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On *p* and *q*-Horn's Matrix Function of Two Complex Variables

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Abstract

The main aim of this paper is to define and study of a new Horn's matrix function, say, the *p* and *q*-Horn's matrix function of two complex variables. The radius of regularity on this function is given when the positive integers *p* and *q* are greater than one, an integral representation of ${}^{p}\mathrm{H}_2^{q}(A,A',B,B';C;z,w)$ is obtained, recurrence relations are established. Finally, we obtain a higher order partial differential equation satisfied by the *p* and *q*-Horn's matrix function.

Keywords: Hypergeometric Matrix functions, *p* and *q*-Horn's Matrix Function, Contiguous Relations, Matrix Functions, Matrix Differential Equation, Differential Operator

1. Introduction

Many special functions encountered in mathematical physics, theoretical physics, engineering and probability theory are special cases of hypergeometric functions [1]. Hypergeometric series in one and more variables occur naturally in a wide variety of problems in applied mathematics, statistics [2-4], and operations research and so on [5]. In [6,7], the hypergeometric matrix function has been introduced as a matrix power series and an integral representation. Moreover, Jódar and Cortés introduced, studied the hypergeometric matrix function $F(A,B;C;z)$, the hypergeometric matrix differential equation in [8] and the explicit closed form general solution of it has been given in [9]. Upadhyaya and Dhami have earlier studied the generalized Horn's functions of matrix arguments with real positive definite matrices as arguments [10] and this function H_7 also [11], while the author has earlier studied the Horn's matrix function H_2 of two complex variables under differential operators [7]. In [12, 13], extension to the matrix function framework of the classical families of *p*-Kummer's matrix functions and *p* and *q*-Appell matrix functions have been proposed.

Our purpose here is to introduce and study an extension of the matrix functions of two variables. This paper is organized as follows: Section 2 contains the definition of the *p* and *q*-Horn's matrix function of two variables, its radius of regularity and integral relation of the *p* and *q*-Horn's matrix function is given. Some matrix recurrence relations are established in Section 3. Finally, the effect of differential operator on this function is investigated and *p* and *q*-Horn's matrix partial differential equation are obtained in Section 4.

Throughout this paper D_0 will denote the complex plane cut along the negative real axis. The spectrum of a matrix A in $C^{N\times N}$, denoted by $\sigma(A)$ is the set of its eigenvalues of A. If A is a matrix in $C^{N\times N}$, its two-norm denoted by $\|A\|_2$ is defined by [14]

$$\|A\|_2 = \sup_{x\neq 0}\frac{\|Ax\|_2}{\|x\|_2}$$

where for a vector y in C^N, $\|y\|_2 = \left(y^T y\right)^{\frac{1}{2}}$ is the Euclidean norm of *y*.

If *f*(*z*) and *g*(*z*) are holomorphic functions of complex variables *z*, defined in an open set Ω of the complex plane, and if A and B are a matrix in $C^{N\times N}$ with $\sigma(A)\subset\Omega$ and $\sigma(B)\subset\Omega$ also and if $AB=BA$, then from the properties of the matrix functional calculus [15], it follows that

$$f(A)g(B) = g(B)f(A). \tag{1.1}$$

The reciprocal gamma function denoted by $\Gamma^{-1}(z)=\dfrac{1}{\Gamma(z)}$ is an entire function of the complex variable z. Then for any matrix A in $C^{N\times N}$, the image of $\Gamma^{-1}(z)$ acting on *A* denoted by $\Gamma^{-1}(A)$ is a welldefined matrix. Furthermore, if

$$A+nI \text{ is invertible for every non negative integer } n \tag{1.2}$$

where I is the identity matrix in $C^{N\times N}$, then $\Gamma(A)$ is invertible, its inverse coincides with $\Gamma^{-1}(A)$ and one gets [8]

$$\begin{aligned}(A)_n &= A(A+I)\cdots(A+(n-1)I)\\ &= \Gamma(A+nI)\Gamma^{-1}(A); n\geq 1; (A)_0 = I.\end{aligned} \tag{1.3}$$

Jódar and Cortés have proved in [16], that

$$\Gamma(A) = \lim_{n\to\infty}(n-1)![(A)_n]^{-1}n^A. \tag{1.4}$$

Let P and Q be two positive stable matrices in $C^{N\times N}$. The gamma matrix function $\Gamma(P)$ and the beta matrix function $B(P,Q)$ have been defined in [16], as follows

$$\Gamma(P) = \int_0^\infty e^{-t}t^{P-I}\mathrm{d}t; t^{P-I} = e^{(P-I)\ln t} \tag{1.5}$$

and

$$B(P,Q) = \int_0^1 t^{P-I}(1-t)^{Q-I}\mathrm{d}t. \tag{1.6}$$

Let P and Q be commuting matrices in $C^{N\times N}$ such that the matrices $P+nI$, $Q+nI$ and $P+Q+nI$ are invertible for every integer $n\geq 0$. Then according to [8], we have

$$B(P,Q) = \Gamma(P)\Gamma(Q)\left[\Gamma\left(P+Q\right)\right]^{-1}. \tag{1.7}$$

2. Definition of *p* and *q*-Horn's Matrix Function

Suppose that p and q are positive integers. The p and q-Horn's matrix function ${}^{p}\mathrm{H}_2^{q}(A,A',B,B';C;z,w)$ of two complex variables is written in the form

$$\begin{aligned}&{}^{p}\mathrm{H}_2^{q}(A,A',B,B';C;z,w)\\ &= \sum_{m,n=0}^{\infty}\frac{(A)_{m-n}(A')_m(B)_n(B')_n[(C)_m]^{-1}}{(pm)!(qn)!}z^m w^n\end{aligned} \tag{2.1}$$

where $U_{m,n}(z,w) = V_{m,n}z^m w^n$ and

$$V_{m,n} = \frac{(A)_{m-n}(A')_m(B)_n(B')_n[(C)_m]^{-1}}{(pm)!(qn)!}.$$

For simplicity, we can write the ${}^{p}\mathrm{H}_2^{q}(A,A',B,B';C;z,w)$ in the form ${}^{p}\mathrm{H}_2^{q}$, ${}^{p}\mathrm{H}_2^{q}(A+I,A',B,B';C;z,w)$ in the form ${}^{p}\mathrm{H}_2^{q}(A\pm),\cdots$, ${}^{p}\mathrm{H}_2^{q}(A,A',B,B';C+I;z,w)$ in the form ${}^{p}\mathrm{H}_2^{q}(C\pm)$.

We begin the study of this function by calculating its radius of regularity R of such function for this purpose we recall relation (1.3.10) of [17,18] and keeping in mind that $1\leq\sigma_{m,n}\leq 2^{\frac{m+n}{2}}$. We define the radius of regularity of the function ${}^{p}\mathrm{H}_2^{q}(A,A',B,B';C;z,w)$ as

$$\begin{aligned}\frac{1}{R} &= \limsup_{m+n\to\infty}\left(\frac{\|V_{m,n}\|}{\sigma_{m,n}}\right)^{\frac{1}{m+n}}\\
&= \limsup_{m+n\to\infty}\left(\frac{\left\|(A)_{m-n}(A')_m(B)_n(B')_n[(C)_m]^{-1}\right\|}{(pm)!(qn)!\sigma_{m,n}}\right)^{\frac{1}{m+n}}\\
&= \limsup_{m+n\to\infty}\left\|\left(\frac{(m-n)^{-A}(A)_{m-n}}{(m-n-1)!}(m-n-1)!(m-n)^A\frac{m^{-A'}(A')_m}{(m-1)!}(m-1)!m^{A'}\frac{n^{-B}(B)_n}{(n-1)!}(n-1)!n^B\frac{n^{-B'}(B')_n}{(n-1)!}(n-1)!\right.\right.\\
&\qquad\left.\left. n^{B'}\frac{m^C[(C)_m]^{-1}}{(m-1)!}(m-1)!m^{-C}\right)\right\|^{\frac{1}{m+n}}\left(\frac{1}{(pm)!(qn)!\sigma_{m,n}}\right)^{\frac{1}{m+n}}\\
&= \limsup_{m+n\to\infty}\left\|\left(\frac{\Gamma(C)\Gamma^{-1}(A)\Gamma^{-1}(A')\Gamma^{-1}(B)\Gamma^{-1}(B')}{(m-n)^A m^{A'-C}n^{B+B'}}\right)\right\|^{\frac{1}{m+n}}\left(\frac{(m-n-1)!(n-1)!(n-1)!}{(pm)!(qn)!\sigma_{m,n}}\right)^{\frac{1}{m+n}}\end{aligned}$$

where $\sigma_{m,n} = \begin{cases}\left(\dfrac{m+n}{m}\right)^{\frac{m}{2}}\left(\dfrac{m+n}{n}\right)^{\frac{n}{2}}, & m,n\neq 0;\\ 1, & m,n=0.\end{cases}$

Using Stirling formula and take $m=\mu n$ is a positive integer, then

$$\frac{1}{R} \le \limsup_{n\to\infty} \left\| \left([n(\mu-1)]^{A} (\mu n)^{A'-C} n^{B+B'} \right) \right\|^{\frac{1}{n(\mu+1)}} \left(\frac{(\mu n-n-1)!(n-1)!(n-1)!}{(p\mu n)!(qn)!} \right)^{\frac{1}{n(\mu+1)}}$$

$$= \limsup_{n\to\infty} \left(\frac{(\mu n-n-1)!(n-1)!(n-1)!}{(p\mu n)!(qn)!} \right)^{\frac{1}{n(\mu+1)}} = \limsup_{n\to\infty} \left(\frac{\sqrt{2\pi(\mu n-n-1)}\left(\frac{\mu n-n-1}{e}\right)^{\mu n-n-1}}{\sqrt{2\pi p\mu n}\left(\frac{p\mu n}{e}\right)^{p\mu n}} \frac{2\pi(n-1)\left(\frac{n-1}{e}\right)^{2(n-1)}}{\sqrt{2\pi q n}\left(\frac{qn}{e}\right)^{qn}} \right)^{\frac{1}{n(\mu+1)}}$$

$$= \limsup_{n\to\infty} \left(\frac{(\mu n-n-1)^{\frac{\mu-1}{\mu+1}} (n-1)^{\frac{2}{\mu+1}}}{n^{\frac{q+p\mu}{\mu+1}} q^{\frac{q}{\mu+1}} (p\mu)^{\frac{p\mu}{\mu+1}}} \right) = 0.$$

Summarizing, the following result has been established. As a conclusion, we get the following result.

Theorem 2.1. Let A, A', B, B' and C be matrices in $C^{N\times N}$ such that $C+mI$ are invertible for all integer $m\ge 0$. Then, the *p* and *q*-Horn's matrix function is an entire function in the case that, at least, one of the integers *p* and *q* are greater than one.

If $p=q=1$, then the function is convergence in $|z|\le r$, $|w|\le s$ and $(r+1)s=1$ in [5,19].

Integral form of the *p* and *q*-Horn Matrix Function

Suppose that A' and C are matrices in the space $C^{N\times N}$ of the square complex matrices, such that $A'C=CA'$, A', C and $C-A'$ are positive stable matrices.

By (1.3), (1.4) and (1.7) one gets

$$\begin{aligned}&(A')_m\left[(C)_m\right]^{-1}\\ &=\Gamma\left(A'+mI\right)\Gamma(C)\Gamma^{-1}(A')\Gamma^{-1}\left(C+mI\right) \qquad (2.2)\\ &=\Gamma^{-1}(A')\Gamma^{-1}\left(C-A'\right)\Gamma(C)\int_0^1 t^{A'+(m-1)I}\left(1-t\right)^{C-A'-I}\,\mathrm{d}t.\end{aligned}$$

Substituting from (2.1) and (2.2), we see that

$$\begin{aligned}&{}^{p}\mathrm{H}_2^{q}(A,A',B,B';C;z,w)\\ &=\sum_{m,n=0}^{\infty}\frac{(A)_{m-n}(B)_n(B')_n}{(pm)!(qn)!}z^m w^n\\ &\quad\cdot\Gamma^{-1}(A')\Gamma^{-1}(C-A')\Gamma(C)\int_0^1 t^{A'+(m-1)I}(1-t)^{C-A'-I}\,\mathrm{d}t\\ &=\Gamma^{-1}(A')\Gamma^{-1}(C-A')\Gamma(C)\\ &\quad\cdot\int_0^1 t^{A'-I}\left(1-t\right)^{C-A'-I}\,{}_3^{p}\mathrm{F}_0^{q}(A,B,B';\text{-};zt,w)\,\mathrm{d}t.\end{aligned}$$

Therefore, the following result has been established.

Theorem 2.2. Let A, A', B, B' and C be matrices in $C^{N\times N}$. Then the *p* and *q*-Horn's matrix function of two complex variables satisfies the following integral form

$$\begin{aligned}&{}^{p}\mathrm{H}_2^{q}(A,A',B,B';C;z,w)\\ &=\Gamma^{-1}(A')\Gamma^{-1}(C-A')\Gamma(C) \qquad (2.3)\\ &\quad\cdot\int_0^1 t^{A'-I}\left(1-t\right)^{C-A'-I}\,{}_3^{p}\mathrm{F}_0^{q}(A,B,B';\text{-};zt,w)\,\mathrm{d}t\end{aligned}$$

where

$${}_3^{p}\mathrm{F}_0^{q}(A,B,B';\text{-};zt,w)=\sum_{m,n=0}^{\infty}\frac{(A)_{m-n}(B)_n(B')_n}{(pm)!(qn)!}(zt)^m w^n .$$

3. Matrix Recurrence Relations

Some recurrence relation are carried out on the *p* and *q*-Horn's matrix function. In this connection the following contiguous functions relations follow, directly by increasing or decreasing one in original relation

$$\begin{aligned}&{}^{p}\mathrm{H}_2^{q}(\mathrm{A}+)\\ &=\sum_{m,n=0}^{\infty}\frac{(A+I)_{m-n}(A')_m(B)_n(B')_n[(C)_m]^{-1}}{(pm)!(qn)!}z^m w^n\\ &=\sum_{m,n=0}^{\infty}A^{-1}(A+(m-n)I)\\ &\quad\cdot\frac{(A+I)_{m-n}(A')_m(B)_n(B')_n[(C)_m]^{-1}}{(pm)!(qn)!} \qquad (3.1)\\ &=\sum_{m,n=0}^{\infty}A^{-1}(A+(m-n)I)U_{m,n}(z,w)\end{aligned}$$

and

$$\begin{aligned}{}^{p}\mathrm{H}_2^{q}(\mathrm{A}-)&=\sum_{m,n=0}^{\infty}\frac{(A-I)_{m-n}(A')_m(B)_n(B')_n[(C)_m]^{-1}}{(pm)!(qn)!}z^m w^n\\ &=\sum_{m,n=0}^{\infty}(A-I)[(A+(m-n-1)I)]^{-1}U_{m,n}(z,w).\end{aligned} \qquad (3.2)$$

Similarly

$$ {}^{p}\mathrm{H}_2^q(A'+) = \sum_{m,n=0}^{\infty} A'^{-1}\left(A'+mI\right)U_{m,n}(z,w), $$

$$ {}^{p}\mathrm{H}_2^q(A'-) = \sum_{m,n=0}^{\infty} \left(A'-I\right)\left[\left(A'+\left(m-1\right)I\right)\right]^{-1}U_{m,n}(z,w), $$

$$ {}^{p}\mathrm{H}_2^q(B+) = \sum_{m,n=0}^{\infty} B^{-1}\left(B+nI\right)U_{m,n}(z,w), $$

$$ {}^{p}\mathrm{H}_2^q(B-) = \sum_{m,n=0}^{\infty} \left(B-I\right)\left[\left(B+\left(n-1\right)I\right)\right]^{-1}U_{m,n}(z,w), $$

$$ {}^{p}\mathrm{H}_2^q(B'+) = \sum_{m,n=0}^{\infty} B'^{-1}\left(B'+nI\right)U_{m,n}(z,w), $$

$$ {}^{p}\mathrm{H}_2^q(B'-) = \sum_{m,n=0}^{\infty} \left(B'-I\right)\left[\left(B'+\left(n-1\right)I\right)\right]^{-1}U_{m,n}(z,w), $$

$$ {}^{p}\mathrm{H}_2^q(\mathrm{C}+) = \sum_{m,n=0}^{\infty} C^{-1}\left(C+mI\right)U_{m,n}(z,w), $$

$$ {}^{p}\mathrm{H}_2^q(\mathrm{C}-) = \sum_{m,n=0}^{\infty} \left(C-I\right)\left[\left(C+\left(m-1\right)I\right)\right]^{-1}U_{m,n}(z,w). \tag{3.3} $$

4. The *p* and *q*-Horn's Matrix Function under the Differential Operator

Consider the differential operator D on the p and q-Horn's matrix function of two complex variables, defined in [7, 17] as

$$ D = \begin{cases} d_1 + d_2, & m,n \geq 1 \\ 1, & \text{otherwise} \end{cases} $$

where $d_1 = z\dfrac{\partial}{\partial z}$ and $d_2 = w\dfrac{\partial}{\partial w}$. This operator has the property $Dz^m w^n = (m+n)z^m w^n$.

For the p and q-Horn's matrix function the following relations hold

$$ \begin{aligned} &(DI+A)\ {}^{p}\mathrm{H}_2^q \\ &= \sum_{m,n=0}^{\infty}\left(A+\left(m+n\right)I\right) \\ &\quad\cdot\frac{(A)_{m-n}(A')_m(B)_n(B')_n[(C)_m]^{-1}}{(pm)!(qn)!}z^m w^n \\ &= A\,{}^{p}\mathrm{H}_2^q(A+) + 2d_2\,{}^{p}\mathrm{H}_2^q \end{aligned} \tag{4.1} $$

and

$$ \begin{aligned} &\left(d_1 I + A'\right)\ {}^{p}\mathrm{H}_2^q \\ &= \sum_{m,n=0}^{\infty}\left(A'+mI\right)\frac{(A)_{m\text{-}n}(A')_m(B)_n(B')_n[(C)_m]^{-1}}{(pm)!(qn)!}z^m w^n \\ &= A'\,{}^{p}\mathrm{H}_2^q(A'+). \end{aligned} \tag{4.2} $$

By the same way, we have

$$ \begin{aligned} &\left(d_2 I + B\right){}^{p}\mathrm{H}_2^q = B\,{}^{p}\mathrm{H}_2^q(B+), \\ &\left(d_2 I + B'\right){}^{p}\mathrm{H}_2^q = B'\,{}^{p}\mathrm{H}_2^q(B'+), \\ &\left(d_1 I + C - I\right){}^{p}\mathrm{H}_2^q = (C-I)\,{}^{p}\mathrm{H}_2^q(C-). \end{aligned} \tag{4.3} $$

From (4.1), (4.2) and (4.3), we get

$$ \begin{aligned} \left(A-A'-B\right){}^{p}\mathrm{H}_2^q &= A\,{}^{p}\mathrm{H}_2^q(A+) + 2d_2\,{}^{p}\mathrm{H}_2^q \\ &\quad - A'\,{}^{p}\mathrm{H}_2^q(A'+) - B\,{}^{p}\mathrm{H}_2^q(B+), \\ \left(A-A'-B\right){}^{p}\mathrm{H}_2^q &= A\,{}^{p}\mathrm{H}_2^q(A+) + 2d_2\,{}^{p}\mathrm{H}_2^q \\ &\quad - A'\,{}^{p}\mathrm{H}_2^q(A'+) - B'\,{}^{p}\mathrm{H}_2^q(B'+). \end{aligned} \tag{4.4} $$

From (4.1), (4.3) and (4.4), we have

$$ \begin{aligned} \left(A-B-C\right){}^{p}\mathrm{H}_2^q &= A\,{}^{p}\mathrm{H}_2^q(A+) + 2d_2\,{}^{p}\mathrm{H}_2^q \\ &\quad -(C-I)\,{}^{p}\mathrm{H}_2^q(C-) + {}^{p}\mathrm{H}_2^q - B\,{}^{p}\mathrm{H}_2^q(B+), \\ \left(A-B'-C\right){}^{p}\mathrm{H}_2^q &= A\,{}^{p}\mathrm{H}_2^q(A+) + 2d_2\,{}^{p}\mathrm{H}_2^q \\ &\quad -\left(C-I\right){}^{p}\mathrm{H}_2^q(C-) + {}^{p}\mathrm{H}_2^q - B'\,{}^{p}\mathrm{H}_2^q(B'+). \end{aligned} \tag{4.5} $$

Also from (4.2), (4.3) and (4.4), we see that

$$ \begin{aligned} &\left(A'-C\right){}^{p}\mathrm{H}_2^q \\ &= A'\,{}^{p}\mathrm{H}_2^q(A'+) - \left(C-I\right){}^{p}\mathrm{H}_2^q(C-) - {}^{p}\mathrm{H}_2^q, \\ &\left(B-B'\right){}^{p}\mathrm{H}_2^q = B\,{}^{p}\mathrm{H}_2^q(B+) - B'\,{}^{p}\mathrm{H}_2^q(B'+), \\ &\left(A'-C-B+B'\right){}^{p}\mathrm{H}_2^q \\ &= A'\,{}^{p}\mathrm{H}_2^q(A'+) - \left(C-I\right){}^{p}\mathrm{H}_2^q(C-) \\ &\quad - {}^{p}\mathrm{H}_2^q - B\,{}^{p}\mathrm{H}_2^q(B+) + B'\,{}^{p}\mathrm{H}_2^q(B'+). \end{aligned} \tag{4.6} $$

Now, we append this section by introducing the differential operator $d_1 = z\dfrac{\partial}{\partial z}$ and $d_2 = w\dfrac{\partial}{\partial w}$ to the entire functions in successive manner as follows;

$$\left[d_1\left(d_1-\frac{1}{p}\right)\left(d_1-\frac{2}{p}\right)\cdots\left(d_1-\frac{p-1}{p}\right)+d_2\left(d_2-\frac{1}{q}\right)\left(d_2-\frac{2}{q}\right)\cdots\left(d_2-\frac{q-1}{q}\right)\right]\ {}^{\mathrm{p}}\mathrm{H}_2^{\mathrm{q}}$$

$$=\sum_{m=1,n=0}^{\infty} m\left(m-\frac{1}{p}\right)\left(m-\frac{2}{p}\right)\cdots\left(m-\frac{p-1}{p}\right)\frac{(A)_{m-n}(A')_m(B)_n(B')_n[(C)_m]^{-1}}{(pm)!(qn)!}z^m w^n$$

$$+\sum_{m=0,n=1}^{\infty} n\left(n-\frac{1}{q}\right)\left(n-\frac{2}{q}\right)\cdots\left(n-\frac{q-1}{q}\right)\frac{(A)_{m-n}(A')_m(B)_n(B')_n[(C)_m]^{-1}}{(pm)!(qn)!}z^m w^n$$

$$=\frac{1}{p^p}\sum_{m=1,n=0}^{\infty} mp\left(\frac{pm-1}{p}\right)\left(\frac{pm-2}{p}\right)\cdots\left(\frac{pm-p+1}{p}\right)\frac{(A)_{m-n}(A')_m(B)_n(B')_n[(C)_m]^{-1}}{(pm)!(qn)!}z^m w^n$$

$$+\frac{1}{q^q}\sum_{m=0,n=1}^{\infty} nq\left(\frac{qn-1}{q}\right)\left(\frac{qn-2}{q}\right)\cdots\left(\frac{qn-q+1}{q}\right)\frac{(A)_{m-n}(A')_m(B)_n(B')_n[(C)_m]^{-1}}{(pm)!(qn)!}z^m w^n$$

$$=\frac{1}{p^p}\sum_{m=1,n=0}^{\infty}\frac{(A)_{m-n}(A')_m(B)_n(B')_n[(C)_m]^{-1}}{(pm-p)!(qn)!}z^m w^n+\frac{1}{q^q}\sum_{m=0,n=1}^{\infty}\frac{(A)_{m-n}(A')_m(B)_n(B')_n[(C)_m]^{-1}}{(pm)!(qn-q)!}z^m w^n$$

$$=\frac{1}{p^p}\sum_{m,n=0}^{\infty}\frac{(A)_{m-n+1}(A')_{m+1}(B)_n(B')_n[(C)_{m+1}]^{-1}}{(pm)!(qn)!}z^{m+1}w^n+\frac{1}{q^q}\sum_{m,n=0}^{\infty}\frac{(A)_{m-n-1}(A')_m(B)_{n+1}(B')_{n+1}[(C)_m]^{-1}}{(pm)!(qn)!}z^m w^{n+1}$$

$$=\frac{z}{p^p}\sum_{m,n=0}^{\infty}(A+(m-n+1)I)(A'+mI)\left[\left(C+mI\right)\right]^{-1}\frac{(A)_{m-n+1}(A')_{m+1}(B)_n(B')_n[(C)_{m+1}]^{-1}}{(pm)!(qn)!}z^m w^n$$

$$+\frac{w}{q^q}\sum_{m,n=0}^{\infty}\left[\left(A+\left(m-n-1\right)I\right)\right]^{-1}(B+nI)(B'+nI)$$

$$\frac{(A)_{m-n+1}(A')_m(B)_{n+1}(B')_{n+1}\left[(C)_m\right]^{-1}}{(pm)!(qn)!}z^m w^n$$

$$=\frac{z}{p^p}AA'\left[(C)\right]^{-1}\,{}^{p}\mathrm{H}_2^q(A+,A'+,B,B';C+;z,w)+\frac{w}{q^q}\left[\left(A-I\right)\right]^{-1}BB'\,{}^{p}\mathrm{H}_2^q(A-,A',B+,B'+;C;z,w)$$

i.e.,

$$\left[d_1\left(d_1-\frac{1}{p}\right)\left(d_1-\frac{2}{p}\right)\cdots\left(d_1-\frac{p-1}{p}\right)+d_2\left(d_2-\frac{1}{q}\right)\left(d_2-\frac{2}{q}\right)\cdots\left(d_2-\frac{q-1}{q}\right)\right]\ {}^{p}\mathrm{H}_2^{q}$$

$$=\frac{z}{p^p}AA'\left[(C)\right]^{-1}\,{}^{p}\mathrm{H}_2^q(A+,A'+,B,B';C+;z,w)+\frac{w}{q^q}\left[\left(A-I\right)\right]^{-1}BB'\,{}^{p}\mathrm{H}_2^q(A-,A',B+,B'+;C;z,w)$$

We can written the ${}^{p}\mathrm{H}_2^q(A,A',B,B';C;z,w)$, then

$$\left[d_1\left(d_1-\frac{1}{p}\right)\left(d_1-\frac{2}{p}\right)\cdots\left(d_1-\frac{p-1}{p}\right)\left(d_1I+C-I\right)+d_2\left(d_2-\frac{1}{q}\right)\left(d_2-\frac{2}{q}\right)\cdots\left(d_2-\frac{q-1}{q}\right)\left(d_2I-I\right)\right]\ {}^{\mathrm{p}}\mathrm{H}_2^{\mathrm{q}}$$

$$=\frac{1}{p^p}\sum_{m=1,n=0}^{\infty}\frac{\left(C+\left(m-1\right)I\right)(A)_{m-n}(A')_m(B)_n(B')_n[(C)_m]^{-1}}{(pm-p)!(qn)!}z^m w^n+\frac{1}{q^q}\sum_{m=0,n=1}^{\infty}\frac{(n-1)(A)_{m-n}(A')_m(B)_n(B')_n[(C)_m]^{-1}}{(pm)!(qn-q)!}z^m w^n$$

$$=\left[\frac{z}{p^p}\left(DI+A\right)\left(d_1I+A'\right)-\frac{2z}{p^p}d_2\left(d_1I+A'\right)+\frac{w}{q^q}\left(DI+A\right)d_2I-\frac{w}{q^q}\left(d_1I+A'\right)d_2I\right]\ {}^{\mathrm{p}}\mathrm{H}_2^{\mathrm{q}}.$$

Therefore, the following result has been established.

Theorem 4.1. Let A, A', B, B' and C be matrices in $C^{N\times N}$. Then the ${}^{p}\mathrm{H}_2^q(A,A',B,B';C;z,w)$ is a solution for the following differential equation

$$\left[d_1\left(d_1-\frac{1}{p}\right)\left(d_1-\frac{2}{p}\right)\cdots\left(d_1-\frac{p-1}{p}\right)(d_1I+C-I)+d_2\left(d_2-\frac{1}{q}\right)\left(d_2-\frac{2}{q}\right)\cdots\left(d_2-\frac{q-1}{q}\right)(d_2I-I)\right.$$
$$\left.-\frac{z}{p^p}(DI+A)(d_1I+A')+\frac{2z}{p^p}d_2(d_1I+A')-\frac{w}{q^q}(DI+A)d_2I+\frac{w}{q^q}(d_1I+A)d_2I\right]\ {}^{p}\mathrm{H}_2^{q} \tag{4.7}$$
$$=0.$$

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