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Approximate Analytical Solutions for the Nonlinear Brinkman-Forchheimer-Extended Darcy Flow Model

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Abstract

New approximate analytical solutions for steady flow in parallel-plates channels filled with porous materials governed by non-linear Brinkman-Forchheimer extended Darcy model for three different physical situations are presented. These results are compared with those obtained from an implicit finite-difference solution of the corresponding time dependent flow problem. It is seen that the time dependent flow solutions yield the almost same steady state values as obtained by using the new approximate analytical solutions

Keywords: Non-Linear, Darcy, Time Dependent Flow, Steady State

1. Introduction

The behaviour of fluid flow in porous media has achieved considerable attention due to its important practical applications. Applications include packed-bed catalytic reactor, geothermal reservoir, drying of porous solids, shell-side flow model in shell-and tube heat exchanger, petroleum resources and many others. Reference [1] and [2] studied the mixed convection flow problems in porous media based on the model of Darcy law. For flow through the porous media with solid boundary, [3] proposed the classical boundary term in addition to Darcy's law. This Brinkman-extended Darcy model analyzed the no slip boundary condition at the wall and showed that although the wall shear resistance has little influence on the pressure drop, it has drastic effects on stream-wise velocity component and heat transfer rate at the interface between porous media and solid boundary.

In many modern applications, porous media are characterized by high velocities, *i.e*., the Reynolds number based on mean pore size is grater than unity. In such cases, it is necessary to account for deviation from linearity in the momentum equation for porous media. This deviation is accounted for by the Forchheimer term representing the quadratic drag which is essential for large particle Reynolds numbers. From the physical point quadratic drag appears in the momentum equation for porous media because of large filtration velocities, the form drag due to the solid obstacles becomes comparable with the surface drag due to friction [4].

Reference [5] also presented a closed form solution of the Brinkman-Forchheimer-extended Darcy momentum equation and the associated heat transfer equation for the case of fully developed flow with uniform heat flux at the boundary. They assumed a boundary-layer-type developed flow and as a consequence their solution is inaccurate when the inertia parameter is small and Darcy number approaches and exceeds the value of unity. Reference [6] reconsidered the analysis presented in [5] without invoking their boundary-layer assumption and derived a more general theoretical solution.

Recent results on the model (Brinkman-Forchheimer extended Darcy) presented are in [7-10]. In all the results presented above, no attempt was made to solve the nonlinear equations analytically.

In the present work, flow formation in a parallel plate channels filled with a fluid saturated porous media is analyzed analytically and numerically. The flow is described by the Brinkman-Forchheimer extended Darcy equation.

2. Mathematical Model

The physical problem under consideration consists of a steady laminar fully developed flow between two infinitely long horizontal parallel plates filled with porous material. The flow formation is caused either by pressure gradient or (and) by the movement of one of the bounding plates. The fluid is assumed to be Newtonian with uniform properties and the porous medium is isotropic and homogeneous. The x'-axis is taken along one of the plate while y'-axis is normal to it. Under the above mentioned assumption and using the dimensionless parameters given in the nomenclature, the equation of motion in porous media which accounts for the boundary and non-linear inertia term is

$$\gamma \frac{\mathrm{d}^2 u}{\mathrm{d}y^2} - \frac{u}{Da} - \frac{C}{\sqrt[4]{Da^n}} u^n + G = 0 , \quad (1)$$

The first term in the left-hand side of Equation (1) is the Brinkman term, second is the Darcy and third is the Forchheimer term $(n = 2)$, hence the momentum transfer in the porous media is governed by steady Brinkman-Forchheimer extended Darcy model.

The boundary conditions in dimensionless form are:

$$\begin{aligned} u &= B \quad \text{at} \quad y = 0 \\ u &= 0 \quad \text{at} \quad y = 1 \end{aligned} \quad (2)$$

The above equations have been rendered in dimensionless form by using the non-dimensional parameters defined in nomenclature.

3. Analytical Solutions

By introducing the assumption $\alpha = u^{n-1}$ into Equation (1) it becomes

$$\gamma \frac{\mathrm{d}^2 u}{\mathrm{d}y^2} - \frac{u}{Da} - \frac{C\alpha}{\sqrt[4]{Da^n}} u + G = 0 \quad (3)$$

Equation (3) has the solutions in Subsections 3.1-3.3.

3.1. Couette Flow [*G* = 0.0 and *B* = 1.0]

$$u(y) = \frac{\mathrm{Sinh}(\lambda(1.-y))}{\mathrm{Sinh}(\lambda)} \quad (4)$$

where $\lambda = \frac{1}{\sqrt{\gamma}} \sqrt{\frac{1}{Da} + \frac{C\alpha}{\sqrt[4]{Da^n}}}$

3.2. Pressure Driven Flow [*B* = 0 and *G* ≠ 0.0]

$$u(y) = \frac{G}{\gamma\lambda^2} \left[\frac{\mathrm{Sinh}(\lambda y) - \mathrm{Sinh}(1.-y)}{\mathrm{Sinh}(\lambda)} + 1. \right] \quad (5)$$

3.3. Generalized Couette Flow [*B* = 1.0 and *G* ≠ 0.0]

$$u(y) = \frac{\mathrm{Sinh}(\lambda(1.-y))}{\mathrm{Sinh}(\lambda)} - \frac{G}{\gamma\lambda^2} \left[\frac{\mathrm{Sinh}(\lambda(1.-y))}{\mathrm{Sinh}(\lambda)} + \frac{\mathrm{Sinh}(\lambda y)}{\mathrm{Sinh}(\lambda)} - 1. \right] \quad (6)$$

The Equations (4) to (6) can be used to find the values of the dimensionless velocity u as a function of dimensionless distance y in the interval [0,1] at the iteration $(i+1)$ in terms of value of λ at the iteration i. It should be noted here that λ is function of α which really stands for $u_i(y)$. Thus Equations (4) to (6) can be written in the following algorithmic form

$$u_{i+1}(y) = F(y, u_i(y)) \quad (7)$$

4. Numerical Solution

The analytical solutions of the previous section are valid for steady state momentum transfer in porous medium containing Darcy, Brinkman and Forchheimer terms. To explore the limits of validity of these analytical solutions and to extend our investigation to time dependent momentum transfer in porous medium, numerical solution of the time dependent problem is obtained using implicit finite difference approach.

Consider the dimensionless form of time dependent momentum equation

$$\frac{\partial u}{\partial t} = \gamma \frac{\partial^2 u}{\partial y^2} - \frac{u}{Da} - \frac{C}{\sqrt[4]{Da^n}} u^n + G = 0 \quad (8)$$

The first term in the right-hand side of Equation (8) is the Brinkman term, second is the Darcy and third is the Forchheimer term $(n = 2)$, hence the momentum transfer in the porous media is governed by time dependent Brinkman-Forchheimer extended Darcy model.

The initial and boundary conditions in dimensionless form for the present problem are:

$$\begin{aligned} &u = 0, \quad \text{for all } y \text{ when } t \le 0, \\ &\text{at } y = 0: \quad u = B, \\ &\text{at } y = 1: \quad u = 0, \quad \text{for } t > 0 \end{aligned} \quad (9)$$

The equations above also have been rendered in dimensionless form by using the non-dimensional parameters defined in nomenclature.

The numerical solution of Equation (8) using the initial and boundary conditions (9) is obtained by discretization of the momentum Equation (8) into the finite deference equation at the grid points (i, j). They are in order as follows:

$$\frac{u(i,j) - u(i,j-1)}{\Delta t} = \gamma \frac{u(i+1,j) - 2u(i,j) + u(i,j-1)}{(\Delta y)^2} - \frac{u(i,j)}{Da} - \frac{C(u(i,j))^n}{\sqrt[4]{Da^n}} + G \quad (10)$$

Here the index *i* refers to *y* and *j* to *t*. The partial time derivative is approximated by the backward difference formula, while the second-order partial space derivative is approximated by the central difference formula. The above equation is solved by Thomas algorithm by manipulating into a system of linear algebraic equations in the tri-diagonal form.

In each time step, the process of numerical integration for every dependent variable starts from the first neighboring grid point of the plate at $y = 0$ and proceeds towards the another plate using the tri-diagonal form of the finite difference Equation (10) until it reaches at immediate grid point of the plate at $y = 1$.

In each time step the velocity field is obtained. The process of computation is advanced until a steady state is approached by satisfying the following convergence criterion:

$$\frac{\sum\left|A_{i,j+1} - A_{i,j}\right|}{M\left|A\right|_{\max}} < 10^{-6} \quad (12)$$

with respect to velocity field.

Here $A_{i,j}$ represents the velocity field, M is the number of interior grid points and $\left|A\right|_{\max}$ is the maximum absolute value of $A_{i,j}$.

In the numerical computation special attention is needed to specify Δt to get a steady state solution as rapidly as possible, yet small enough to avoid instabilities.

It is set, which is suitable for present computation, as

$$\Delta t = Stabr \times \left(\Delta y\right)^2 \quad (13)$$

The parameter $Stabr$ is determined by numerical experimentation in order to achieve convergence and stability of the solution procedure. Numerical experiments show that the value 2 is suitable for numerical computations.

5. Results and Discussion

For the Brinkman-Forchheimer extension of Darcy equation to model the flow in a porous media (n = 2), B = 1.0, γ = 1.0, C = 0.52, Da = 0.01, and G = + 10.0, 0.0 and –10.0, the solutions of Equation (1) have been compared with the implicit finite-difference solution of Equation (8) in **Tables 1**, **2** and **3**, for Couette flow, pressure driven flow and generalized Couette flow respectively.

Table 1. *B* = 1.0, *γ* = 1.0, *Da* = 0.01, G = 0.0 & *C* = 0.52.

y	ANALYTICAL SOLUTION	NUMERICAL SOLUTION (IMPLICIT FINITE-DIFFERENCE SOLUTION)
0.0	1.00000	1.00000
0.1	0.36443	0.36457
0.2	0.13439	0.13381
0.3	0.04959	0.04923
0.4	0.01828	0.01813
0.5	0.00673	0.00668
0.6	0.00248	0.00246
0.7	0.00091	0.00090
0.8	0.00033	0.00032
0.9	0.00011	0.00010
1.0	0.00000	0.00000

Table 2. *B* = 0.0, *γ* = 1.0, *Da* = 0.01 & *C* = 0.52.

G = 10.0	*y*	ANALYTICAL SOLUTION	NUMERICAL SOLUTION (IMPLICIT FINITE-DIFFERENCE SOLUTION)
	0.0	0.00000	0.00000
	0.1	0.06306	0.06296
	0.2	0.08611	0.08606
	0.3	0.09450	0.09448
	0.4	0.09745	0.09744
	0.5	0.09817	0.09816
	0.6	0.09745	0.09744
	0.7	0.09450	0.09448
	0.8	0.08611	0.08606
	0.9	0.06306	0.06296
	1.0	0.00000	0.00000

G = –10.0			
	0.0	0.00000	0.00000
	0.1	–0.06335	–0.06331
	0.2	–0.08676	–0.08671
	0.3	–0.09536	–0.09532
	0.4	–0.09840	–0.09837
	0.5	–0.09915	–0.09912
	0.6	–0.09840	–0.09837
	0.7	–0.09536	–0.09532
	0.8	–0.08676	–0.08671
	0.9	–0.06335	–0.06331
	1.0	0.00000	0.00000

Table 3. ***B*** **= 1.0,** ***γ*** **= 1.0,** ***Da*** **= 0.01 &** ***C*** **= 0.52.**

G = 10.0	*y*	ANALYTICAL SOLUTION	NUMERICAL SOLUTION (IMPLICIT FINITE-DIFFERENCE SOLUTION)
	0.0	1.00000	1.00000
	0.1	0.42608	0.42646
	0.2	0.21942	0.21891
	0.3	0.14532	0.14313
	0.4	0.11546	0.11527
	0.5	0.10478	0.10470
	0.6	0.09988	0.09983
	0.7	0.09539	0.09536
	0.8	0.08643	0.08637
	0.9	0.06316	0.06306
	1.0	0.00000	0.00000
G = –10.0			
	0.0	1.00000	1.00000
	0.1	0.30250	0.30234
	0.2	0.04874	0.04808
	0.3	–0.04517	–004548
	0.4	–0.07984	–0.07992
	0.5	–0.09229	–0.09229
	0.6	–0.09587	–0.09584
	0.7	–0.09443	–0.09439
	0.8	–0.08643	–0.08637
	0.9	–0.06324	–0.06320
	1.0	0.00000	0.00000
Gr = 0.0			
	0.0	1.00000	1.00000
	0.1	0.36443	0.36457
	0.2	0.13439	0.13381
	0.3	0.04959	0.04923
	0.4	0.01828	0.01813
	0.5	0.00673	0.00668
	0.6	0.00248	0.00246
	0.7	0.00091	0.00090
	0.8	0.00033	0.00032
	0.9	0.00011	0.00010
	1.0	0.00000	0.00000

From the results presented in Tables, it can be noticed that the approximate analytical solutions presented in this work, though it is simple, it gives good and accurate results, and hence it can be efficiently used to solve this class of nonlinear differential equation models.

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Nomenclature

$\frac{dp}{dx}$ = axial pressure gradient

G = dimensionless pressure gradient, $\left(-\frac{H^3}{\rho\nu^2}\frac{dp}{dx}\right)$

H = total width of the channel

y' = dimensional co-ordinate

y = dimensionless co-ordinate, $\left(\frac{y'}{H}\right)$

n = index

C^* = inertia coefficient

C = dimensionless inertia coefficient, $\left(C^*\nu^{n-2}H^{\left[3-\frac{3n}{2}\right]}\right)$

U_0 = motion of the channel wall at $y' = 0$

B = dimensionless motion of the channel wall at $y' = 0$

K = permeability of the porous medium

Da = Darcy number, $\left(\frac{K}{H^2}\right)$

u' = velocity of the fluid

u = dimensionless velocity of the fluid, $\left(\frac{u'H}{\nu}\right)$

t' = dimensional time

t = dimensionless time, $\left(\frac{t'\nu}{H^2}\right)$

Greek symbols

ν_{eff} = effective kinematics viscosity of porous medium

ν = kinematics viscosity of fluid

γ = ratio of kinematics viscosity

G = dimensionless axial pressure gradient