Applied Mathematics, **2011, 2, 1317-1322**
doi:10.4236/am.2011.211184 Published Online November 2011 (http://www.SciRP.org/journal/am)

On Generalized φ-Recurrent Sasakian Manifolds

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Received July 22, 2011; *revised Septembe*r 13, 2011; *accepted September* 20, 2011

Abstract

The object of the present paper is to introduce the notion of *generalized φ-recurrent Sasakian manifold* and study its various geometric properties with the existence of such notion. Among others we study generalized concircularly φ-recurrent Sasakian manifolds. The existence of generalized φ-recurrent Sasakian manifold is given by a proper example.

Keywords: Locally φ-Symmetric Sasakian Manifold, φ-Recurrent Sasakian Manifold, Generalized φ-Recurrent Sasakian Manifold, Scalar Curvature

1. Introduction

Let M be an n-dimensional connected Riemannian manifold with Riemannian metric g and Levi-Civita connection ∇. M is called locally symmetric if its curvature tensor is parallel with respect to ∇. During the last five decades, the notion of locally symmetric manifold has been weakend many authors in different directions such as recurrent manifold by Walker [1], semi-symmetric manifold by Szabó [2], pseudo-symmetric manifold by Chaki [3], pseudo-symmetric manifold by Deszcz [4], weakly symmetric manifold by Tamássy and Binh [5], weakly symmetric manifold by Selberg [6]. However, the notion of pseudo-symmetry by Chaki and Deszcz are different and that of weak symmetry by Selberg and Tamássy and Binh are also different. As a weaker version of local symmetry, in 1977 Takahashi [7] introduced the notion of local φ-symmetry on a Sasakian manifold. By extending the notion of local φ-symmetry of Takahashi [7], De *et al*. [8] introduced and studied the notion of φ-recurrent Sasakian manifold. It may be mentioned that locally φ-symmetric and φ-recurrent LP-Sasakian, $(LCS)_n$ and (k, μ)-contact metric manifolds are respectively studied in [9-13].

Again, in 1979 Dubey [14] introduced the notion of generalized recurrent manifold and then such a manifold is studied by De and Guha [15]. The manifold M, $n > 2$, is called generalized recurrent [14] if its curvature tensor R of type (1,3) satisfies the condition

$$\nabla R = A \otimes R + B \otimes G, \tag{1}$$

where A and B are nowhere vanishing unique 1-forms defined by $A(\cdot) = g(\cdot, \rho_1)$, $B(\cdot)=g(\cdot, \rho_2)$ and G is a tensor of type (1,3) given by

$$G(X,Y)Z = g(Y,Z)X - g(X,Z)Y \tag{2}$$

for all vector fields $X,Y,Z \in \chi(M)$; $\chi(M)$ being the Lie algebra of all smooth vector fields on M and ∇ is the Levi-Civita connection. Again M, $n > 2$, is called generalized Ricci-recurrent manifold [16] if its Ricci tensor S of type (0, 2) satisfies the condition

$$\nabla S = A \otimes S + B \otimes g \tag{3}$$

where A and B are nowhere vanishing unique 1-forms.

The object of the present paper is to introduce a type of non-flat Sasakian manifolds called *generalized φ-recurrent Sasakian manifold*, which includes both the notion of local φ-symmetry of Takahashi [7] and also φ-recurrence of De *et al*. [8] as particular cases. The paper is organized as follows. Section 2 deals with some preliminaries of Sasakian manifolds. Section 3 is devoted to the study of generalized φ-recurrent Sasakian manifolds and it is shown that such a manifold is generalized Ricci-recurrent [16]. In Section 4, we study generalized concircularly φ-recurrent Sasakian manifolds and it is shown that in a generalized concircularly φ-recurrent Sasakian manifold the vector field ρ_2 associted with the 1-form B and the characterstic vector field ξ are co-directional. We also introduce the notion of *super generalized Ricci-recurrent manifolds* and proved that a generalized concircularly φ-recurrent Sasakian manifold is such one. Also the existence of generalized φ-recurrent Sasakian manifold is ensured by a proper example in the last Section.

2. Sasakian Manifolds

An $n = (2m+1)$-dimensional C^∞ manifold M is said to be a contact manifold if it carries a global 1-form η such that $\eta \wedge (d\eta)^m \neq 0$ everywhere on the manifold. Given a contact form η, it is well-known that there exists a unique vector field ξ, called the characteristic vector field of η, satisfying $\eta(\xi) = 1$ and $d\eta(X, \xi) = 0$ for any vector field X on M. A Riemannian metric g is said to be an associated metric if there exists a tensor field φ of type (1,1) such that

$$\varphi^2 = -I + \eta \otimes \xi,\ \eta(\cdot) = g(\cdot, \xi),\ \mathrm{d}\eta(\cdot,\cdot) = g(\cdot, \varphi\cdot) \quad (4)$$

$$\varphi\xi = 0, \eta \circ \varphi = 0, g(\varphi\cdot, \cdot) = -g(\cdot, \varphi\cdot) \quad (5)$$

$$g(\varphi\cdot, \varphi\cdot) = g(\cdot,\cdot) - \eta \otimes \eta. \quad (6)$$

Then the structure (φ, ξ, η, g) on M is called a contact metric stucture and the manifold M equipped with such a stucture is called a contact metric manifold [17].

Given a contact metric manifold M we define a (1,1) tensor field h by $h = \frac{1}{2}£_\xi \phi$, where £ denotes the operator of Lie differentiation. Then h is symmetric. The vector field ξ is a Killing vector field with respect to g if and only if $h = 0$. A contact metric manifold M for which ξ is a Killing vector is said to be a K-contact manifold. A contact structure on M gives rise to an almost complex structure J on the product $M \times \mathbb{R}$ defined by

$$J\left(X, f\frac{\mathrm{d}}{\mathrm{d}t}\right) = \left(\phi X - f\xi, \eta(X)\frac{\mathrm{d}}{\mathrm{d}t}\right),$$

where f is a real valued function, is integrable, then the structure is said to be normal and the manifold M is a Sasakian manifold. Equivalently, a contact metric manifold is Sasakian if and only if

$$R(X,Y)\xi = \eta(Y)X - \eta(X)Y \quad (7)$$

holds for all X, Y where R denotes the curvature tensor of the manifold.

In an n-dimensional Sasakian manifold M the following relations hold [17-19]:

$$R(\xi, X)Y = (\nabla_X\varphi)(Y) = g(X,Y)\xi - \eta(Y)X = -R(X,\xi)Y, \quad (8)$$

$$\nabla_X\xi = -\varphi X, (\nabla_X\eta)(Y) = g(X, \varphi Y), \quad (9)$$

$$\eta(R(X,Y)Z) = g(Y,Z)\eta(X) - g(X,Z)\eta(Y), \quad (10)$$

$$S(X,\xi) = (n-1)\eta(X), S(\xi,\xi) = (n-1), \quad (11)$$

$$S(\varphi X, \varphi Y) = S(X,Y) - (n-1)\eta(X)\eta(Y), \quad (12)$$

$$(\nabla_W S)(Y,\xi) = S(Y, \varphi W) - (n-1)g(Y, \varphi W), \quad (13)$$

$$(\nabla_W R)(X,Y)\xi = g(W, \varphi Y)X - g(W, \varphi X)Y + R(X,Y)\varphi W, \quad (14)$$

$$(\nabla_W R)(X,\xi)Z = g(X,Z)\varphi W - g(Z, \varphi W)X + R(X, \varphi W)Z, \quad (15)$$

for all vector fields $X, Y, Z, W \in \chi(M)$.

Definition 1. [7] A Sasakian manifold is said to be locally φ-symmetric if

$$\varphi^2\left((\nabla_W R)(X,Y)Z\right) = 0 \quad (16)$$

for all vector fields X, Y, Z, W orthogonal to ξ.

Definition 2. [8] A Sasakian manifold is said to be φ-recurrent if there exists a nowhere vanishing unique 1-form A such that

$$\varphi^2\left((\nabla_W R)(X,Y)Z\right) = A(W)R(X,Y)Z \quad (17)$$

for all vector fields $X, Y, Z, W \in \chi(M)$.

Especially, if the 1-form A vanishes and the vector fields are horizontal, then the manifold turns to be a locally φ-symmetric Sasakian manifold [7].

3. Generalized φ-Recurrent Sasakian Manifolds

Definition 3. An n-dimensional, $n \geq 3$, Sasakian manifold M is said to be a generalized φ-recurrent if its curvature tensor satisfies the relation

$$\varphi^2\left((\nabla_W R)(X,Y)Z\right) = A(W)\varphi^2\left(R(X,Y)Z\right) + B(W)\varphi^2\left(G(X,Y)Z\right) \quad (18)$$

for all $X, Y, Z, W \in \chi(M)$, where A and B are nowhere vanishing unique 1-forms such that $A(X) = g(X, \rho_1)$, $B(X) = g(X, \rho_2)$ and $G(X,Y)Z$ is defined in (2).

We consider a Sasakian manifold M, $n \geq 3$, which is generalized φ-recurrent. Then by virtue of (4), (18) yields

$$(\nabla_W R)(X,Y)Z = \eta\left((\nabla_W R)(X,Y)Z\right)\xi + A(W)\left[R(X,Y)Z - \eta\left(R(X,Y)Z\right)\xi\right] + B(W)\left[G(X,Y)Z - \eta\left(G(X,Y)Z\right)\xi\right], \quad (19)$$

from which it follows that

$$g\left((\nabla_W R)(X,Y)Z, U\right) = \eta\left((\nabla_W R)(X,Y)Z\right)\eta(U) + A(W)\left[g\left(R(X,Y)Z, U\right) - \eta\left(R(X,Y)Z\right)\eta(U)\right] + B(W)\left[g\left(G(X,Y)Z, U\right) - \eta\left(G(X,Y)Z\right)\eta(U)\right]. \quad (20)$$

Let $\{e_i : i = 1, 2, \cdots, n\}$ be an orthonormal basis of the tangent space at any point of the manifold. Then putting

$X = U = e_i$ in (20) and taking summation over i, $1 \le i \le n$, and using (15), (10), (8) and (2), we get

$$\begin{aligned}(\nabla_W S)(Y,Z) &= A(W)S(Y,Z) \\ &+ \{(n-2)B(W) - A(W)\}g(Y,Z) \\ &\cdot \{A(W)+B(W)\}\eta(Y)\eta(Z).\end{aligned} \tag{21}$$

Setting $Z = \xi$ in (19) and using (7), (2), (14) and (10) we obtain

$$(\nabla_W R)(X,Y)\xi = \{A(W)+B(W)\}[\eta(Y)X - \eta(X)Y]. \tag{22}$$

From (14) and (22), we obtain

$$\begin{aligned}&g(W,\varphi Y)X - g(W,\varphi X)Y + R(X,Y)\varphi W \\ &= \{A(W)+B(W)\}[\eta(Y)X - \eta(X)Y].\end{aligned} \tag{23}$$

Taking inner product of (23) with Z and then taking contraction over X and Z, we get

$$S(Y,\varphi W) = (n-1)[\{A(W)+B(W)\}\eta(Y) - g(W,\varphi Y)]. \tag{24}$$

Putting $Y = \xi$ in (24) we get

$$A(W) + B(W) = 0 \quad \text{for all } W. \tag{25}$$

This leads to the following.

Theorem 1. *In a generalized φ-recurrent Sasakian manifold M*, $n \ge 3$, *the associated 1-forms A and B are related by the relation* $A + B = 0$.

In view of (25), (21) turns into

$$(\nabla_W S)(Y,Z) = A(W)S(Y,Z) + b(W)g(Y,Z), \tag{26}$$

where $b(W) = (n-3)A(W)$. This leads to the following.

Theorem 2. A *generalized φ-recurrent Sasakian manifold M*, $n \ge 3$, *is generalized Ricci-recurrent.*

4. Generalized Concircularly φ-Reccurent Sasakian Manifolds

The concircular transformation on a Riemannian manifold is a transformation under which geodesic circles remains invariant [20]. The concircular curvature tensor $\widetilde{C}$ of type (1,3) is given by [20]

$$\widetilde{C}(X,Y)Z = R(X,Y)Z - \frac{r}{n(n-1)}G(X,Y)Z. \tag{27}$$

If the concircular curvature tensor $\widetilde{C}$ satisfies the relation (18), then the manifold is said to be generalized concircularly φ-recurrent Sasakian manifold. We also note that since conformal and projective curvature tensors are trace free, there do not exist any generalized conformally and projectively φ-reccurent Sasakian manifolds.

Let us consider a generalized concircularly φ-recurrent Sasakian manifold M, $n \ge 3$. Hence the defining condition of a generalized concircularly φ-recurrent Sasakian manifold, yields by virtue of (27) that

$$\begin{aligned}&\varphi^2((\nabla_W R)(X,Y)Z) - A(W)\varphi^2(R(X,Y)Z) \\ &-B(W)\varphi^2(G(X,Y)Z) \\ &= \frac{rA(W) - dr(W)}{n(n-1)} \\ &\cdot[g(Y,Z)X - \eta(X)g(Y,Z)\xi - g(X,Z)Y + \eta(Y)g(X,Z)\xi].\end{aligned} \tag{28}$$

This leads to the following.

Theorem 3. *A generalized concircularly φ-recurrent Sasakian manifold M*, $n \ge 3$, *is generalized φ-recurrent if and only if*

$$\begin{aligned}&\frac{rA(W) - dr(W)}{n(n-1)}[g(Y,Z)X - \eta(X)g(Y,Z)\xi \\ &-g(X,Z)Y + \eta(Y)g(X,Z)\xi] = 0.\end{aligned} \tag{29}$$

Now taking inner product of (29) with U we have

$$\begin{aligned}&\frac{rA(W) - dr(W)}{n(n-1)}[g(Y,Z)g(X,U) - \eta(X)g(Y,Z)\eta(U) \\ &-g(X,Z)g(Y,U) + \eta(Y)g(X,Z)\eta(U)] = 0.\end{aligned}$$

Taking contraction over X and U we get

$$\{rA(W) - dr(W)\}[(n-2)g(Y,Z) + \eta(Y)\eta(Z)] = 0.$$

Again taking contraction over Y and Z we get

$$\{rA(W) - dr(W)\}[n(n-2)+1] = 0.$$

which implies that

$$A(W) = \frac{1}{r}dr(W) \quad \text{for all } W \text{ and } r \ne 0$$

i.e., $\rho_1 = \frac{1}{r} grad r$, where $A(W) = g(W,\rho_1)$.

This leads to the following.

Theorem 4. *If a generalized concircularly φ-recurrent Sasakian manifold M*, $n \ge 3$, *is a generalized φ-recurrent Sasakian manifold, then the associated vector field corresponding to the 1-form A is given by* $\rho_1 = \frac{1}{r} grad r$, *r being the non-zero and non-constant scalar curvature of the manifold.*

Now by virtue of (4), it follows from (28) that

$$\begin{aligned}&(\nabla_W R)(X,Y)Z = \eta((\nabla_W R)(X,Y)Z)\xi \\ &+A(W)[R(X,Y)Z - \eta(R(X,Y)Z)\xi] \\ &+B(W)[G(X,Y)Z - \eta(G(X,Y)Z)\xi] \\ &-\frac{rA(W) - dr(W)}{n(n-1)}[g(Y,Z)X - \eta(X)g(Y,Z)\xi \\ &-g(X,Z)Y + \eta(Y)g(X,Z)\xi].\end{aligned} \tag{30}$$

Taking inner product of (30) with U and then contracting over X and U, and then using (2), (15), (10) and (8) we get

$$\begin{aligned}&(\nabla_W S)(Y,Z)=A(W)S(Y,Z)\\&+\left[(n-2)B(W)-A(W)\right]g(Y,Z)\\&+\frac{dr(W)}{n(n-1)}\left[(n-2)g(Y,Z)+\eta(Y)\eta(Z)\right]\\&+A(W)\left[\left\{1-\frac{r}{n(n-1)}\right\}\eta(Y)\eta(Z)-\frac{(n-2)r}{n(n-1)}g(Y,Z)\right]\\&+B(W)\eta(Y)\eta(Z).\end{aligned} \tag{31}$$

Again taking contraction over Y and Z in (31), we get

$$\mathrm{d}r(W)=\{r-n(n-1)\}A(W)+n(n-1)^2B(W). \tag{32}$$

From (32), we can state the following.

Theorem 5. *In a generalized concircularly φ-recurrent Sasakian manifold M*, $n\geq 3$, *the associated 1-forms A and B are related by the relation* (32).

Corollary 1. *In a generalized concircularly φ-recurrent Sasakian manifold M*, $n\geq 3$, *with constant scalar curvature, the associated 1-forms A and B are related by*

$$\{r-n(n-1)\}A+n(n-1)^2B=0.$$

Now using (32) in (31) we get

$$\begin{aligned}&(\nabla_W S)(Y,Z)=A(W)S(Y,Z)\\&+\{n(n-2)B(W)-(n-1)A(W)\}g(Y,Z)\\&+nB(W)\eta(Y)\eta(Z).\end{aligned} \tag{33}$$

From (33), it follows that the Ricci tensor S satisfies the condition

$$\nabla S=\alpha\otimes S+\beta\otimes g+\gamma\otimes \pi, \tag{34}$$

where $\alpha(W)=A(W)$,

$\beta(W)=n(n-2)B(W)-(n-1)A(W)$, $\gamma(W)=nB(W)$

and $\pi=\eta\otimes\eta$.

By extending the notion of generalized Ricci-recurrent manifold [16], we introduce the notion of super generalized Ricci-recurrent manifold defined as follows.

Definition 4. An n-dimensional Riemannian manifold M, $n>2$, is called a super generalized Ricci-recurrent if its Ricci tensor S of type (0,2) satisfies the relation

$$\nabla S=\alpha\otimes S+\beta\otimes g+\gamma\otimes \pi,$$

where α, β, γ are nowhere vanishing unique 1-forms and $\pi=\eta\otimes\eta$.

From (34), we can state the following:

Theorem 6. A *generalized concircularly φ-recurrent Sasakian manifold M*, $n\geq 3$, *is super generalized Ricci-recurrent manifold.*

Now taking contraction of (33) over W and Z, we get

$$\begin{aligned}\frac{1}{2}\mathrm{d}r(Y)&=S(Y,\rho_1)+n(n-2)B(Y)\\&-(n-1)A(Y)+n\eta(Y)B(\xi).\end{aligned}$$

By virtue of (32), the above relation takes the form

$$\begin{aligned}S(Y,\rho_1)&=\frac{r-(n-1)(n-2)}{2}A(Y)\\&+\frac{n(n^2-4n+5)}{2}B(Y)-n\eta(Y)B(\xi).\end{aligned} \tag{35}$$

From (35), we can state the following.

Theorem 7. *In a generalized concircularly φ-recurrent Sasakian manifold M*, $n\geq 3$, *the Ricci tensor in the direction of* ρ_1 *is given by* (35).

Now setting $Z=\xi$ in (33) and then using (13) and (11) we get

$$S(Y,\varphi W)=(n-1)g(Y,\varphi W)+n(n-1)B(W)\eta(Y). \tag{36}$$

Replacing Y by φY in (36) and using (12) and (6) we have

$$S(Y,W)=(n-1)g(Y,W). \tag{37}$$

Replacing W by φW in (36) and then using (4) we get

$$S(Y,W)=(n-1)g(Y,W)-n(n-1)B(\varphi W)\eta(Y). \tag{38}$$

From (37) and (38) we have

$$B(\varphi W)=0,$$

which implies that

$$B(W)=\eta(W)B(\xi).$$

This leads to the following.

Theorem 8. *In a generalized concircularly φ-recurrent Sasakian manifold M*, $n\geq 3$, *the vector field* ρ_2 *associated with the* 1-*form B and the characterstic vector field ξ are codirectional.*

5. Example of Generalized φ-Recurrent Sasakian Manifold

Example 1. We consider a 3-dimensional manifold $M=\{(x,y,z)\in\mathbb{R}^3: y\neq 0\}$, where (x, y, z) are the standard coordinates in $\mathbb{R}^3$. Let $\{E_1,E_2,E_3\}$ be a linearly independent global frame on M given by

$$E_1=-2\frac{\partial}{\partial x},E_2=x\frac{\partial}{\partial z}-y^2\frac{\partial}{\partial y},E_3=\frac{\partial}{\partial z}.$$

Let g be the Riemannian metric defined by

$$g(E_1,E_3)=g(E_2,E_3)=g(E_1,E_2)=0,$$
$$g(E_1,E_1)=g(E_2,E_2)=g(E_3,E_3)=1.$$

Let η be the 1-form defined by $\eta(U)=g(U,E_3)$ for any $U\in\chi(M)$. Let φ be the (1,1) tensor field defined by $\varphi E_1=-E_2, \varphi E_2=E_1$ and $\varphi E_3=0$. Then using the linearity of φ and g we have $\eta(E_3)=1$,
$\varphi^2 U=-U+\eta(U)E_3$ and $g(\varphi U,\varphi W)=g(U,W)$ $-\eta(U)\eta(W)$ for any $U,W\in\chi(M)$. Thus for $E_3=\xi$, (φ,ξ,η,g) defines an almost contact metric structure on M. Let ∇ be the Riemannian connection of g. Then we have

$$[E_1,E_2]=-2E_3,\ [E_1,E_3]=0,\ [E_2,E_3]=0.$$

Using Koszul formula for the Riemannian metric g, we can easily calculate

$$\nabla_{E_1}E_1=0, \nabla_{E_1}E_2=-E_3, \nabla_{E_1}E_3=E_2,$$

$$\nabla_{E_2}E_1=E_3, \nabla_{E_2}E_2=0, \nabla_{E_2}E_3=-E_1,$$

$$\nabla_{E_3}E_1=E_2, \nabla_{E_3}E_2=-E_1, \nabla_{E_3}E_3=0.$$

From the above it can be easily seen that (φ,ξ,η,g) is a Sasakian structure on M. Consequently $M^3(\varphi,\xi,\eta,g)$ is a Sasakian manifold. Using the above relations, we can easily calculate the components of the curvature tensor as follows:

$$R(E_1,E_2)E_1=3E_2, R(E_1,E_2)E_2=-3E_1,$$
$$R(E_1,E_2)E_3=0,$$

$$R(E_1,E_3)E_1=3E_3, R(E_1,E_3)E_2=0, R(E_1,E_3)E_3=E_1,$$

$$R(E_2,E_3)E_1=0, R(E_2,E_3)E_2=-E_3, R(E_2,E_3)E_3=E_2$$

and the components which can be obtained from these by the symmetry properties.

Since $\{E_1,E_2,E_3\}$ forms a basis of the Sasakian manifold, any vector field $X,Y,Z\in\chi(M)$ can be written as

$$X=a_1E_1+b_1E_2+c_1E_3,\quad Y=a_2E_1+b_2E_2+c_2E_3,$$

$$X=a_3E_1+b_3E_2+c_3E_3,$$

where $a_i,b_i,c_i\in\mathbb{R}^+$ (the set of all positive real numbers), i = 1, 2, 3. Then

$$\begin{aligned}R(X,Y)Z=&\left[c_3(a_1c_2-a_2c_1)-3b_3(a_1b_2-a_2b_1)\right]E_1\\&+\left[3a_3(a_1b_2-a_2b_1)+c_3(b_1c_2-b_2c_1)\right]E_2\\&-\left[a_3(a_1c_2-a_2c_1)+b_3(b_1c_2-b_2c_1)\right]E_3\end{aligned}\tag{39}$$

and

$$\begin{aligned}G(X,Y)Z=&(a_2a_3+b_2b_3+c_2c_3)(a_1E_1+b_1E_2+c_1E_3)\\&-(a_1a_3+b_1b_3+c_1c_3)(a_2E_1+b_2E_2+c_2E_3).\end{aligned}\tag{40}$$

By virtue of (39) we have the following:

$$\begin{aligned}(\nabla_{E_1}R)(X,Y)Z=&4\{c_3(a_1b_2-a_2b_1)+b_3(a_1c_2-a_2c_1)\}E_1\\&-4a_3(a_1c_2-a_2c_1)E_2-4a_3(a_1b_2-a_2b_1)E_3,\end{aligned}\tag{41}$$

$$\begin{aligned}(\nabla_{E_2}R)(X,Y)Z=&4b_3(b_1c_2-b_2c_1)E_1\\&+4c_3(a_1b_2-a_2b_1)-a_3(b_1c_2-b_2c_1)E_2\\&-4b_3(a_1b_2-a_2b_1)E_3,\end{aligned}\tag{42}$$

$$(\nabla_{E_3}R)(X,Y)Z=0.\tag{43}$$

From (39) and (40), we get

$$\begin{aligned}\varphi^2(R(X,Y)Z)&=p_1E_1+p_2E_2\\\varphi^2(G(X,Y)Z)&=q_1E_1+q_2E_2,\end{aligned}\tag{44}$$

where

$$\begin{aligned}p_1&=3b_3(a_1b_2-a_2b_1)-c_3(a_1c_2-a_2c_1)\\p_2&=-3a_3(a_1b_2-a_2b_1)-c_3(b_1c_2-b_2c_1)\\q_1&=a_2(b_1b_3+c_1c_3)-a_1(b_2b_3+c_2c_3)\\q_2&=b_2(a_1a_3+c_1c_3)-b_1(a_2a_3+c_2c_3).\end{aligned}$$

Also from (41)-(43), we obtain

$$\varphi^2((\nabla_{E_i}R)(X,Y)Z)=u_iE_1+v_iE_2 \quad \text{for } i=1,2,3,\tag{45}$$

where

$$\begin{aligned}u_1&=-4\{c_3(a_1b_2-a_2b_1)+b_3(a_1c_2-a_2c_1)\}\\v_1&=4a_3(a_1c_2-a_2c_1)\\u_2&=-4b_3(b_1c_2-b_2c_1)\\v_2&=4\{a_3(b_1c_2-b_2c_1)-c_3(a_1b_2-a_2b_1)\}\\u_3&=0, v_3=0.\end{aligned}$$

Let us now consider the components of the 1-forms as

$$\begin{aligned}A(E_i)&=\frac{q_2u_i-q_1v_i}{p_1q_2-p_2q_1} &&\text{for } i=1,2\\&=0 &&\text{for } i=3\end{aligned}\tag{46}$$

and

$$\begin{aligned}B(E_i)&=\frac{p_1v_i-p_2u_i}{p_1q_2-p_2q_1} &&\text{for } i=1,2\\&=0 &&\text{for } i=3\end{aligned}\tag{47}$$

where $p_1q_2-p_2q_1\neq0$, $q_2u_i-q_1v_i\neq0$ and $p_1v_i-p_2u_i\neq0$ for i = 1, 2.

From (18), we have

$$\begin{aligned}\varphi^2((\nabla_{E_i}R)(X,Y)Z)=&A(E_i)\varphi^2(R(X,Y)Z)\\&+B(E_i)\varphi^2(G(X,Y)Z)\end{aligned}\tag{48}$$

for $i = 1, 2, 3$. By virtue of (44)-(47), it can be easily shown that the manifold satisfies the relation (48). Hence the manifold under consideration is a generalized φ-recurrent Sasakian manifold, which is not φ-recurrent. This leads to the following.

Theorem 9. *There exists a* 3*-dimensional generalized* φ*-recurrent Sasakian manifold*, *which is neither* φ*-symmetric nor* φ*-recurrent.*

6. References

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