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The Equivalence between the Mann and Ishikawa Iterations for Generalized Contraction Mappings in a Cone

L. Jones Tarcius Doss, T. Esakkiappan
Department of Mathematics, Anna University, Chennai, India
E-mail: *tesakki@gmail.com*
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Abstract

In this paper, equivalence between the Mann and Ishikawa iterations for a generalized contraction mapping in cone subset of a real Banach space is discussed.

Keywords: Mann Iteration, Ishikawa Iterations, Generalized Contraction, Cone

1. Introduction

Generally, the iteration techniques of W.R.Mann [1] and Shiro Ishikwa [2] are used to find the approximation of fixed point of a contraction mapping. These iterations are quite useful even for the cases of where Picard iteration fails. In this paper, we see the equivalence between these Mann and Ishikawa iterations for a generalized contraction mapping in a cone. First, we recall the definition of a cone (refer Huang Long-guang and Zhang Xian [3]) and some of its properties.

Definition 1.1: Let E be a real Banach space and a subset P of E is said to be a cone if satisfies the following:

1) $P \neq \varnothing$, P is closed and $P \neq \{0\}$;
2) $ax + by \in P$ for every $x, y \in P$ and $a, b \geq 0$;
3) $P \cap (-P) = \{0\}$.

The partial ordering $\leq$ with respect to the cone P is defines by $x \leq y$ if and only if $y - x \in P$. We shall write $x < y$ to indicate that $x \leq y$ but $x \neq y$. Further $x \ll y$ will stand for $y - x \in intP$, where $intP$ denotes interior of P. We now define the generalized contraction mapping. Let E be a real Banach space, P a nonempty convex cone subset of E. Let T a self map of P with the property that

$$\|Tx - Ty\| \leq \phi(M(x, y)) \tag{1.1}$$

where ϕ and $M(x, y)$ satisfy the following:

1) $\phi : [0, \infty) \to [0, \infty)$ is a real-valued, nondecreasing, right continuous function; (1.2)

2) $\phi(t) < t$ for each $t > 0$; (1.3)

3) ϕ is nondecreasing on $(0, \infty)$; (1.4)

4) $g(t) := t/(t - \varphi(t))$ is nonincreasing on $(0, \infty)$; (1.5)

5) $M(x, y) := \max\{\|x - y\|, \|x - Tx\|, \|y - Ty\|, \|x - Ty\|, \|y - Tx\|\}$ (1.6)

T satisfying above conditions is said to be a Generalized contraction. Below, we see the definition of the two iteration schemes due to Mann [1] and Ishikawa [2]. Further, these two iterations are applied to a class of generalized contraction mapping which is mentioned just above.

Let $x_0 = u_0 \in P$. The Mann iteration is defined by

$$u_{n+1} = (1 - \alpha_n)u_n + \alpha_n Tu_n. \tag{1.7}$$

The Ishikawa iteration is defined by

$$\begin{aligned} x_{n+1} &= (1 - \alpha_n)x_n + \alpha_n Ty_n, \\ y_n &= (1 - \beta_n)x_n + \beta_n Tx_n, \end{aligned} \tag{1.8}$$

where $\{\alpha_n\} \subset (0,1), \{\beta_n\} \subset [0,1)$. Clearly, the sequences $\{u_n\}, \{x_n\}$ and $\{y_n\}$ are in P because $x_0 = u_0 \in P$ and $\{\alpha_n\} \subset (0,1)$ and $\{\beta_n\} \subset [0,1)$ and from the definition of cone.

Let $\{w_n\}$ be a sequence in P which is a subset of a real Banach space. We say that $\{w_n\}$ converges to w and write $\lim_{n\to\infty} w_n = w$ if $\lim_{n\to\infty} \|w_n - w\| = 0$ where $\|.\|$ is the norm associated with E.

The main aim of this paper is to show that the convergence of Mann iteration is equivalent to the convergence of Ishikawa iteration in the cone P.

Below, we sate two results without proof which are very much useful for our analysis. for proof, one may refer [4] and [5] respectively.

Lemma 1 [4]

Let $\{a_n\}$ be a nonnegative sequence which satisfies the following inequality:

$$a_{n+1} \le (1-\lambda_n)a_n + \sigma_n, \qquad (1.9)$$

where $\lambda_n \in (0,1)$ for all $n \ge n_0, \sum_{n=1}^{\infty} \lambda_n = \infty$, and $\sigma_n = o(\lambda_n)$. Then $\lim_{n\to\infty} a_n = 0$.

Lemma 2 [5] Let P be a nonempty closed convex subset of a Banach space E, and T a self-map of P satisfying (1.1). Let $\{\alpha_n\}$ satisfy the conditions $\alpha_n > 0$ for all $n \ge 0$ and $\sum_{n=1}^{\infty} \alpha_n = \infty$. Then the sequences $\{x_n\}, \{y_n\}, \{u_n\}, \{Tx_n\}, \{Ty_n\}$ and $\{Tu_n\}$ are bounded.

Clearly, the sequences $\{u_n\}, \{x_n\}$ and $\{y_n\}$ are in P because $x_0 = u_0 \in P$ and $\{\alpha_n\} \subset (0,1)$ and $\{\beta_n\} \subset [0,1)$ and from the definition of cone. Here, P is a closed and convex subset of E which also follows from the definition of cone. Therefore, the above lemma can be verified for P.

2. Main Result

In this section, we discuss the main result which gives the equivalence of Mann and Ishikawa iterations in the cone. The analysis is similar to the work of Rhoades and Soltuz [6].

THEOREM 2.1

Let P be a cone subset of a Banach space E, and T a self-map of P satisfying (1.1)-(1.6). Let $\{\alpha_n\}$ satisfy the conditions $\alpha_n > 0$ for all $n \ge 0$ and $\sum_{n=1}^{\infty} \alpha_n = \infty$. Denote by x^* the unique fixed point of *T*. Then for $u_0 = x_0 \in P$, the following are equivalent:

1) the Mann iteration (1.7) converges to x^*;

2) the Ishikawa iteration (1.8) converges to x^*.

Proof: By Lemma 2, both Mann and Ishikawa iterations are bounded. we have to prove the equivalence between (1.7) and (1.8). We need to prove that

$$\lim_{n\to\infty} \|x_n - u_n\| = 0. \qquad (2.1)$$

Set

$$r_n = \max\Big\{\sup\big(\|x_n - Ty_j\| : j \ge n\big) \cup \sup\big(\|u_n - Tu_j\| : j \ge n\big) \cup \sup\big(\|x_n - Tu_j\| : j \ge n\big) \cup \sup\big(\|u_n - Ty_j\| : j \ge n\big)\Big\} \qquad (2.2)$$

We then have the following

$$\begin{aligned}\|x_n - Ty_j\| &\le (1-\alpha_{n-1})\|x_{n-1} - Ty_j\| + \alpha_{n-1}\|Ty_{n-1} - Ty_j\| \\ &\le (1-\alpha_{n-1})r_{n-1} + \alpha_{n-1}\phi\big(M(y_{n-1}, y_j)\big) \\ &\le (1-\alpha_{n-1})r_{n-1} + \alpha_{n-1}\phi(r_{n-1});\end{aligned}$$

$$\begin{aligned}\|u_n - Tu_j\| &\le (1-\alpha_{n-1})\|u_{n-1} - Tu_j\| + \alpha_{n-1}\|Tu_{n-1} - Tu_j\| \\ &\le (1-\alpha_{n-1})r_{n-1} + \alpha_{n-1}\phi\big(M(u_{n-1}, u_j)\big) \\ &\le (1-\alpha_{n-1})r_{n-1} + \alpha_{n-1}\phi(r_{n-1});\end{aligned}$$

$$\begin{aligned}\|x_n - Tu_j\| &\le (1-\alpha_{n-1})\|x_{n-1} - Tu_j\| + \alpha_{n-1}\|Ty_{n-1} - Tu_j\| \\ &\le (1-\alpha_{n-1})r_{n-1} + \alpha_{n-1}\phi\big(M(y_{n-1}, u_j)\big) \\ &\le (1-\alpha_{n-1})r_{n-1} + \alpha_{n-1}\phi(r_{n-1});\end{aligned}$$

$$\begin{aligned}\|u_n - Ty_j\| &\le (1-\alpha_{n-1})\|u_{n-1} - Ty_j\| + \alpha_{n-1}\|Tu_{n-1} - Ty_j\| \\ &\le (1-\alpha_{n-1})r_{n-1} + \alpha_{n-1}\phi\big(M(u_{n-1}, y_j)\big) \\ &\le (1-\alpha_{n-1})r_{n-1} + \alpha_{n-1}\phi(r_{n-1}).\end{aligned}$$

From the definition of r_n and all above inequalities imply that,

$$\begin{aligned}&r_n \le (1-\alpha_{n-1})r_{n-1} + \alpha_{n-1}\phi(r_{n-1}) \Leftrightarrow \\ &0 \ge \alpha_{n-1}\left[r_{n-1} - \phi(r_{n-1})\right] \le r_{n-1} - r_n.\end{aligned} \qquad (2.3)$$

Therefore, $\{r_n\}$ is monotone non-increasing in n and positive, *i.e.*, bounded below. Hence, there exists $\lim_{n\to\infty} r_n$, denoted by $r \ge 0$. We wish to show that $r = 0$.

Suppose not that, $r > 0$. From (2.3), we get the following,

$$\frac{\alpha_{n-1} r_{n-1}}{g(r_{n-1})} \le r_{n-1} - r_n$$

$$\alpha_{n-1} \le \frac{g(r_{n-1})}{r_{n-1}}(r_{n-1} - r_n) \le \frac{g(r)}{r}(r_{n-1} - r_n).$$

In general, we have that

$$\alpha_k \le \frac{g(r)}{r}(r_k - r_{k+1}).$$

Therefore, on summing we obtain,

$$\sum_{k=0}^{n} \alpha_k \le \frac{g(r)}{r}\sum_{k=0}^{n}(r_k - r_{k+1}) = \frac{g(r)}{r}(r_0 - r_{n+1}).$$

The right-hand side is bounded and the left-hand side is unbounded, which leads to a contradiction. Thus $r = o$.

Therefore, we have

$$\lim_{n\to\infty}\|x_n - Tu_n\| = 0 \qquad \lim_{n\to\infty}\|u_n - Ty_n\| = 0 \qquad (2.4)$$

$$\lim_{n\to\infty}\|x_n - Ty_n\| = 0 \qquad \lim_{n\to\infty}\|u_n - Tu_n\| = 0 \qquad (2.5)$$

We now show that both the iteration schemes are equivalent. Suppose the Mann iteration converges,then we have

$$\begin{aligned}\|x_{n+1} - u_{n+1}\| &\le (1-\alpha_n)\|x_n - u_n\| + \alpha_n\|Ty_n - Tu_n\| \\ &\le (1-\alpha_n)\|x_n - u_n\| + \alpha_n\big(\|Ty_n - x_n\| + \|x_n - Tu_n\|\big).\end{aligned}$$

Using (2.4), (2.5), Lemma 1 and above equations with the following

$$\lambda_n := \|x_n - u_n\|,\ \sigma_{n:} = \alpha_n\left(\|Ty_n - x_n\| + \|x_n - Tu_n\|\right),$$
$$\sigma_{n:} = o(\alpha_n)$$

we have $\lim_{n\to\infty} \lambda_n = 0$, that is (2.1) holds.

Then, the relation

$$\|x_n - x^*\| \le \|x_n - u_n\| + \|x^* - u_n\| \to 0.$$

This implies that Ishikawa iteration also converges. Suppose the Ishikawa iteration converges, then we have

$$\|x_{n+1} - u_{n+1}\| \le (1-\alpha_n)\|x_n - u_n\| + \alpha_n\|Ty_n - Tu_n\|$$
$$\le (1-\alpha_n)\|x_n - u_n\| + \alpha_n\left(\|Ty_n - u_n\| + \|u_n - Tu_n\|\right).$$

Using (2.4), (2.5), Lemma 1 and above equations with the following

$$\lambda_n := \|x_n - u_n\|, \sigma_n := \alpha_n\left(\|Ty_n - u_n\| + \|u_n - Tu_n\|\right),$$
$$\sigma_n = o(\alpha_n),$$

we have $\lim_{n\to\infty} \lambda_n = 0$, that is (2.1) holds.

Then, the relation

$$\|u_n - x^*\| \le \|x_n - u_n\| + \|x_n - x^*\| \to 0.$$

This implies that Mann iteration converges. Hence the theorem.

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