Scientific Research

Existence and Uniqueness of Solution for a Fractional Order Integro-Differential Equation with Non-Local and Global Boundary Conditions

Mehran Fatemi[1], Nihan Aliev[1], Sedaghat Shahmorad[2]
[1]*Department of Mathematics, Baku State University, Baku, Azerbaijan*
[2]*Department of Applied Mathematics, University of Tabriz, Tabriz, Iarn*
E-mail: *fatemi.mehran@yahoo.com, shahmorad@tabrizu.ac.ir*

Abstract

In this paper, we prove an important existence and uniqueness theorem for a fractional order Fredholm – Volterra integro-differential equation with non-local and global boundary conditions by converting it to the corresponding well known Fredholm integral equation of second kind. The considered problem in this paper has been solved already numerically in [1].

1. Introduction

Let's consider a problem under boundary condition containing non-local and global terms for a fractional order integro-differential equation

$$D^q y(x) = f(x) + \int_a^x K_1(x,t) y(t) dt + \int_a^b K_2(x,t) y(t) dt,$$
$$q \in (m-1, m], \ x \in (a,b),$$
(1)

$$\sum_{j=1}^m \left(\gamma_{ij} y^{(j-1)}(a) + \eta_{ij} y^{(j-1)}(b) \right) + \lambda \int_a^b H_i(t) y(t) dt = d_i,$$
$$i = \overline{1,m},$$
(2)

where $m - q \in [0,1)$, $f(x)$, $K_1(x,t)$, $K_2(x,t)$ and $H_i(t)$, $i = \overline{1,m}$ are continuous, real-valued functions, γ_{ij}, η_{ij}, λ_i and d_i, $i = \overline{1,m}$, $j = \overline{1,m}$ are real constants, and boundary conditions (2) are linearly independent.

2. Existence and Uniqueness of Solution

Theorem. Let *the functions* $f(x)$, $K_j(x,t)$, $j = 1, 2$ *and* $H_i(t)$, $i = \overline{1,m}$ *are continuous*, γ_{ij}, η_{ij}, λ_i *and* d_i, $i.j = \overline{1,m}$ *are real constant, the boundary conditions* (2) *are linearly independent and condition* (15) *is satisfied. Then the boundary value problem* (1)-(2) *has unique solution.*

Proof: Acting in Equation (1) by fractional order derivative operator D^{m-q} [2], we get

$$D^{m-q} D^q y(x) = D^{m-q} f(x) D^{m-q} \left(\int_a^x K_1(x,t) y(t) dt \right) + D^{m-q} \int_a^b K_2(x,t) y(t) dt,$$

since $D^{m-q} D^q y(x) = D^m y(x)$, then we get the equation

$$D^m y(x) = F(x) + \int_a^x M_1(x,t) y(t) dt + \int_a^b M_2(x,t) y(t) dt,$$
(3)

where

$$\begin{cases} F(x) = D^{m-q}f(x) = \dfrac{\mathrm{d}}{\mathrm{d}x} \int\limits_{a}^{x} \dfrac{(x-\xi)^{q-m}}{(q-m)!} f(\xi)\mathrm{d}\xi, \\[4mm] M_1(x,t) = \dfrac{\mathrm{d}}{\mathrm{d}x} \int\limits_{t}^{x} \dfrac{(x-\xi)^{q-m}}{(q-m)!} K_1(\xi,t)\mathrm{d}\xi, \\[4mm] M_2(x,t) = \dfrac{\mathrm{d}}{\mathrm{d}x} \int\limits_{a}^{x} \dfrac{(x-\xi)^{q-m}}{(q-m)!} K_2(\xi,t)\mathrm{d}\xi. \end{cases} \quad (4)$$

Now, we write Equation (3) in the general from

$$D^m y(x) = G(x,y), \qquad (3.1)$$

and accept that $G(x,y)$ is known, then the fundamental solution (see [3]) is in the form

$$Y(x-\eta) = \dfrac{(x-\eta)^{m-1}}{(m-1)!}\theta(x-\eta). \qquad (5)$$

where

$$\theta(x-\eta) = \begin{cases} 1, & x > \eta, \\ 1/2, & x = \eta, \\ 0, & x < \eta, \end{cases} \qquad (6)$$

is Heaviside's unit function.

Now, we try to get some basic relations. The first of these relations is Lagrange's formula. We multiply both ides of Equation (3) by fundamental solution (5) and integrate the obtained expression on (a,b) (see [4,5]) to get

$$\int\limits_{a}^{b} D^m y(x)Y(x-\eta)\mathrm{d}x = \int\limits_{a}^{b} G(x,y)Y(x-\eta)\mathrm{d}x, \quad (7)$$

where,

$$G(x,y) = F(x) + \int\limits_{a}^{x} M_1(x,t)y(t)\mathrm{d}t + \int\limits_{a}^{b} M_2(x,t)y(t)\mathrm{d}t \qquad (7.1)$$

integrating by parts on the left hand side of expression (7) and taking into account that (5) is a fundamental solution of (3.1), give the first basic relation in the form

$$-\sum\limits_{s=0}^{m-1}(-1)^s\left[D^{m-1-s}y(x)\right]Y_x^{(s)}(x-\eta)\Big|_{x=a}^{b} + \int\limits_{a}^{b} G(x,y)Y(x-\eta)\mathrm{d}x = (-1)^n \begin{cases} y(\eta), & \eta \in (a,b), \\ \dfrac{1}{2}y(\eta), & \eta = a, \eta = b. \end{cases} \qquad (8)$$

Hence, the first expressions for the necessary conditions are obtained in the form

$$\begin{cases} \dfrac{(-1)^m}{2}y(a) = -\sum\limits_{s=0}^{m-1}(-1)^s\left[D^{m-1-s}y(x)\right]Y_x^{(s)}(x-a)\Big|_{x=a}^{b} + \int\limits_{a}^{b} G(x,y)Y(x-a)\mathrm{d}x, \\[4mm] \dfrac{(-1)^m}{2}y(a) = -\sum\limits_{s=0}^{m-1}(-1)^s\left[D^{m-1-s}y(x)\right]Y_x^{(s)}(x-b)\Big|_{x=a}^{b} + \int\limits_{a}^{b} G(x,y)Y(x-b)\mathrm{d}x, \end{cases} \qquad (9)$$

It is easy to see that the second expression in (9) turns into an identity. Indeed, as it is seen from (5)-(6), the integral at the right side of the second condition contains the value of the function $\theta(x-\eta)$, which is zero for $\eta = b$. For $x = a$ the the summation in the second expression contains the Heaviside function which is zero for $s = \overline{0,m-1}$.

Finally, the first summand contains positive degrees of $(x-\eta)$ for $s = \overline{0,m-2}$ these terms become zero at $x = \eta = b$. Here, for $s = m-1$, the the expression of fundamental solution for $s = m-1$ yields the Heaviside function. For $x = b$, $\eta = b$ this becomes $\dfrac{1}{2}$, therefore,

the second one of necessary conditions (9) turns into identity.

Now, we construct the second basic expression to get the second group of necessary conditions. For that, we multiply both sides of (3) by the derivative of (5) and integrate on (a,b) [6,7]:

$$\int\limits_{a}^{b} D^m y(x)Y_x'(x-\eta)\mathrm{d}x = \int\limits_{a}^{b} G(x,y)Y_x'(x-\eta)\mathrm{d}x.$$

Integrating by parts on the left side of the obtained expression and taking into account (5) and (6), we get the second basic relation as follows:

$$-\sum\limits_{s=0}^{m-1}(-1)^s\left[D^{m-1-s}y(x)\right]Y_x^{(1+s)}(x-\eta)\Big|_{x=a}^{b} + \int\limits_{a}^{b} G(x,y)Y_x'(x-\eta)\mathrm{d}x = (-1)^{m-1} \begin{cases} y'(\eta), & \eta \in (a,b), \\ \dfrac{1}{2}y'(\eta), & \eta = a, \eta = b, \end{cases} \qquad (10)$$

and so the second group of the necessary conditions are obtained as

$$
\begin{cases}
\dfrac{(-1)^{m-1}}{2} y'(a) = -\displaystyle\sum_{s=0}^{m-2}(-1)^s\left[D^{m-1-s}y(x)\right]Y_x^{(1+s)}(x-a)\Big|_{x=a}^{b} + \int_a^b G(x,y)Y_x'(x-a)\mathrm{d}x, \\[4mm]
\dfrac{(-1)^{m-1}}{2} y'(b) = -\displaystyle\sum_{s=0}^{m-2}(-1)^s\left[D^{m-1-s}y(x)\right]Y_x^{(1+s)}(x-b)\Big|_{x=a}^{b} + \int_a^b G(x,y)Y_x'(x-b)\mathrm{d}x.
\end{cases}
\tag{11}
$$

Similar to the second expression of (9), we can show that the second expression of (11) turns into identity. If we continue this process, in order to get the *m*-th basic relation, we multiply (3) the $(m-1)$-th order derivative of (5) and integrate on (a,b) to get:

$$
\int_a^b D^m y(x)Y_x^{(m-1)}(x-\eta)\mathrm{d}x = \int_a^b G(x,y)Y_x^{(m-1)}(x-\eta)\mathrm{d}x.
$$

Here, once integrating by parts on the left side of the obtained expression gives

$$
D^{m-1}y(x)Y_x^{(m-1)}(x-\eta)\Big|_{x=a}^{b} - \int_a^b D^m y(x)Y_x^m(x-\eta)\mathrm{d}x
$$
$$
= \int_a^b G(x,y)Y_x^{(m-1)}(x-\eta)\mathrm{d}x.
$$

Thus, if we take into account that (5) is the fundamental solution, the last relation (*m*-th) will be as follows:

$$
-D^{m-1}y(x)Y_x^{(m-1)}(x-\eta)\Big|_{x=a}^{b} + \int_a^b G(x,y)Y_x^{(m-1)}(x-\eta)\mathrm{d}x = -\begin{cases} y^{m-1}(\eta), & \eta\in(a,b), \\[2mm] \dfrac{1}{2}y^{(m-1)}(\eta), & \eta=a,\eta=b. \end{cases}
\tag{12}
$$

Therefore, the last group of necessary conditions will be in the form:

$$
-\frac{1}{2}y^{(m-1)}(a) = -D^{m-1}y(x)\cdot Y_x^{(m-1)}(x-a)\Big|_{x=a}^{b} + \int_a^b G(x,y)Y_x^{(m-1)}(x-a)\mathrm{d}x,
$$
$$
-\frac{1}{2}y^{(m-1)}(b) = -D^{m-1}y(x)\cdot Y_x^{(m-1)}(x-b)\Big|_{x=a}^{b} + \int_a^b G(x,y)Y_x^{(m-1)}(x-b)\mathrm{d}x,
\tag{13}
$$

here, as above, the second necessary condition turns into identity.

Now, we join to the given *m* linearly independent boundary condition (7), the necessary conditions in (9), (11) and etc. (13) that are not identities, and write the system of 2*m* linear algebraic equations obtained with respect to the boundary values of the unknown function in the following way.

$$
\begin{cases}
\gamma_{11}y(a)+\gamma_{12}y'(a)+\cdots+\gamma_{1m}y^{(m-1)}(a)+\eta_{11}y(b)+\eta_{12}y'(b)+\cdots+\eta_{1m}y^{(m-1)}(b)=d_1-\lambda_1\displaystyle\int_a^b H_1(t)y(t)\mathrm{d}t, \\[2mm]
\qquad\qquad\vdots \\[2mm]
\gamma_{m1}y(a)+\gamma_{m2}y'(a)+\cdots+\gamma_{mm}y^{(m-1)}(a)+\eta_{m1}y(b)+\eta_{m2}y'(b)+\cdots+\eta_{mm}y^{(m-1)}(b)=d_m-\lambda_m\displaystyle\int_a^b H_m(t)y(t)\mathrm{d}t, \\[2mm]
(-1)^m y(a)+y'(a)+\cdots+y^{(m-1)}(a)-(-1)^m y(b)-(-1)^{m-1}Y^{(m-2)}(b-a)y'(b)-\cdots-(-1)Y(b-a)y^{(m-1)}(b) \\[2mm]
\qquad\qquad\qquad\qquad = \displaystyle\int_a^b G(x,y)Y(x-a)\mathrm{d}x, \\[2mm]
y(a)+(-1)^{m-1}y'(a)+y''(a)+\cdots+y^{(m-1)}(a)-y(b)-(-1)^{m-1}y'(b)-(-1)^{m-2}Y^{(m-2)}(b-a)y''(b)-\cdots \\[2mm]
\qquad\qquad -(-1)y'(b-a)y^{(m-1)}(b)=\displaystyle\int_a^b G(x,y)Y'(x-a)\mathrm{d}x, \\[2mm]
\qquad\qquad\vdots \\[2mm]
(-1)y^{(m-1)}(a)-y(b)-\cdots-y^{(m-2)}(b)-(-1)Y^{(m-1)}(b-a)y^{(m-1)}(b)=\displaystyle\int_a^b G(x,y)Y'^{(m-1)}(x-a)\mathrm{d}x,
\end{cases}
\tag{14}
$$

For solving the system (14) by the Cramer's rule, it is necessary that its basic determinant differ from zero.

Accept that the following condition is satisfied

$$\Delta = \begin{vmatrix} \gamma_{11} & \gamma_{12} & \cdots & \gamma_{1m} & \eta_{11} & \eta_{12} & \cdots & \eta_{1m} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \gamma_{m1} & \gamma_{m2} & \cdots & \gamma_{mm} & \eta_{m1} & \eta_{m2} & \cdots & \eta_{mm} \\ (-1)^m & 0 & \cdots & 0 & -(-1)^m & -(-1)^{m-1} Y^{(m-2)}(b-a) & \cdots & -(-1)(b-a) \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & (-1) & 0 & 0 & \cdots & -(-1)Y^{m-1}(b-a) \end{vmatrix} \neq 0, \tag{15}$$

Then, from system (14), we get

$$\begin{cases} y^{(k-1)}(a) = \dfrac{1}{\Delta}\sum_{s=1}^{m}\left[d_s - \lambda_s \int_a^b H_s(t)y(t)dt\right]\Delta^{(s,k)} + \dfrac{1}{\Delta}\sum_{s=1}^{m}\left[\int_a^b G(x,y)Y^{(s-1)}(x-a)dx\right]\Delta^{(m+s,k)}, \\ y^{(k-1)}(b) = \dfrac{1}{\Delta}\sum_{s=1}^{m}\left[d_s - \lambda_s \int_a^b H_s(t)y(t)dt\right]\Delta^{(s,m+k)} + \dfrac{1}{\Delta}\sum_{s=1}^{m}\left[\int_a^b G(x,y)Y^{(s-1)}(x-a)dx\right]\Delta^{(m+s,m+k)}, \end{cases} \tag{16}$$

where $\Delta^{(p,q)}$ denotes the cofactor of the elements at the intersection of *p*-th row and *q*-th column of the determi-

nant Δ. Calculate the following expression:

$$\int_a^b G(x,y)Y^s(x-a)dx = \int_a^b\left[F(x)+\int_a^x M_1(x,t)y(t)dt+\int_a^b M_2(x,t)y(t)dt\right]Y^{(s)}(x-a)dx$$

$$= \int_a^b F(x)Y^s(x-a)dx + \int_a^b Y^{(s)}(x-a)dx\int_a^x M_1(x,t)y(t)dt + \int_a^b Y^{(s)}(x-a)dx\int_a^b M_2(x,t)y(t)dt$$

$$= \int_a^b F(x)Y^s(x-a)dx + \int_a^b y(t)dt\int_a^b Y^{(s)}(x-a)M_1(x,t)dx + \int_a^b y(t)dt\int_a^b Y^{(s)}(x-a)M_2(x,t)dx.$$

Then, we get:

$$\int_a^b G(x,y)Y^{(s)}(x-a)dx = \tilde{F}_s + \int_a^b \tilde{M}_s(t)y(t)dt, \tag{17}$$

and so,

$$\begin{cases} \tilde{F}_s = \int_a^b F(x)Y^{(s)}(x-a)dx, \\ \tilde{M}_s(t) = \int_a^b Y^{(s)}(x-a)M_2(x,t)dx + \int_a^b Y^{(s)}(x-a)M_1(x,t)dx, \end{cases} \tag{18}$$

Finally, coming back to (8), we take into account (16) and (17) and write the second kind Fredholm type inte-

gral equation [8] for which the boundary value problem (1)-(2) is reduced to:

$$y(\eta) = A(\eta) + \int_a^b B(\eta,t)y(t)dt, \tag{19}$$

where

$$A(\eta) = \sum_{s=0}^{m-1}(-1)^{s+m+1}Y^{(s)}(b-\eta)\sum_{k=1}^{m}\dfrac{d_k}{\Delta}\Delta^{(k,2m-s)} + \sum_{s=0}^{m-1}(-1)^{s+m+1}Y^{(s)}(b-\eta)\sum_{k=1}^{m}\dfrac{\Delta^{(m+k,2m-s)}}{\Delta}\tilde{F}_{k-1}$$

$$+ \sum_{s=0}^{m-1}(-1)^{s+m+1}Y^{(s)}(a-\eta)\sum_{k=1}^{m}\dfrac{d_k}{\Delta}\Delta^{(k,m-s)} + \sum_{s=0}^{m-1}(-1)^{s+m}Y^{(s)}(a-\eta)\sum_{k=1}^{m}\dfrac{\Delta^{(m+k,m-s)}}{\Delta}\tilde{F}_{k-1} + \int_a^b F(x)Y(x-\eta)dx, \tag{20}$$

$$B(\eta,t) = \sum_{s=0}^{m-1}(-1)^{s+m} Y^{(s)}(b-\eta)\sum_{k=1}^{m}\frac{d_k}{\Delta}\Delta^{(k,2m-s)}H_k(t) + \sum_{s=0}^{m-1}(-1)^{s+m+1}Y^{(s)}(b-\eta)\sum_{k=1}^{m}\frac{\Delta^{(m+k,2m-s)}}{\Delta}\tilde{M}_{k-1}(t)$$

$$+ \sum_{s=0}^{m-1}(-1)^{s+m+1}Y^{(s)}(a-\eta)\sum_{k=1}^{m}\frac{d_k}{\Delta}\Delta^{(k,m-s)}H_k(t) + \sum_{s=0}^{m-1}(-1)^{s+m}Y^{(s)}(a-\eta)\sum_{k=1}^{m}\frac{\Delta^{(m+k,m-s)}}{\Delta}\tilde{M}_{k-1}(t) \qquad (21)$$

$$+ \int_{a}^{b}Y(x-\eta)M_1(x,t)\mathrm{d}x + \int_{a}^{b}Y(x-\eta)M_2(x,t)\mathrm{d}x.$$

By the hypothesis of theorem on the functions $f(x)$, $K_j(x,t)$, $j = 1, 2$ and $H_i(t)$, $i = \overline{1,m}$ the integral Equation (19) *has unique solution and so* in all conducted operations we can come back and we conclude that the solution of (19) is the unique solution of boundary value problem (1)-(2).

3. References

[1] D. Nazari and S. Shahmorad, "Application of Fractional Differential Transform Method to the Fractional Order Integro-Differential Equations with Nonlocal Boundary Conditions," *Journal of Computational and Applied Mathematics*, Vol. 234, No. 3, 2010, pp. 883-891. doi:10.1016/j.cam.2010.01.053

[2] S. G. Samko, A. A. Kilbas and O. I. Marichev, "Fractional Integrals and Derivatives," Theory and Applications, Cordon and Breach, Yverdon, 1993.

[3] C. J. Tranter, "Integral Transforms in Mathematical Physics," Methuen, London and New York, 1949.

[4] V. S. Vladimirov, "Equation of Mathematical Physics," Mir Publication, Moscow, 1984.

[5] G. E. Shilov, "Mathematical Analysis. The Second Special Course," Nauka, Moscow, 1965.

[6] S. M. Hosseini and N. A. Aliev, "Sufficient Conditions for the Reduction of a BVP for PDE with Non-Local and Global Boundary Conditions to Fredholm Integral Equations (on a Rectangular Domain)," *Applied Mathematics and Computation*, Vol. 147, No. 3, 2004, pp. 669-685.

[7] F. Bahrami, N. Aliev and S. M. Hosseini, "A Method for the Reduction of Four Imensional Mixed Problems with General Boundary Conditions to a System of Second Kind Fredholm Integral Equations," *Italian Journal of Pure and Applied Mathematics*, No. 17, 2005, pp. 91-104.

[8] N. Aliev and M. Jahanshehi, "Solution of Poissoins Equation with Global, Local and Non-Local Boundary Conditions," *International Journal of Mathematical Education in Science and Technology*, Vol. 33, No. 2, 2002, pp. 241-247. doi:10.1080/00207390110097551