

Propagation of Torsional Surface Waves under the Effect of Irregularity and Initial Stress

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Abstract

The present paper has been framed to study the influence of irregularity, initial stress and porosity on the propagation of torsional surface waves in an initially stressed anisotropic poro-elastic layer over a semi-infinite heterogeneous half space with linearly varying rigidity and density due to irregularity at the interface. The irregularity has been taken in the half-space in the form of a parabola. It is observed that torsional surface waves propagate in this assumed medium. In the absence of irregularity the velocity of torsional surface wave has been obtained. Further, it has been seen that for a layer over a homogeneous half space, the velocity of torsional surface waves coincides with that of Love waves.

Keywords: Irregularity, Torsional Surface Waves, Anisotropic, Initial Stress

1. Introduction

The study of surface waves in a half-space is important to seismologists due to its possible applications in Geophysical prospecting and in understanding the cause and estimation of damage due to earthquakes. Surface waves carry the greatest amount of energy from shallow shocks and are of primary cause of destruction that can result from earthquakes. The propagation of surface waves in detail are well documented in the text book literature (Achenbach, J. D. [1], Ewing, W. M., Jardetzky, W. S., Press, F. [2], Bath, M. [3]). One type of surface wave may be available in non-homogeneous earth known as torsional surface waves. These waves are horizontally polarized but give a twist to the medium when it propagates. Although much information is available on the propagation of surface waves such as Rayleigh waves, Love waves and Stonely waves etc., the torsional wave has not drawn much attention and very little literature is available on the propagation of this wave. Some papers have been published on the propagation of torsional waves in elastic medium with different types of inhomogeneity. Lord Rayleigh [4] in his remarkable paper showed that the isotropic homogeneous elastic half-space does not allow a torsional surface wave to propagate, in this connection Georgiadis *et al.* [5] have examined the torsional surface wave in a linear gradient-elastic half-space. Meissner [6] pointed out that in an inhomogeneous elastic half-space with quadratic variation of shear

modulus and density varying linearly with depth, torsional surface waves do exist. Vardoulakis, I. [7] has studied the problem on torsional surface waves in inhomogeneous elastic media. Also S. Dey *et al.* [8] studied the propagation of torsional waves in a homogeneous substratum over a heterogeneous half-space.

The study of porous medium in recent time has acquired prime interest. The layer of the earth usually of such materials and the medium is generally dealt under the name of poro-elastic medium. Investigation on propagation of waves in liquid saturated porous solids are relevant to geophysical prospecting methods, survey techniques are very useful in oil industry. While deriving the mechanics of such medium it is assumed that the pore sizes are small and macroscopically speaking their average distribution is uniform. The role of pore water in seismology has been emphasized in many studies. Biot [9] has established the theory of the propagation of elastic waves in a porous elastic solid saturated by a viscous fluid. Under the assumption of dynamic coupling between solid and fluid mass, Biot [10] has developed the mathematical theory for the propagation of elastic waves in a fluid saturated porous medium. Based on this theory, many problems of surface waves in poro-elastic materials have been studied in the past years by Buckingham, M. J. [11], Sharma, M. D. and Gogna, M. L. [12], Sharma, M. D., Kumar, R. and Gogna, M. L. [13], Sharma, M. D., Kumar, R. and Gogna, M. L. [14].

The development of initial stresses in the medium is

due to many reasons, for example resulting from the difference of temperature, process of quenching, shot peening and cold working, slow process of creep, differential external forces, gravity variations etc. These stresses have a pronounced influence on the propagation of waves as shown by Biot [15]. The earth is also an initially stressed medium. It is therefore of much interest to study the influence of these stresses on the propagation of torsional surface wave. The study of surface waves in an initially stressed medium is of interest not for theoretical taste only but for practical purposes too. Based on the pioneering work of Biot [15] on pre-stressed solids, various studies of body and surface wave propagation in the pre-stressed solids have been carried out by many researchers such as Chattopadhyay *et al.* [16], Kar, B. K. and Kalyani, V. K. [17] and Dey, S. and Addy, S. K. [18].

The study of wave propagation in elastic medium with different irregularities is of great importance to seismologists as well as to geophysicists to understand and predict the seismic behavior at the different margins of earth. This fact leads us towards this study. The present paper discusses the possibility of existence of torsional surface wave in an initially stressed anisotropic poro-elastic layer over a semi-infinite heterogeneous half space with linearly varying rigidity and density due to irregularity at the interface. The irregularity has been taken in the half-space in the form of a parabola. It is observed that torsional surface waves propagate in this assumed medium. Irregularity and initial stress play an important role in the seismic wave propagation. This paper has been framed out to show the effect of irregularity at the interface in the parabolic form and initial stress on the propagation of torsional surface wave. As the porosity parameter decreases, the half-space will become an elastic solid with less pores and the velocity of torsional surface waves decreases and ultimately vanishes when the medium is elastic solid. The effect of irregularity and the initial stresses are very prominent on the propagation of torsional surface waves. The presence of initial compressive stress reduces the velocity of torsional surface waves. As the initial compressive stress increases, the velocity of propagation decreases. It is also observed that in the limiting case if the porous medium changes to a liquid layer then a torsional surface wave don't exist. It is interesting to note that in a poro-elastic medium over a homogeneous half-space the torsional wave mode changes to Love wave mode. Parabolic irregularity is mostly found in the earth so the present study can help the seismologists to understand and predict the seismic behaviour at different margins of the earth.

2. Formulation of the Problem

Let us Consider a model which consists of a water satu-

rated porous layer M_1 of thickness H with anisotropy of Weiskopf type under compressive initial stresses

$P' = -s'_{rr}$ along the radial direction and with one parabolic irregularity on the interface between the layer and a semi-infinite non-homogeneous elastic half-space M_2 as shown in **Figure 1**. The heterogeneity has been considered both in density and rigidity. We assume that the irregularity is of the form of a parabola with span of length $2m$ and depth h .

Assuming the origin of the cylindrical co-ordinate system at the middle point of the interface irregularity and the z -axis downward positive, the following variation in rigidity and density has been taken

1) for the layer, $\mu = \mu_0, \rho = \rho_0$

2) for the half-space, $\mu = \mu_1(1+az), \rho = \rho_1(1+bz)$ where μ and ρ are rigidity and density of the media respectively and a, b are constants having dimensions that are inverse of length.

The equation of irregularity has been taken as

$$z = \varepsilon F(r), \text{ where } F(r) = \begin{cases} h \left(1 - \frac{r^2}{m^2} \right) & \text{for } |r| \leq m \\ 0 & \text{for } |r| > m \end{cases}$$

and $\varepsilon = \frac{h}{2m}$ and $\varepsilon \ll 1$.

3. Solution of the Problem

3.1. Solution for Porous Layer

The dynamical equations of initially stressed poroelastic medium is obtained by suitably coupling Biot's [19] dynamical equations of an initially stressed medium with the equations of poro-elasticity given by Weiskopf [20] and Biot [9,10]. Those are

$$\frac{\partial s'_{rr}}{\partial r} + \frac{1}{r} \frac{\partial s'_{r\theta}}{\partial \theta} + \frac{\partial s'_{rz}}{\partial z} + \frac{s'_{rr} - s'_{\theta\theta}}{r} - P' \frac{\partial \omega'_\theta}{\partial z} = \frac{\partial^2}{\partial t^2} (\rho_{rr} u'_r + \rho_{r\theta} U'_r) \tag{1}$$

$$\frac{\partial s'_{r\theta}}{\partial r} + \frac{1}{r} \frac{\partial s'_{\theta\theta}}{\partial \theta} + \frac{\partial s'_{\theta z}}{\partial z} + \frac{2}{r} s'_{r\theta} - P' \frac{\partial \omega'_z}{\partial r} = \frac{\partial^2}{\partial t^2} (\rho_{rr} v'_\theta + \rho_{r\theta} V'_\theta) \tag{2}$$

$$\frac{\partial s'_{rz}}{\partial r} + \frac{1}{r} \frac{\partial s'_{\theta z}}{\partial \theta} + \frac{\partial s'_{zz}}{\partial z} + \frac{1}{r} s'_{rz} - P' \frac{\partial \omega'_\theta}{\partial r} = \frac{\partial^2}{\partial t^2} (\rho_{rr} w'_z + \rho_{r\theta} W'_z) \tag{3}$$

and

$$\frac{\partial s'}{\partial r} = \frac{\partial^2}{\partial t^2} (\rho_{r\theta} u'_r + \rho_{\theta\theta} U'_r) \tag{4}$$

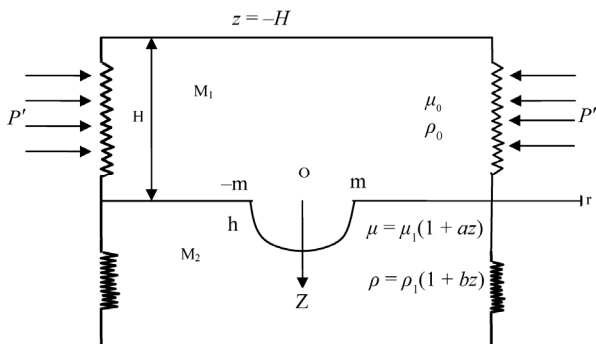


Figure 1. Geometry of the problem.

$$\frac{\partial s'_\theta}{\partial \theta} = \frac{\partial^2}{\partial t^2} (\rho_{r\theta} v'_\theta + \rho_{\theta\theta} V'_\theta) \quad (5)$$

$$\frac{\partial s'_z}{\partial z} = \frac{\partial^2}{\partial t^2} (\rho_{r\theta} w'_z + \rho_{\theta\theta} W'_z) \quad (6)$$

where s'_{ij} = incremental stress components;
 (u'_r, v'_θ, w'_z) = components of the displacement vector of the solid;
 (U'_r, V'_θ, W'_z) = components of the displacement vector of the liquid;
 s' = stress in the liquid,
 and

$$\omega'_r = \frac{1}{2r} \left[\frac{\partial w'_z}{\partial \theta} - r \frac{\partial v'_\theta}{\partial z} \right], \omega'_\theta = \frac{1}{2} \left[\frac{\partial u'_r}{\partial z} - \frac{\partial w'_z}{\partial r} \right] \quad (7)$$

$$\omega'_z = \frac{1}{2r} \left[\frac{\partial (rv'_\theta)}{\partial r} - \frac{\partial v'_\theta}{\partial \theta} \right]$$

are the components of the rotational vector ω' .
 The stress-strain relations for the water saturated anisotropic porous layer under normal initial stress P' are

$$\begin{aligned} s'_{rr} &= (A + P')e_{rr} + (A - 2N + P')e_{\theta\theta} + (F + P')e_{zz} + Q_\varepsilon \\ s'_{\theta\theta} &= (A - 2N)e_{rr} + Ae_{\theta\theta} + Fe_{zz} + Q_\varepsilon \\ s'_{zz} &= Fe_{rr} + Fe_{\theta\theta} + Ce_{zz} + Q_\varepsilon \\ s'_{r\theta} &= 2Ne_{r\theta}, s'_{\theta z} = 2Ge_{\theta z}, s'_{rz} = 2Ge_{rz} \end{aligned} \quad (8)$$

where A, F, C, N and G are elastic constants of the medium. N and G are, in particular, shear moduli of the anisotropic layer in the radial and the z -direction respectively.

$$\begin{aligned} e_{rr} &= \frac{\partial u'_r}{\partial r}, e_{\theta\theta} = \frac{1}{r} \frac{\partial v'_\theta}{\partial \theta} + \frac{u'_r}{r}, e_{zz} = \frac{\partial w'_z}{\partial z} \\ e_{r\theta} &= \frac{1}{2} \left[\frac{1}{r} \frac{\partial u'_r}{\partial \theta} + \frac{\partial v'_\theta}{\partial r} - \frac{v'_\theta}{r} \right], e_{\theta z} = \frac{1}{2} \left[\frac{\partial v'_\theta}{\partial z} + \frac{1}{r} \frac{\partial w'_z}{\partial \theta} \right] \\ e_{rz} &= \frac{1}{2} \left[\frac{\partial w'_z}{\partial r} + \frac{\partial u'_r}{\partial z} \right] \end{aligned} \quad (9)$$

Further, Q_ε being the measure of coupling between

the volume change of the solid and the liquid is a positive quantity. s' is the stress vector due to the liquid. The stress vector s' is related to the fluid pressure P by the relation

$$-s' = fP \quad (10)$$

where f is porosity of the layer.

The mass coefficients $\rho_{rr}, \rho_{r\theta}$ and $\rho_{\theta\theta}$ are related to the densities ρ, ρ_s, ρ_w of the layer, the solid and the water respectively by

$$\rho_{rr} + \rho_{r\theta} = (1 - f)\rho_s, \rho_{r\theta} + \rho_{\theta\theta} = f\rho_w \quad (11)$$

so that the mass density of the aggregate is

$$\rho' = \rho_s + f(\rho_w - \rho_s) \quad (12)$$

The above relation shows that in case the fluid of lighter density (ρ_w) is filled up in the solid matrix of density (ρ_s) then the density of the aggregate (ρ') will be less than the density of the solid (ρ_s), there may be the case of heavier fluid such as mercury, molten metal etc. filled in the solid matrix, when the density of the aggregate will be more than that of solid. Further this relation shows that as the porosity factor f decreases from 1 to 0 i.e. as the volume of pores decreases, the density of the aggregate tends to the density of the solid. It has been shown by Biot that the mass coefficients obey the following inequalities also

$$\rho_{rr} > 0, \rho_{\theta\theta} > 0, \rho_{r\theta} < 0, \rho_{rr}\rho_{\theta\theta} - \rho_{r\theta}^2 > 0 \quad (13)$$

For torsional surface waves propagating along the radial direction having displacement of the particles along θ direction we have,

$$\left. \begin{aligned} u'_r &= 0, w'_z = 0, v'_\theta = v'(r, z, t) \\ U'_r &= 0, W'_z = 0, V'_\theta = V'(r, z, t) \end{aligned} \right\} \quad (14)$$

The above displacements will produce $e_{\theta z}$ and $e_{r\theta}$ strain components and the other strain components will be zero. Hence the stress-strain relations are

$$s'_{\theta z} = 2Ge_{\theta z}, s'_{r\theta} = 2Ne_{r\theta} \quad (15)$$

Using (15) in (1) to (6) the equations of motion, which are not automatically satisfied, are

$$\frac{\partial s'_{r\theta}}{\partial r} + \frac{\partial s'_{\theta z}}{\partial z} + \frac{2}{r} s'_{r\theta} - P' \frac{\partial \omega'_z}{\partial r} = \frac{\partial^2}{\partial t^2} (\rho_{rr} v'_\theta + \rho_{r\theta} V'_\theta) \quad (16)$$

$$\frac{\partial^2}{\partial t^2} (\rho_{r\theta} v'_\theta + \rho_{\theta\theta} V'_\theta) = 0 \quad (17)$$

Using stress-strain relations (8), (16) may be written as

$$\begin{aligned} \left(N - \frac{P'}{2} \right) \left(\frac{\partial^2 v'_\theta}{\partial r^2} - \frac{v'_\theta}{r^2} + \frac{1}{r} \frac{\partial v'_\theta}{\partial r} \right) + G \frac{\partial^2 v'_\theta}{\partial z^2} \\ = \frac{\partial^2}{\partial t^2} (\rho_{rr} v'_\theta + \rho_{r\theta} V'_\theta) \end{aligned} \quad (18)$$

From (17), we have

$$\rho_{r\theta}v'_\theta + \rho_{\theta\theta}V'_\theta = d''(\text{say})$$

$$\therefore V'_\theta = \frac{d'' - \rho_{r\theta}v'_\theta}{\rho_{\theta\theta}}$$

Now,

$$\frac{\partial^2}{\partial t^2}(\rho_{rr}v'_\theta + \rho_{r\theta}V'_\theta) = d' \frac{\partial^2(v'_\theta)}{\partial t^2}, \tag{19}$$

where $d' = \rho_{rr} - \frac{\rho_{r\theta}^2}{\rho_{\theta\theta}}$

Using (19) in (18), we get

$$\left(N - \frac{P'}{2}\right) \left(\frac{\partial^2 v'_\theta}{\partial r^2} - \frac{v'_\theta}{r^2} + \frac{1}{r} \frac{\partial v'_\theta}{\partial r}\right) + G \frac{\partial^2 v'_\theta}{\partial z^2} = d' \frac{\partial^2 v'_\theta}{\partial t^2} \tag{20}$$

From Equation (20) it is clear that the velocity of the

shear wave along radial(r) direction is $\left[\frac{N - \frac{P'}{2}}{d'}\right]^{\frac{1}{2}}$ and

that along z-direction is $\left(\frac{G}{d'}\right)^{\frac{1}{2}}$. Now the shear wave

velocity β' in the porous medium along radial direction may be expressed as,

$$\beta' = \left[\frac{1 - \zeta}{d'} c_\beta^2\right]^{\frac{1}{2}}, \text{ where } c_\beta = \sqrt{\frac{N}{\rho'}}$$

is the velocity of the shear wave in the corresponding initial stress free non-porous anisotropic elastic medium along the radial

direction, $\zeta = \frac{P'}{2N}$ is the non-dimensional parameter due

to the initial stress P' and the density ratio,

$$d = \frac{d'}{\rho'} = \frac{1}{\rho'} \left(\rho_{rr} - \frac{\rho_{r\theta}^2}{\rho_{\theta\theta}}\right) = \gamma_{11} - \frac{\gamma_{12}^2}{\gamma_{22}}$$

where $\gamma_{11} = \frac{\rho_{rr}}{\rho'}$, $\gamma_{12} = \frac{\rho_{r\theta}}{\rho'}$, $\gamma_{22} = \frac{\rho_{\theta\theta}}{\rho'}$ are the non-dimensional parameters for the material of the porous layer as obtained by Biot.

Introducing the non-dimensional radial and depth coordinates we have

$$R = \frac{r}{L}, \xi = \xi_0 + \frac{z}{L},$$

ξ_0 being a constant and $\frac{1}{L}$ being wave number where

$k = \frac{1}{L}$, the Equation (20) may be written as

$$\frac{\partial^2 v'_\theta}{\partial R^2} + \frac{1}{R} \frac{\partial v'_\theta}{\partial R} - \frac{v'_\theta}{R^2} + \frac{G}{N - \frac{P'}{2}} \frac{\partial^2 v'_\theta}{\partial \xi^2} = \frac{d'L^2}{N - \frac{P'}{2}} \frac{\partial^2 v'_\theta}{\partial t^2} \tag{21}$$

We assume a solution of Equation (21) of the form $v'_\theta = v'_2(\xi) J_1(R) e^{i\omega t}$, where $J_1(R)$ is the Bessel function of the first kind and $v'_2(\xi)$ is the solution of the following equation

$$\frac{d^2 v'_2}{d\xi^2} + q^2 v'_2 = 0 \tag{22}$$

$$\text{where } q^2 = \frac{\left(\frac{\omega^2 L^2 d'}{N - \frac{P'}{2}} - 1\right)}{\frac{G}{N - \frac{P'}{2}}}$$

The solution of Equation (22) may be taken as

$$v'_2(\xi) = A e^{iq\xi} + B e^{-iq\xi}$$

Therefore, for the torsional surface wave propagating in the radial direction, the solution of Equation (21) may be taken as

$$v'_\theta = J_1(R) (A e^{iq\xi} + B e^{-iq\xi}) e^{i\omega t} \tag{23}$$

where $q^2 = d v \left(\frac{c^2}{c_\beta^2} - \frac{1 - \zeta}{d}\right)$, where $v = \frac{N}{G}$, $c = \omega L$ is the phase velocity of the torsional surface wave.

3.2. Solution for Non-Homogeneous Elastic Half Space

The lower medium is considered as non-homogeneous elastic half space. The Equation of motion may be written as

$$\frac{\partial \sigma_{r\theta}}{\partial r} + \frac{2}{r} \sigma_{r\theta} + \frac{\partial \sigma_{z\theta}}{\partial z} = \rho \frac{\partial^2 v}{\partial t^2} \tag{24}$$

where $\sigma_{r\theta}$ and $\sigma_{z\theta}$ are stress components in the half space, $v(r, \theta, z)$ is the displacement and ρ is the density of the material of the half space.

The non-homogeneity in the medium are taken as

$$\mu = \mu_1(1 + az), \rho = \rho_1(1 + bz) \tag{25}$$

where, μ_1 and ρ_1 are the values of μ and ρ respectively at $z = 0$, and a, b are constants having dimensions that are inverse of length.

Using the stress-strain relations

$$\sigma_{r\theta} = \mu \left[\frac{\partial v}{\partial r} - \frac{v}{r}\right], \sigma_{z\theta} = \mu \frac{\partial v}{\partial z} \tag{26}$$

and the relations (25), the Equation of motion (24) may be written as

$$\frac{\partial^2 v}{\partial r^2} + \frac{1}{r} \frac{\partial v}{\partial r} - \frac{v}{r^2} + \frac{\partial^2 v}{\partial z^2} + \frac{a}{1+az} \frac{\partial v}{\partial z} = \frac{\rho_1(1+bz)}{\mu_1(1+az)} \frac{\partial^2 v}{\partial t^2} \quad (27)$$

Introducing the non-dimensional co-ordinates

$$R = \frac{r}{L} \text{ and } \xi = \xi_0 + \frac{z}{L},$$

Equation (27) takes the form as

$$\begin{aligned} \frac{\partial^2 v}{\partial R^2} + \frac{1}{R} \frac{\partial v}{\partial R} - \frac{v}{R^2} + \frac{\partial^2 v}{\partial \xi^2} + \frac{La}{[1+a(\xi-\xi_0)L]} \frac{\partial v}{\partial \xi} \\ = \frac{\rho_1 L^2}{\mu_1} \frac{1+b(\xi-\xi_0)L}{1+a(\xi-\xi_0)L} \frac{\partial^2 v}{\partial t^2} \end{aligned} \quad (28)$$

We assume a solution of Equation (28) of the form

$$v = v_2(\xi) J_1(R) e^{i\omega t},$$

where v_2 is the solution of the following equation

$$\begin{aligned} \frac{d^2 v_2}{d\xi^2} + \frac{aL}{1+a(\xi-\xi_0)L} \frac{dv_2}{d\xi} \\ + \left[\frac{c^2}{c_1^2} \frac{1+b(\xi-\xi_0)L}{1+a(\xi-\xi_0)L} - 1 \right] v_2 = 0 \end{aligned} \quad (29)$$

In the above, $c = \omega L$ is the phase velocity of the torsional surface wave and $c_1 = \sqrt{\frac{\mu_1}{\rho_1}}$, velocity of shear wave in an initially stress-free elastic medium and $J_1(R)$ is the Bessel function of the first kind.

Substituting $v_2(\xi) = \frac{L^{-\frac{1}{2}} \phi(\xi)}{\left[\frac{1}{L} + a(\xi - \xi_0) \right]^{\frac{1}{2}}}$ in the Equa-

tion (29) to eliminate the term $\frac{dv_2}{d\xi}$,

we obtain

$$\begin{aligned} \phi''(\xi) + \left[\frac{a^2}{4 \left\{ \frac{1}{L} + a(\xi - \xi_0) \right\}^2} + \frac{c^2}{c_1^2} \frac{1+b(\xi-\xi_0)L}{L+ a(\xi-\xi_0)L} - 1 \right] \phi(\xi) \\ = 0 \end{aligned} \quad (30)$$

Using $\phi(\xi) = \psi(\eta)$ in (30), where

$$\eta = \frac{2p}{a} \left[\frac{1}{L} + a(\xi - \xi_0) \right] \text{ we get}$$

$$\psi''(\eta) + \left[\frac{1}{4\eta^2} + \frac{s}{\eta} - \frac{1}{4} \right] \psi(\eta) = 0 \quad (31)$$

where $s = \frac{c^2(a-b)L}{2c_1^2 pa^2 L^2}$, $p^2 = 1 - \frac{b}{a} \frac{c^2}{c_1^2}$

Equation (31) is Whittaker's equation.

We are interested in the solution of Equation (31) which is bounded and vanishes as $z \rightarrow \infty$, therefore we search for the solution which gives $v(z) \rightarrow 0$ as $z \rightarrow \infty$. This condition is equivalent to $\lim_{\eta \rightarrow \infty} \psi(\eta) \rightarrow 0$.

Therefore the solution of Equation (31) satisfying the above condition may be written as

$$v = \frac{D_1 W_{s,0}(\eta)}{L^{\frac{1}{2}} \left(\frac{a\eta}{2p} \right)^{\frac{1}{2}}} J_1(R) e^{i\omega t} \quad (32)$$

4. Boundary Conditions and Dispersion Equation

The boundary conditions are as follows

- 1) $\mu_0 \left(\frac{\partial v'_\theta}{\partial z} \right) = 0$ at $z = -H$
- 2) $v'_\theta = v$ at $z = \varepsilon F(r)$
- 3) $(\sigma_{r\theta})_0 = (\sigma_{r\theta})_1$ at $z = \varepsilon F(r)$

where

$$\begin{aligned} (\sigma_{r\theta})_0 &= l\mu_0 \frac{\partial v'_\theta}{\partial \theta} + n\mu_0 \frac{\partial v'_\theta}{\partial z}, \\ (\sigma_{r\theta})_1 &= l\mu_1 \frac{\partial v}{\partial \theta} + n\mu_1 \frac{\partial v}{\partial z} \end{aligned}$$

and $(l, 0, n)$ are components of unit normal (to the interface at $z = 0$).

$$l = 0, n = 1 \text{ at } z = -H$$

$$l = -\frac{\varepsilon F'}{\sqrt{1 + \varepsilon^2 F'^2}}, n = \frac{1}{\sqrt{1 + \varepsilon^2 F'^2}};$$

$$\text{at } z = \varepsilon F(r)$$

where $F' = \frac{dF}{dr}$

The boundary condition (3) may be written as

$$\begin{aligned} \mu_0 \left(\frac{\partial v'_\theta}{\partial r} - \frac{v'_\theta}{r} \right) \left(-\frac{\varepsilon F'}{\sqrt{1 + \varepsilon^2 F'^2}} \right) + \frac{1}{\sqrt{1 + \varepsilon^2 F'^2}} \mu_0 \frac{\partial v'_\theta}{\partial z} \\ = \mu_1 \left(\frac{\partial v}{\partial r} - \frac{v}{r} \right) \left(-\frac{\varepsilon F'}{\sqrt{1 + \varepsilon^2 F'^2}} \right) + \frac{1}{\sqrt{1 + \varepsilon^2 F'^2}} \mu_1 \frac{\partial v}{\partial z} \end{aligned}$$

at $z = \varepsilon F(r)$

Now, using boundary conditions (1), (2) and (3) respectively we get

$$Ae^{iq\left(\xi_0 - \frac{H}{L}\right)} - Be^{-iq\left(\xi_0 - \frac{H}{L}\right)} = 0 \tag{33}$$

$$Ae^{iq\xi} + Be^{-iq\xi} = \frac{D_1 W_{s,0}(\eta)}{(1 + a\varepsilon F(r))^{\frac{1}{2}}} \tag{34}$$

$$AM_1 e^{iq\xi} - BN_1 e^{-iq\xi} = D_1 K_1 \tag{35}$$

where

$$K_1 = \frac{\mu_1}{\mu_0} \left[\frac{W_{s,0}(\eta)}{(1 + a\varepsilon F(r))^{1/2}} \left(\frac{2rh\varepsilon}{m^2} \right) \left\{ \frac{1}{L} J_1' \left(\frac{r}{L} \right) - \frac{1}{r} J_1(R) \right\} + \left(\frac{2p}{aL} \right)^s J_1(R) e^{-p/aL(1+a\varepsilon F(r))} (1 + az)^{s-\frac{1}{2}} \left\{ \frac{a^2 L (s-0.5)^2 (s+0.5)(3.5-s)}{8p (1+a\varepsilon F(r))^2} - \frac{a^3 L^2 (s-0.5)^2 (s-1.5)^2 (2.5-s)}{8p^2 (1+a\varepsilon F(r))^3} + \frac{a(s-0.5)(s+1.5)}{2(1+a\varepsilon F(r))} - \frac{p}{L} - \frac{a^2 L (s-0.5)^3}{2p (1+a\varepsilon F(r))^2} \right\} \right] \tag{36}$$

Now, eliminating A , B and D_1 from Equations (33), (34) and (35) we get

$$\begin{vmatrix} e^{iq\left(\xi_0 - \frac{H}{L}\right)} & -e^{-iq\left(\xi_0 - \frac{H}{L}\right)} & 0 \\ e^{iq\xi} & e^{-iq\xi} & -\frac{W_{s,0}(\eta)}{(1 + a\varepsilon F(r))^{\frac{1}{2}}} \\ M_1 e^{iq\xi} & N_1 - e^{-iq\xi} & -K_1 \end{vmatrix} = 0 \tag{37}$$

Expanding the Whittaker function $W_{s,0}(\eta)$ up to linear terms in η , Equation (37) reduces to

$$\tan(kqH) = \frac{\mu_1 p}{\mu_0 q} \frac{\left[1 - \frac{aL}{2p}(s-0.5)(s+1.5) + \frac{a^2 L^2}{8p^2}(s-0.5)^2(s+2.5)(s-1.5) - \frac{a^3 L^3}{8p^3}(s-0.5)^2(s-1.5)^2(s-2.5) \right]}{1 - \frac{aL}{2p}(s-0.5)^2 + \frac{a^2 L^2}{8p^2}(s-0.5)^2(s-1.5)^2}$$

which is the same velocity equation in initially stressed anisotropic heterogeneous poroelastic medium as obtained by Dey & Sarkar (2002).

5.2. Case II

If $\varepsilon \rightarrow 0, a \rightarrow 0, b \rightarrow 0$ i.e. when half space is free

$$\tan kq(\varepsilon F(r) + H) = \frac{1}{q} \left[R_1 - \frac{\mu_1 R_2}{\mu_0 R_3} \right] \tag{38}$$

where

$$R_1 = \frac{-2rh\varepsilon}{m^2} \left(1 - \frac{\mu_1}{\mu_0} \right) \left(\frac{R}{4} + \frac{R^3}{96} + \frac{R^5}{1536} \right),$$

$$R_2 = \left(\frac{a^2 L^2 (s-0.5)^2 (s+0.5)(3.5-s)}{8p (1+a\varepsilon F(r))^2} - \frac{a^3 L^3 (s-0.5)^2 (s-1.5)^2 (2.5-s)}{8p^2 (1+a\varepsilon F(r))^3} + \frac{aL(s-0.5)(s+1.5)}{2(1+a\varepsilon F(r))} - p - \frac{a^2 L^2 (s-0.5)^3}{2p (1+a\varepsilon F(r))^2} \right),$$

$$R_3 = \left(1 - \frac{aL (s-0.5)^2}{2p (1+\varepsilon F(r))} + \frac{a^2 L^2 (s-0.5)^2 (s-1.5)^2}{8p^2 (1+\varepsilon F(r))^2} \right).$$

5. Particular Cases

If the medium is non-porous then $f \rightarrow 0$ and $\rho_s \rightarrow \rho'$ which leads to $\gamma_{11} + \gamma_{12} \rightarrow 1$ and $\gamma_{11} + \gamma_{12} \rightarrow 0$ and hence $\left(\gamma_{11} - \frac{\gamma_{12}^2}{\gamma_{22}} \right) \rightarrow 1$ giving $d \rightarrow 1$.

Again if $f \rightarrow 1$ then $\rho_w \rightarrow \rho'$ and the medium becomes fluid. In this case, the velocity of shear wave in the medium can not exist when $d = \left(\gamma_{11} - \frac{\gamma_{12}^2}{\gamma_{22}} \right) \rightarrow 0$.

Therefore, for a non-porous solid $d = 1$, for fluid $d = 0$ and for porous solid filled with liquid $0 < d < 1$.

5.1. Case I

If $\varepsilon \rightarrow 0, h \rightarrow 0$ i.e. in the absence of irregularity in the half space, then (38) reduces to

from irregularity and also half space is homogeneous then from (38) we get

$$\tan(kqH) = \frac{\mu_1 p}{\mu_0 q},$$

where $p^2 = 1 - \frac{c^2}{c_1^2}, q^2 = d\nu \left(\frac{c^2}{c_\beta^2} - \frac{1-\zeta}{d} \right)$

which is the dispersion equation of Love wave in an initially stressed poro-elastic medium.

5.3. Case III

If $P' \neq 0, a \rightarrow 0, b \rightarrow 0$ i.e. in the presence of normal initial stress and when the half-space is homogeneous then Equation (38) reduces to

$$\tan[kq(\varepsilon F(r) + H)] = \frac{\mu_1 p}{\mu_0 q},$$

where $p^2 = 1 - \frac{c^2}{c_1^2}$ and $q^2 = d\nu \left(\frac{c^2}{c_\beta^2} - \frac{1-\zeta}{d} \right)$

which is the velocity equation of Love wave in this case.

5.4. Case IV

If $P' = 0, a \rightarrow 0, b \rightarrow 0$ i.e. in the absence of normal initial stress and when the half-space is homogeneous then from (38) we get

$$\tan[kq(\varepsilon F(r) + H)] = \frac{\mu_1 p}{\mu_0 q},$$

where $p^2 = 1 - \frac{c^2}{c_1^2}$ and $q^2 = d\nu \left(\frac{c^2}{c_\beta^2} - \frac{1}{d} \right)$

5.5. Case V

If $\varepsilon \rightarrow 0, a \rightarrow 0, b \rightarrow 0, \zeta \rightarrow 0, d \rightarrow 1$ and $N=G$ i.e. when the half-space is homogeneous and free from irregularity and also when the layer is initial stress free, isotropic and non-porous then (38) reduces to

$$\tan(kqH) = \frac{\mu_1 p}{\mu_0 q},$$

where

$$p^2 = 1 - \frac{c^2}{c_1^2}, \quad q^2 = \frac{c^2}{c_\beta^2} - 1$$

$$\text{or, } \tan \left[\left(\frac{c^2}{c_\beta^2} - 1 \right)^{\frac{1}{2}} kH \right] = \frac{\mu_1 \left(1 - \frac{c^2}{c_1^2} \right)^{\frac{1}{2}}}{\mu_0 \left(\frac{c^2}{c_\beta^2} - 1 \right)^{\frac{1}{2}}}$$

which is the well-known classical result of Love wave.

6. Numerical Computation and Discussion

To study the consolidated effect of porosity, irregularity and initial stresses on the propagation of the torsional surface waves, the numerical computation of velocity

c^2/c_β^2 has been made from (38) under different values of $d, aL, bL, N/G, h/H, \mu_1/\mu_0, r/m, h/m, km, \varepsilon, \zeta$ and $c_\beta^2/c_1^2 = 0.5$. The results are presented in Figures 2 to 5.

Figure 2 gives the dispersion curves for increasing values of kH at different sizes of the irregularity and compares the results in the medium with and without initial stresses. The curve numbers 1, 2 and 3 are for an initially stress-free medium, and the curve numbers 4, 5 and 6 are for the initially stressed medium with $\zeta = 0.4$. The figure shows that the effect of the irregularity and the initial stresses are very prominent on the propagation of torsional surface waves. The presence of initial com-

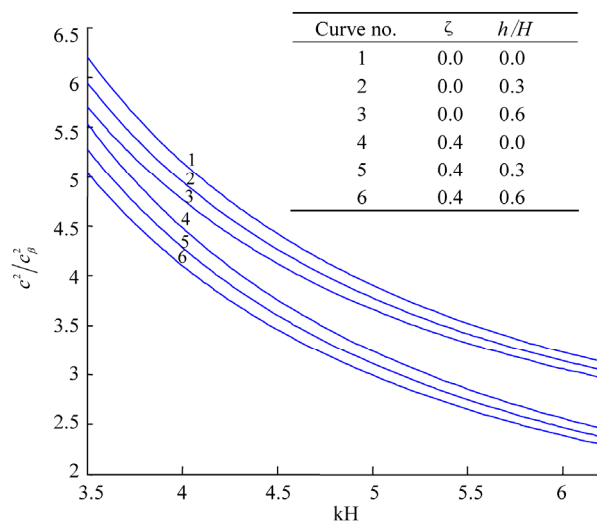


Figure 2. Torsional wave dispersion curves at different initial stresses and at different sizes of irregularity $\mu_1/\mu_0 = 0.4, N/G = 0.3, d = 0.6, a/k = 0.1, b/k = 0.1, \varepsilon = 0.1, r/m = 0.02, h/m = 0.03, km = 0.01$.

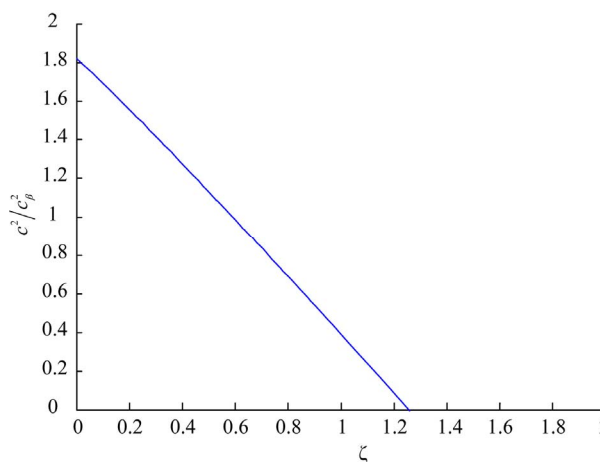


Figure 3. Variation of velocity of torsional wave with respect to initial compressive stress for $kH = 0.72, N/G = 2, d = 0.6, aL = 0.1, bL = 0.1, h/H = 0.3, \varepsilon = 0.1, r/m = 0.01, h/m = 0.02, km = 0.01, \mu_1/\mu_0 = 0.4$.

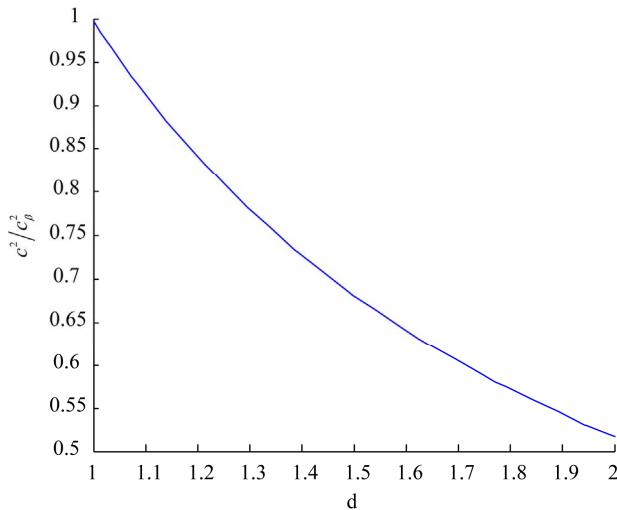


Figure 4. Variation of velocity of torsional wave with respect to porosity for $kH = 0.72$, $N/G = 2$, $\zeta = 0.6$, $aL = 0.1$, $bL = 0.1$, $h/H = 0.3$, $\varepsilon = 0.1$, $r/m = 0.01$, $h/m = 0.02$, $km = 0.01$, $\mu_1/\mu_0 = 0.4$.

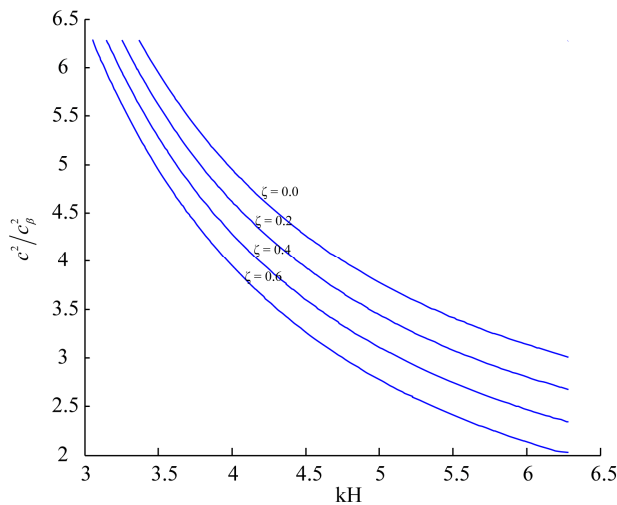


Figure 5. Effect of initial stress on the velocity of torsional wave for $\mu_1/\mu_0 = 0.4$, $N/G = 0.3$, $d = 0.6$, $a/k = 0.1$, $b/k = 0.1$, $\varepsilon = 0.1$, $r/m = 0.02$, $h/m = 0.03$, $km = 0.01$, $h/H = 0.3$.

pressive stress reduces the velocity of torsional surface waves. The presence of the liquid-filled pore in the medium also reduces the velocity.

Figure 3 gives a variation of velocity of torsional surface waves for the variation of ζ . The curve confirm that as the initial stress parameter increases, the velocity decreases.

Figure 4 gives a variation of velocity of torsional surface waves for the variation of porosity. The curve confirm that as the porosity increases, the velocity decreases.

Figure 5 gives the dispersion curves at different values of kH . This shows that as kH increases, the velocity drops down. Hence it is expected that after certain

thickness of the layer, the torsional surface wave will not propagate in the medium. This figure also confirms that as the initial compressive stress increases, the velocity of propagation decreases.

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Appendix

H : Thickness of the layer

μ : Rigidity of the medium

ρ : Density of the medium

a, b : Constants having dimensions that are inverse of length

s'_{ij} : Incremental stress components

e_{ij} : Components of strains

u'_r, v'_θ, w'_z : Displacement components of the solid in the radial, circumferential and axial directions respectively.

U'_r, V'_θ, W'_z : Displacement components of the liquid in the radial, circumferential and axial directions respectively.

s' : Stress in the liquid

$\omega'_r, \omega'_\theta, \omega'_z$: Components of the rotational vector ω' in radial, circumferential and axial directions respectively.

ω : Circular frequency

k : Wave number

f : Porosity of the layer

c : Phase velocity of the torsional surface wave

c_1 : Velocity of shear wave in an initially stress-free elastic medium.

R : Dimensionless quantity.

A, B and D_1 : Arbitrary constant