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Inverse Eigenvalue Problem for Generalized Arrow-Like Matrices

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Abstract

This paper researches the following inverse eigenvalue problem for arrow-like matrices. Give two characteristic pairs, get a generalized arrow-like matrix, let the two characteristic pairs are the characteristic pairs of this generalized arrow-like matrix. The expression and an algorithm of the solution of the problem is given, and a numerical example is provided.

Keywords: Generalized Arrow-Like Matrices, Characteristic Value, Inverse Problem, Unique

1. Introduction

The Inverse eigenvalue problem for matrices in the problems involved in the field of structural design, pattern recognition, parameter recognition, automatic control and so on, it has a good engineering background, and its research has obvious significance [1]. Many experts and scholars have addressed more extensively and in-depth studied, get a lot of conclusions about inverse eigenvalue problem for Jacobi matrices [2], but there is less research about the inverse eigenvalue problem for arrow- like matrices [3,4]. This paper researches the following inverse eigenvalue problem for generalized arrow-like matrices.

Generalized arrow-like matrices refer to the matrix as follows:

$$J=\begin{pmatrix} a_1 & b_1 & \cdots & b_{m-1} & b_m & & & & \\ c_1 & a_2 & & & & & & & \\ \vdots & & \ddots & & & & & & \\ c_{m-1} & & & a_m & & & & & \\ c_m & & & & a_{m+1} & b_{m+1} & & & \\ & & & & c_{m+1} & a_{m+2} & \ddots & & \\ & & & & & \ddots & \ddots & b_{n-1} \\ & & & & & & c_{n-1} & a_n \end{pmatrix} \quad (1)$$

When $m=1$, J becomes generalized Jacobi matrix [1]; when $m=n$, J is an arrow-like matrice. This article studies the following characteristic value inverse:

Question IEPGAM. Given two real numbers $\lambda,\mu(\lambda\neq\mu)$ and two nonzero real vectors $x=(x_1,x_2,\cdots,x_n)^{\mathrm{T}}\in R^n$, $y=(y_1,y_2,\cdots,y_n)^{\mathrm{T}}\in R^n$.

Find the $n\times n$ real generalized arrow-like matrice J, such that $Jx=\lambda x, Jy=\mu y$.

The expression and an algorithm of the solution of the problem is given in Section 2, and a numerical example is provided in Section 3.

2. The Solution of Question IEPGAM

Because (λ,x) and (μ,y) are two characteristic pairs of the generalized arrow-like matrices J,

In it,

Let:

$$x_0=x_{n+1}=y_0=y_{n+1}=b_0=b_n=c_0=c_n=0, \quad (2)$$

$$D_i=\begin{vmatrix} x_i & y_i \\ x_{i+1} & y_{i+1}\end{vmatrix}(i=0,1,\cdots,n), \quad (3)$$

$$E_i=\begin{vmatrix} x_1 & x_i \\ y_1 & y_i\end{vmatrix}(i=2,3,\cdots,m+1) \quad (4)$$

So

$$a_1x_1+b_1x_2+\cdots+b_{m-1}x_m+b_mx_{m+1}=\lambda x_1, \quad (5\text{-}1)$$

$$c_1x_1+a_2x_2=\lambda x_2, \quad (5\text{-}2)$$

$$\cdots$$

$$c_{m-1}x_1+a_mx_m=\lambda x_m, \quad (5\text{-}m)$$

$$c_mx_1+a_{m+1}x_{m+1}+b_{m+1}x_{m+2}=\lambda x_{m+1}, \quad (5\text{-}m+1)$$

$$c_{m+1}x_{m+1}+a_{m+2}x_{m+2}+b_{m+2}x_{m+3}=\lambda x_{m+2}, \quad (5\text{-}m+2)$$

$$\cdots$$

$$c_{n-2}x_{n-2}+a_{n-1}x_{n-1}+b_{n-1}x_n=\lambda x_{n-1}, \quad (5\text{-}n\text{–}1)$$

$$c_{n-1}x_{n-1}+a_nx_n=\lambda x_n. \quad (5\text{-}n)$$

$$a_1y_1+b_1y_2+\cdots+b_{m-1}y_m+b_my_{m+1}=\mu y_1, \quad (6\text{-}1)$$

$$c_1y_1+a_2y_2=\mu y_2, \quad (6\text{-}2)$$

$$\cdots$$

$$c_{m-1}y_1+a_my_m=\mu y_m, \quad (6\text{-}m)$$

$$c_my_1+a_{m+1}y_{m+1}+b_{m+1}y_{m+2}=\mu y_{m+1}, \quad (6\text{-}m\text{+}1)$$

$$c_{m+1}y_{m+1}+a_{m+2}y_{m+2}+b_{m+2}y_{m+3}=\mu y_{m+2}, \quad (6\text{-}m\text{+}2)$$

$$\cdots$$

$$c_{n-2}y_{n-2}+a_{n-1}y_{n-1}+b_{n-1}y_n=\mu y_{n-1}, \quad (6\text{-}n\text{–}1)$$

$$c_{n-1}y_{n-1}+a_ny_n=\mu y_n. \quad (6\text{-}n)$$

- For inverse $b_i, c_i (i=m+1,m+2,\cdots,n-1)$, $a_i (i=m+2,m+3,\cdots,n)$.

From (5) and (6), we can get

$$c_{i-1}x_{i-1}+a_ix_i+b_ix_{i+1}=\lambda x_i (i=m+2,m+3,\cdots,n), \quad (7)$$

$$c_{i-1}y_{i-1}+a_iy_i+b_iy_{i+1}=\mu y_i (i=m+2,m+3,\cdots,n). \quad (8)$$

In order to eliminate a_i, multiply by y_i on both sides of (7), multiply by x_i on both sides of (8), then cut on both sides, we can get

$$b_iD_i=(\mu-\lambda)x_iy_i+c_{i-1}D_{i-1} (i=m+2,m+3,\cdots,n). \quad (9)$$

To problem *A*, because $c_i=kb_i, (i=2,3,\cdots,n-1)$, so (9) become

$$b_iD_i=(\mu-\lambda)x_iy_i+kb_{i-1}D_{i-1} \quad (i=m+2,m+3,\cdots,n) \quad (10)$$

Let $i=n$, because $D_n=0$, so

$$b_{n-1}D_{n-1}=\frac{x_ny_n}{k}(\lambda-\mu),$$

Let $i=n-1$, $b_{n-2}D_{n-2}=(\lambda-\mu)\left[\frac{x_ny_n}{k^2}+\frac{x_{n-1}y_{n-1}}{k}\right]$;

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Let $i=m+2$,

$$b_{m+1}D_{m+1}=(\lambda-\mu)\left[\frac{x_ny_n}{k^{n-(m+1)}}+\frac{x_{n-1}y_{n-1}}{k^{n-(m+2)}}+\cdots+\frac{x_{m+2}y_{m+2}}{k}\right]$$

Under normal circumstances,

$$b_jD_j=(\lambda-\mu)\sum_{s=0}^{n-(j+1)}\frac{x_{n-s}y_{n-s}}{k^{n-(s+j)}} (j=m+1,m+2,\cdots,n-1). \quad (11)$$

If $D_j\neq 0$ $(j=m+1,m+2,\cdots,n-1)$, then x_i, y_i can not be zero at the same time ,so

$$b_j=\frac{(\lambda-\mu)}{D_j}\sum_{s=0}^{n-(j+1)}\frac{x_{n-s}y_{n-s}}{k^{n-(s+j)}} (j=m+1,m+2,\cdots,n-1), \quad (12)$$

$$c_j=kb_j, (j=m+1,m+2,\cdots,n-1), \quad (13)$$

$$a_j=\begin{cases}\dfrac{\lambda x_j-c_{j-1}x_{j-1}-b_jx_{j+1}}{x_j}, x_j\neq 0;\\[2ex] \dfrac{\mu y_j-c_{j-1}y_{j-1}-b_jy_{j+1}}{y_j}, y_j\neq 0\end{cases}. \quad (14)$$

$(j=m+2,m+3,\cdots,n)$

- For inverse a_{m+1}, c_m, b_m.

From (5) and the *m* + 1 equation of (6),

$$c_mE_{m+1}=(\lambda-\mu)x_{m+1}y_{m+1}+b_{m+1}D_{m+1}, \quad (15)$$

$$a_{m+1}E_{m+1}=\mu x_1y_{m+1}-\lambda x_{m+1}y_1-b_{m+1}E_{m+2}, \quad (16)$$

$$b_m=\frac{c_m}{k}. \quad (17)$$

- For inverse $c_1, c_i, b_i (i=2,3,\cdots,m-1)$, $a_i (i=2,3,\cdots,m)$.

From (5) and 2 to *m* equation of (6),

$$c_{i-1}x_1+a_ix_i=\lambda x_i (i=2,3,\cdots,m), \quad (18)$$

$$c_{i-1}y_1+a_iy_i=\mu y_i (i=2,3,\cdots,m). \quad (19)$$

From (18) and (19), we can get

$$c_{i-1}E_i=(\lambda-\mu)x_iy_i (i=2,3,\cdots,m), \quad (20)$$

$$a_iE_i=\mu x_1y_i-\lambda x_iy_1 (i=2,3,\cdots,m), \quad (21)$$

$$b_i=\frac{c_i}{k}(i=2,3,\cdots,m-1). \quad (22)$$

- For inverse a_1, b_1.

From (5) and (6), we can get

$$a_1x_1+b_1x_2=\lambda x_1-\sum_{s=2}^{m}b_sx_{s+1}, \quad (23)$$

$$a_1y_1+b_1y_2=\mu y_1-\sum_{s=2}^{m}b_sy_{s+1}. \quad (24)$$

If $D_1\neq 0$, from (23) and (24), then we can get

$$a_1=\frac{\lambda x_1y_2-\mu x_2y_1-\sum_{s=2}^{m}b_s(x_{s+1}y_2-x_2y_{s+1})}{D_1}, \quad (25)$$

$$b_1=\frac{(\mu-\lambda)x_1y_1-\sum_{s=2}^{m}b_s(y_{s+1}x_1-y_1x_{s+1})}{D_1}. \quad (26)$$

According to the above analysis, to question IEPGAM, we can get the follow theorem.

Theorem. If the following conditions are satisfied:

1) $D_1\neq 0$;
2) $D_i\neq 0$ $(i=m+1,m+2,\cdots,n-1)$;
3) $E_i\neq 0$ $(i=2,3,\cdots,m+1)$

Then question IEPGAM has the unique solution, and

$$b_j = \frac{(\lambda-\mu)}{D_j}\sum_{s=0}^{n-(j+1)}\frac{x_{n-s}y_{n-s}}{k^{n-(s+j)}}\,(j=m+1,m+2,\cdots,n-1) \quad (27)$$

$$a_j = \begin{cases} \dfrac{\lambda x_j - c_{j-1}x_{j-1} - b_j x_{j+1}}{x_j}, x_j \neq 0; \\ \dfrac{\mu y_j - c_{j-1}y_{j-1} - b_j y_{j+1}}{y_j}, y_j \neq 0 \end{cases}, \quad (28)$$

$$(j=m+2,m+3,\cdots,n)$$

$$b_m = \frac{(\lambda-\mu)x_{m+1}y_{m+1} + b_{m+1}D_{m+1}}{kE_{m+1}}, \quad (29)$$

$$a_{m+1} = \frac{\mu x_1 y_{m+1} - \lambda x_{m+1}y_1 - b_{m+1}E_{m+2}}{E_{m+1}}, \quad (30)$$

$$b_j = \frac{(\lambda-\mu)x_{j+1}y_{j+1}}{kE_{j+1}}(i=2,3,\cdots,m-1), \quad (31)$$

$$b_1 = \frac{(\mu-\lambda)x_1y_1 - \sum_{s=2}^{m} b_s(y_{s+1}x_1 - y_1x_{s+1})}{D_1} \quad (32)$$

$$a_j = \frac{\mu x_1 y_j - \lambda x_j y_1}{E_j}(j=2,3,\cdots,m), \quad (33)$$

$$a_1 = \begin{cases} \lambda - \dfrac{\sum_{s=1}^{m} b_s x_{s+1}}{x_1}, x_1 \neq 0; \\ \mu - \dfrac{\sum_{s=1}^{m} b_s y_{s+1}}{y_1}, y_j \neq 0 \end{cases} \quad (34)$$

$$c_j = kb_j, (i=2,3,\cdots,n-1), \quad (35)$$

$$c_1 = \frac{(\lambda-\mu)x_2y_2}{E_2}. \quad (36)$$

3. Numerical Examples

Example 1. Give $\lambda=1, \mu=2, k=2, m=2, n=5$, $x=(1,1,1,1,1)^{\mathrm{T}}$, $y=(1,0,2,-1,0)^{\mathrm{T}}$.

It is easy to be calculated

$$D_1=-1\neq 0, D_2=-3\neq 0, D_4=1\neq 0;$$

$$E_2=-1\neq 0, E_3=1\neq 0, E_4=-2.$$

From Theorem, the question IEPGAM has the unique solution. And

$$b_3 = \frac{\lambda-\mu}{D_3}\left[\frac{x_5y_5}{k^2} + \frac{x_4y_4}{k}\right] = -\frac{1}{6},$$

$$b_2 = \frac{1}{kE_3}\left[(\lambda-\mu)x_3y_3 + b_3D_3\right] = -\frac{3}{4},$$

$$b_4 = \frac{\lambda-\mu}{D_4}\left[\frac{x_5y_5}{k}\right] = 0,$$

$$b_1 = \frac{(\mu-\lambda)x_1y_1 - b_2(y_3x_1 - y_1x_3)}{D_1} = -\frac{7}{4};$$

$$c_2 = kb_2 = -\frac{3}{2},$$

$$c_3 = kb_3 = -\frac{1}{3},$$

$$c_4 = kb_4 = 0,$$

$$c_1 = \frac{(\lambda-\mu)x_2y_2}{E_2} = 0;$$

$$a_1 = \lambda - \frac{b_1x_2 + b_2x_3}{x_1} = \frac{7}{2},$$

$$a_2 = \frac{\mu x_1y_2 - \lambda x_2y_1}{E_2} = 1,$$

$$a_3 = \frac{\mu x_1y_3 - \lambda x_3y_1 - b_3E_4}{E_3} = \frac{8}{3},$$

$$a_4 = \frac{\lambda x_4 - c_3x_3 - b_4x_5}{x_4} = \frac{4}{3},$$

$$a_5 = \frac{\lambda x_5 - c_4x_4 - b_5x_6}{x_5} = 1.$$

So

$$J = \begin{pmatrix} \frac{7}{2} & \frac{-7}{4} & \frac{-3}{4} & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ \frac{-3}{2} & 0 & \frac{8}{3} & \frac{-1}{6} & 0 \\ 0 & 0 & \frac{-1}{3} & \frac{4}{3} & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

and $Jx=\lambda x, Jy=\mu y$.

4. References

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