurable and

$$\sup_{0\le t\le 1}\int_0^\pi\int_0^\pi b^2(t,y,x)\mathrm{d}y\mathrm{d}x < \infty.$$

ii) The function $\dfrac{\partial^2 b(t,y,x)}{\partial x^2}$ is measurable,

$b(t,y,0) = b(t,y,\pi) = 0$, and

$$N_1 = \sup_{0\le t\le 1}\left[\int_0^\pi\int_0^\pi\left(\frac{\partial^2 b(t,y,x)}{\partial x^2}\right)^2 \mathrm{d}y\mathrm{d}x\right]^{\frac{1}{2}} < \infty.$$

iii) For the function $h:[0,1]\times R\times R\to R$ the following three conditions are satisfied:

1) For each $t\in[0,1], h(t,.,.)$ is continuous.

2) For each $z\in X_{\frac{1}{2}}, h(.,z,z')$ is measurable.

3) There is a positive number c_1 such that

$$\|g(t,z,z')\|\le c_1\|z\|,$$

for all $(t,z)\in[0,1]\times X_{\frac{1}{2}}$.

iv) $I_k\in C\left(X_{\frac{1}{2}},X_{\frac{1}{2}}\right), k=1,\cdots,$ and there exist constants $d_k, k=1,\cdots,m,$ such that

$$\|I_k(z)\|_{\frac{1}{2}}\le d_k, z\in X_{\frac{1}{2}}.$$

Here we choose $\alpha=\beta=\frac{1}{2}.$ According to paper [21], we know that, if $z\in X_{\frac{1}{2}}$, then z is absolutely continuous , $z'\in X,$ and $z(0)=z(\pi)=0.$ In view of this result, for $(t,z)\in[0,1]\times X_{\frac{1}{2}}, \omega\in\Omega$ (Ω is defined as in Section 3), we can define respectively that

$$F(t,z)(x)=\int_0^\pi b(t,y,x)\left[z(y)+z'(y)\right]\mathrm{d}y.$$

$$G(t,z)(x)=h(t,z(x),z'(x)),$$

and

$$g(\omega(t))=\sum_{i=0}^{p}K_i\omega_1(s_i), \omega\in\Omega,$$

where $K_i:X_{\frac{1}{2}}\to X_{\frac{1}{2}}$ is completely continuous [16] such that $K_i(z)(x)=\int_0^\pi k_i(y,x)z(y)\mathrm{d}y$ and $G:[0,1]\times X_{\frac{1}{2}}\to X$ It is easy to see that

$F:[0,1]\times X_{\frac{1}{2}}\to X_{\frac{1}{2}}, A^{\frac{1}{2}}F:[0,1]\times X_{\frac{1}{2}}\to X_{\frac{1}{2}}\cdots$ In fact, for each $t\in[0,1],$ we have

$$\langle F(t,z),\omega_n\rangle = \frac{1}{n}\sqrt{\frac{2}{\pi}}\left\langle\int_0^\pi\frac{\partial b(t,y,x)}{\partial x}\left[z(y)+z'(y)\right]\mathrm{d}y,\cos(nx)\right\rangle,$$

also,

$$\langle F(t,z),\omega_n\rangle = \frac{-1}{n^2}\sqrt{\frac{2}{\pi}}\left\langle\int_0^\pi\frac{\partial^2 b(t,y,x)}{\partial x^2}\left[z(y)+z'(y)\right]\mathrm{d}y,\sin(nx)\right\rangle.$$

This shows that F and $A^{\frac{1}{2}}F$ both take values in $X_{\frac{1}{2}}$ in terms of properties (a') and (b'), and therefore the function g. Since, for any $x_1,x_2\in X_{\frac{1}{2}}$,

$$\|x_2-x_1\|^2=\sum_{n=0}^{\infty}\langle x_2-x_1,z_n\rangle^2\le\sum_{n=0}^{\infty}n^2\langle x_2-x_1,z_n\rangle^2$$
$$\le\|x_2-x_1\|_{\frac{1}{2}}^2.$$

This inequality alongwith condition (ii) says that (H2) is satisfied. Also G satisfies $(H3)'$ and g satisfies (H4). By (i), $F(t,z)$ is a bounded linear operator on X. Thus $(H1),(H2),(H3)',(H4),(H5)$ are satisfied and the system (1) is controllable on [0,1].

5. Exact Controllability in Infinite Dimensional Space

It has been observed that the example in ([2-11,22]) cannot be recovered as special case of the abstract result. If the semigroup is compact then the assumption (H1) in Section 2 is valid only in finite dimensional space so the applications are restricted to ordinary differential control system but not to partial differential equations (refer [15]). We have tried to overcome to this problem in Section 3 for the inclusion (1). Here we present another way of exact controllability result of the system (4) in infinite dimension space.

LEMMA 5.1 *Let* $\mathcal{PC}([0,\tau]),X_\alpha)$ *be a space formed by normalized piecewise continuous function* $([0,\tau],X_\alpha)$. *Let* $\mathcal{B}\subseteq\mathcal{PC}$ *such tha* $\tilde{\mathcal{B}}_k=\{\tilde{V}_k:V\in\mathcal{B}\}$; *where*

$$\tilde{V}_k(t)=\begin{cases}V(t);\ t\in(t_k,t_{k+1}]\\ V(t_k^+);\ t=t_k\end{cases}$$

The set $\mathcal{B}\subseteq\mathcal{PC}$ is relatively compact if and only if each set $\tilde{\mathcal{B}}_k$ is relatively compact in the space $C([t_k,t_{k+1}];X_\alpha)$.

THEOREM 5.2 *Assume that the function F and G verify the assumptions* $(H1)$ *and* $(H2)'$ *respectively and suppose that the following conditions are fulfilled:*

a1) For every $r>0$ and all $\varepsilon>0$ there are compact sets $U_{\varepsilon,r}^i\subseteq X_\alpha, i=1,2,$ such that $T(\varepsilon)A^\beta F(s,\xi)\in U_{\varepsilon,r}^1$ and $T(\varepsilon)G(s,\xi)\in U_{\varepsilon,r}^2$ for every $F(\xi):J\times\mathcal{B}_r(0,\Omega)$.

b1) Conditions $(H3)$ and $(H4)$ are satisfied. Then there exists a mild solution of the system (3.4).

Proof Consider the system (3.4). As a main portion of the theorem, we prove that N is completely continuous operator.

The mild solution given in Theorem (3.2) can be splitted up into following four parts:

$$N^{(1)}(t) = T(t)\left[x(0) - g(x) - F(0, x(h_1(0)))\right] + F(t, x(h_1(t)))$$

$$N^{(2)}(t) = \int_0^t AT(t-s)F(s, x(h_1(s)))\mathrm{d}s$$

$$N^{(3)}(t)z = \int_0^t T(t-s)G(s, x(h_2(s)))\mathrm{d}s + \int_0^t T(t-s)Bu(s)\mathrm{d}s$$

$$N^{(4)}(t) = \sum_{0 \le t_k \le t} T(t-t_k)I_k(x(t_k^-)), \text{for each } t \in J.$$

Obviously each $N^{(i)}$ is continuous. To prove that N is compact operator we will show separately that $N^{(i)}(\mathcal{B}_r(0,\Omega))$ is relatively compact in Ω for every $x \in \mathcal{B}_r = \mathcal{B}_r(0,\Omega)$.

Step 1: Let $N^{(1)}(\mathcal{B}_r)$ is relatively compact in Ω.

Let $V = N^{(1)}(\mathcal{B}_r)$. Let $T(.)$ is uniformly bounded on $[0,b]$ and continuous for the norm of the operator in $(0,b]$, we can observe that the sets $\tilde{V}_k \subseteq -T(t)F(0, x(h_1(0))) + A^{-\beta}U^1_{\varepsilon,r}$; it follows that $\tilde{V}_k$ is relatively compact for every $t \in [t_k, t_{k+1}], k = 0,1,2,\cdots,m.$

Step 2: Let $N^{(2)}(\mathcal{B}_r)$ is relatively compact in Ω.

We first show that $N^2(\mathcal{B}_r(t))$ is relatively compact for each $t \in J$. For $t = 0$ it is trivial. Assume that $0 < 2\varepsilon < t \le b$ and let $U^1_{\varepsilon,r}$ be the compact set introduced in the hypothesis $(a1)$. Since $A^{\beta}T(.)$ is strongly continuous on $[\varepsilon, b]$, it follows that $U_\varepsilon = \left\{A^{1-\beta}T(s)x : s \in [\varepsilon, b], x \in U^1_{\varepsilon,r}\right\}$ is relatively compact in X_α.

Now using mean value theorem for Bochner integral, we can write

$$\begin{aligned} N^{(2)}x(t) &= \int_0^{t-2\varepsilon} A^{1-\beta}T(t-s-\varepsilon)T(\varepsilon)A^{\beta}F(s, x(h_1(s))\mathrm{d}s \\ &\quad + \int_{t-2\varepsilon}^{t} A^{1-\beta}T(t-s)A^{\beta}F(s, x(h_1(s))\mathrm{d}s \\ &\in (t-2\varepsilon)\overline{co(U_\varepsilon)} + (\mathcal{B}_{r^*})(0, X_\alpha) \end{aligned}$$

for each $x \in (\mathcal{B}_r)$, where $co(U_\varepsilon)$ denotes convex hull of U_ε and

$$r^* = 2^{\beta}C_{1-\beta}\left\{L_1\left(\|x\|_\alpha + 1\right)\right\}\frac{\varepsilon^{\beta}}{\beta}.$$

Thus $N^2(\mathcal{B}_r)(t)$ is relatively compact in X_α.

Next we show that $N^{(2)}(\mathcal{B}_r)$ is equi-continuous.

Let $0 \le t_0 < t \le b$. Then

$$\begin{aligned} &N^{(2)}x(t) - N^{(1)}x(t) \\ &= -\int_0^t AT(t-s)F(s, x(h_1(s)))\mathrm{d}s \\ &\quad + \int_0^{t_0} AT(t_0-s)F(s, x(h_1(s)))\mathrm{d}s \\ &= (I - T(t-t_0))N^{(2)}x(t_0) \\ &\quad - \int_{t_0}^t AT(t-s)F(s, x(h_1(s)))\mathrm{d}s \end{aligned}$$

Since the elements $N^{(2)}x(t_0)$; for $x \in \mathcal{B}_r$; are included in a compact set, it follows that the first term on right hand side converges uniformly to zero as $t \to t_0$. Similarly it follows from $(b1)$ that the function $AT(t-s)F(s, x(h_1(s))), x \in \mathcal{B}_r$ are equi-integrable, which imply that the second term on right hand side also converges uniformly to zero as $t \to t_0$.

This show that $N^{(2)}(\mathcal{B}_r)$ is equi-continuous from the right at t_0. Similarly it can be prove that $N^2(\mathcal{B}_r)$ is equi-continuous from the left at $t_0 > 0$. Thus $N^{(2)}(\mathcal{B}_r)$ is equi-continuous and hence $N^{(2)}(\mathcal{B}_r)$ is relatively compact in Ω.

Step 3: By using same argument as in Step 2 we can prove that the set $N^{(3)}(\mathcal{B}_r)$. is relatively compact in Ω.

Step 4: The relatively compactness of $N^{(4)}(\mathcal{B}_r)$. is consequence of assumption (H4) and Lemma 5.1 Hence the proof.

Remark Throughout Section 5 we have used compactness assumption of Theorem 5.2 (a1) and growth condition (H2) (ii) and (H5). If the maps F and $I_k, k = 1,2,\cdots,m$ satisfy some Lipschitz conditions (H2) (i) and (H5) instead of compactness in (a1) then also we can prove controllability result.

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