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Further Results on Pair Sum Labeling of Trees

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Abstract

Let G be a (p,q) graph. An injective map $f:V(G)\to\{\pm 1,\pm 2,\cdots,\pm p\}$ is called a pair sum labeling if the induced edge function, $f_e:E(G)\to Z-\{0\}$ defined by $f_e(uv)=f(u)+f(v)$ is one-one and $f_e(E(G))$ is either of the form $\{\pm k_1,\pm k_2,\cdots,\pm k_{q/2}\}$ or $\{\pm k_1,\pm k_2,\cdots,k_{(q-1)/2}\}\cup\{k_{(q+1)/2}\}$ according as q is even or odd. A graph with a pair sum labeling is called a pair sum graph. In this paper we investigate the pair sum labeling behavior of some trees which are derived from stars and bistars. Finally, we show that all trees of order nine are pair sum graphs.

Keywords: Path, Star, Bistar, Tree

1. Introduction

The graphs in this paper are finite, undirected and simple. $V(G)$ and $E(G)$ will denote the vertex set and edge set of a graph G. The cardinality of the vertex set of a graph G is called the order of G and is denoted by p. The cardinality of its edge set is called the size of G and is denoted by q. The concept of pair sum labeling has been introduced in[1].The Pair sum labeling behavior of some standard graphs like complete graph, cycle, path, bistar, and some more standard graphs are investigated in [1-3]. Terms not defined here are used in the sense of Harary [4]. All the trees of order ≤8 are pair sum have been proved in [5]. Here we proved that all trees of order nine are pair sum. Let x be any real number. Then $\lfloor x\rfloor$ stands for the largest integer less than or equal to x and $\lceil x\rceil$ stands for the smallest inter greater than or equal to x.

2. Pair Sum Labeling

Definition 2.1: Let G be a (p,q) graph. An injective map $f:V(G)\to\{\pm 1,\pm 2,\cdots,\pm p\}$ is called a pair sum labeling if the induced edge function, $f_e:E(G)\to Z-\{0\}$ defined by is one-one and $f_e(E(G))$ is either of the form $\{\pm k_1,\pm k_2,\cdots,\pm k_{q/2}\}$ or $\{\pm k_1,\pm k_2,...,k_{(q-1)/2}\}\cup\{k_{(q+1)/2}\}$ according as q is even or odd. A graph with a pair sum labeling defined on it is called a pair sum graph.

Notation 2.1: We denote the vertex and edge sets of star $K_{1,n}$ as follows:

$$V(K_{1,n})=\{u,u_i:1\le i\le n\}$$

and

$$E(K_{1,n})=\{uu_i 1\le i\le n\}.$$

Notation 2.2: We denote the vertex and edge sets of bistar $B_{m,n}$ as follows:

$$V(B_{m,n})=\{u,v,u_i,v_i:1\le i\le m,1\le i\le n\}$$

and

$$E(B_{m,n})=\{uv,uu_i,vv_j:1\le i\le m,1\le j\le n\}.$$

Theorem 2.3 [5]: All graphs of order ≤8 are pair sum.

Now we derive some pair sum trees which are used for the final section.

3. Pair Sum Labeling of Star Related Graphs

Here we prove that some trees which are obtained from stars are pair sum.

Theorem 3.1: The trees $G_i,(1\le i\le 5)$ with vertex set and edge set given below are pair sum.

1) $V(G_1)=V(K_{1,n})\cup\{v_i:1\le i\le 6\}$

and

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$$E(G_1) = E(K_{1,n}) \cup \{uv_1, v_1v_2, v_2v_3, v_3v_4, v_4v_5, v_5v_6\}.$$

Then G_1 is a pair sum graph.

2) $V(G_2) = V(K_{1,n}) \cup \{v_i : 1 \le i \le 7\}$

and

$$E(G_2) = E(K_{1,n}) \cup \{uv_6, v_6v_7, v_1v_2, v_2v_3, v_3v_4, v_4v_5, v_5u\}.$$

Then G_2 is a pair sum graph.

3) $V(G_3) = V(K_{1,n}) \cup \{v_i : 1 \le i \le 7\}$

and

$$E(G_3) = E(K_{1,n}) \cup \{uv_5, v_5v_6, v_6v_7, v_1v_2, v_2v_3, v_3v_4, v_4u\}.$$

Then G_3 is a pair sum graph.

4) $V(G_4) = V(K_{1,n}) \cup \{v_iw_i : 1 \le i \le 4\}$

and

$$E(G_4) = E(K_{1,n}) \cup \{uw_1, w_1w_2, uw_3, w_3w_4, v_1v_2, v_2v_3, v_3v_4, v_4u\}.$$

Then G_4 is a pair sum graph.

5) $V(G_5) = V(K_{1,n}) \cup \{v_i, w_i : 1 \le i \le 3\}$

and

$$E(G_5) = E(K_{1,n}) \cup \{uv_1, uv_2, uv_3, v_1w_1, v_2w_2, v_3w_3\}.$$

Then G_5 is a pair sum graph.

Proof 1): Define

$$f : V(G_1) \to \{\pm 1, \pm 2, \cdots, \pm(n+7)\}$$

by

$$f(u) = -1,\ f(v_1) = -7,\ f(v_2) = -5,\ f(v_3) = 1,$$
$$f(v_4) = 3,\ f(v_5) = 5,\ f(v_6) = 7$$
$$f(u_i) = -2i - 4, 1 \le i \le \lceil n/2 \rceil$$

and

$$f\left(u_{\lfloor (n+1)/2 \rfloor + i}\right) = 2i + 6, 1 \le i \le \lceil n/2 \rceil.$$

Then G_1 is a pair sum tree. □

Proof 2): Define a map

$$f : V(G_2) \to \{\pm 1, \pm 2, \cdots, \pm(n+8)\}$$

by

$$f(v_1) = -3,\ f(v_2) = -6,\ f(v_3) = -1,$$
$$f(v_4) = -4,\ f(v_5) = 1,\ f(v_6) = 3,\ f(v_7) = 4,$$
$$f(u) = 2,\ f(u_1) = 7,$$
$$f(u_{1+i}) = 4 + 2i, 1 \le i \le \lfloor n/2 \rfloor$$

and

$$f\left(u_{\lceil (n/2)+i \rceil + i}\right) = -2i - 8, 1 \le i \le \lceil (n-2)/2 \rceil.$$

Then G_2 is a pair sum graph. □

Proof 3): Define a map

$$f : V(G_3) \to \{\pm 1, \pm 2, \cdots, \pm(n+8)\}$$

by

$$f(v_1) = -3,\ f(v_2) = -6,\ f(v_3) = -1,$$
$$f(v_4) = -4,\ f(v_5) = 2,\ f(v_6) = 3,\ f(v_7) = 4,$$
$$f(u) = 1,\ f(u_1) = 7 + i,$$
$$1 \le i \le \lceil n/2 \rceil,\ f\left(u_{\lceil (n/2) \rceil + i}\right) = -i - 10, 1 \le i \le \lfloor n/2 \rfloor.$$

Then G_3 is a pair sum graph. □

Proof 4): Define a map

$$f : V(G_4) \to \{\pm 1, \pm 2, \cdots, \pm(n+9)\}$$

by

$$f(u) = -1,\ f(v_1) = 3,\ f(v_2) = 2,\ f(v_3) = 1,\ f(v_4) = -4,$$
$$f(w_1) = -5,\ f(w_2) = -6,\ f(w_3) = 7,\ f(w_4) = 4.$$

For the other vertices we define,

$$f(u_i) = -5 - 2i, 1 \le i \le \lceil n/2 \rceil$$
$$f\left(u_{\lfloor (n+1)/2 \rfloor + i}\right) = 7 + 2i, 1 \le i \le \lfloor n/2 \rfloor.$$

Obviously f is a pair sum labeling. □

Proof 5): Define

$$f : V(G_5) \to \{\pm 1, \pm 2, \cdots, \pm(n+7)\}$$

by

$$f(u) = -1,\ f(v_1) = 2,\ f(v_2) = 3,\ f(v_3) = 4,$$
$$f(w_1) = -3,\ f(w_2) = -5,\ f(w_3) = -7,$$
$$f(u_i) = 2i + 4, 1 \le i \le \lceil n/2 \rceil$$
$$f\left(u_{\lfloor (n+1) \rfloor + i}\right) = -2 - 2i, 1 \le i \le \lfloor n/2 \rfloor.$$

Obviously f is a pair sum labeling. □

Illustration 1: A pair sum labeling of the tree G_1 with $n = 10$ is

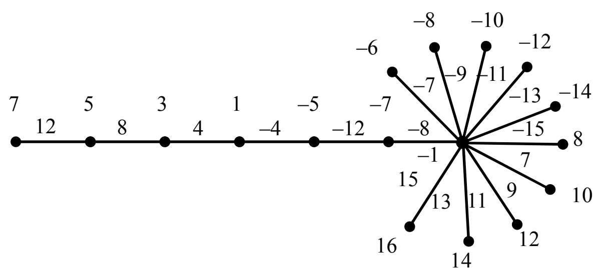


Illustration 2: A pair sum labeling of the tree G_3 with $n=9$ is

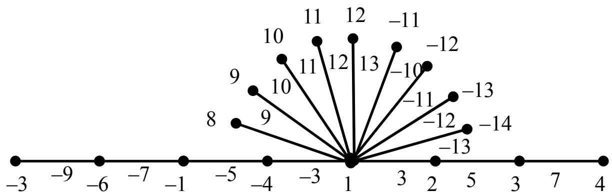
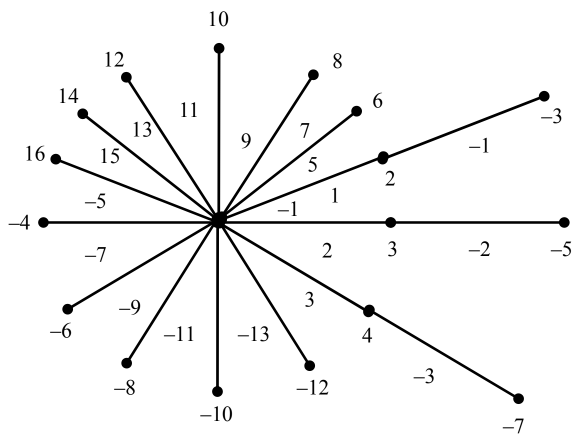


Illustration 3: A pair sum labeling of the tree G_5 with $n=11$ is



4. Bistar Related Graphs

In this section we show that some trees which are obtained from bistar are pair sum.

Theorem 4.1: Let *G* be the tree with

$$V(G)=V(B_{m,n})\cup\{w_i:1\le i\le 6\}$$

and

$$E(G)=E(B_{m,n})\cup\{vw_1,w_1w_2,w_2w_3,vw_4,w_4w_5,w_5w_6\}.$$

Then *G* is a pair sum graph.

Proof: Define

$$f:V(G)\to\{\pm 1,\pm 2,\cdots,\pm(m+n+8)\}$$

and

$$f(w_1)=4,\ f(w_2)=5,\ f(w_3)=6,\ f(w_4)=-2,$$
$$f(w_5)=-7,\ f(w_6)=-4,\ f(u)=2,\ f(v)=-1.$$

Case 1): $m=n$

$$f(u_1)=-3,\ f(u_{1+i})=9+i,1\le i\le n-1,$$
$$f(v_i)=-10-i,1\le i\le n.$$

Case 2): $m>n$

Assign the label to $u_i,v_i\ (1\le i\le n)$ as in case 1). Define

$$f(u_{n+i})=8+n+i,1\le i\le\lceil(m-n)/2\rceil$$

and

$$f\left(u_{\lceil(m+n)/2\rceil+i}\right)=-13-n-i,1\le i\le\lfloor(m-n)/2\rfloor$$

Case 3): $m<n$

Assign the label to $u_i,v_i\ (1\le i\le m)$ as in case 1). Define

$$f(v_{m+i})=11+m+i,1\le i\le\lceil(n-m)/2\rceil$$

$$f\left(v_{\lceil(m+n)/2\rceil+i}\right)=-10-m-i,1\le i\le\lfloor(n-m)/2\rfloor.$$

Then *G* is a pair sum graph. □

Theorem 4.2: If *G* is the tree with

$$V(G)=V(B_{m,n})\cup\{w_i:1\le i\le 3\}$$

and

$$E(G)=E(B_{m,n})\cup\{uw_1,w_1w_2,w_2w_3,w_3v\}/\{uv\}.$$

Then *G* is a pair sum tree.

Proof: Define a function

$$f:V(G)\to\{\pm 1,\pm 2,\cdots,\pm(m+n+5)\}$$

by

$$f(u)=-1,\ f(v)=3,\ f(w_1)=-4,\ f(w_2)=1,$$
$$f(w_3)=2.$$

Case 1): $m=n$.

$$f(u_i)=-5-i,1\le i\le n$$

and

$$f(v_i)=3+i,1\le i\le n.$$

Case 2): $m>n$.

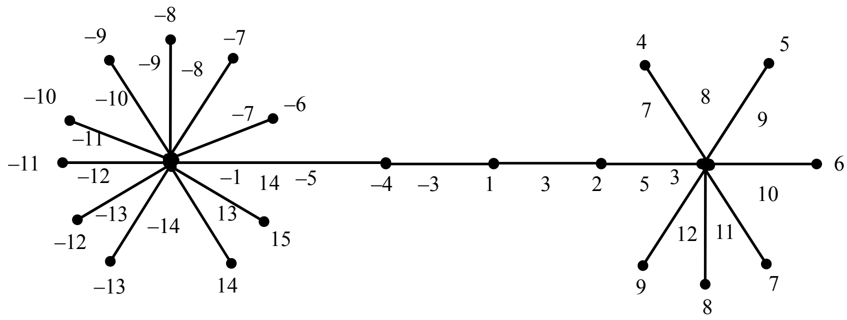
Assign the label to $u_i,v_i\ (1\le i\le n)$ as in case 1). Define

$$f(u_{n+i})=-5-n-i,1\le i\le\lceil(m-n)/2\rceil$$

$$f\left(u_{\lceil(m+n)/2\rceil+i}\right)=7+n+i,1\le i\le\lfloor(m-n)/2\rfloor.$$

Then *G* is a pair sum graph □

Illustration 4: A pair sum labeling of the tree in theorem 4.2 with $m=10$, $n=6$ is given below:



Theorem 4.3: Let G be the tree with

$$V(G)=V(B_{m,n})\cup\{w_i:1\le i\le 4\}$$

and

$$E(G)=E(B_{m,n})\cup\{uw_1,w_1w_2,w_2v,vw_3,w_3w_4\}/\{uv\}.$$

Then G is a pair sum graph.

Proof: Define

$$f:V(G)\to\{\pm1,\pm2,\cdots,\pm(m+n+6)\}$$

by

$$f(u)=-1,\ f(v)=2,\ f(w_1)=-4,\ f(w_2)=1,$$
$$f(w_3)=3,\ f(w_4)=4.$$

Case 1): $m=n$.

$$f(u_1)=-6,\ f(u_{i+1})=-6-i,1\le i\le n-1$$

and

$$f(v_i)=5+i,1\le i\le n.$$

Case 2): $m>n$.

Assign the label to $u_i,v_i\ (1\le i\le n)$ as in case 1). Define

$$f(u_{n+i})=-5-n-i,1\le i\le\lceil(m-n)/2\rceil$$

and

$$f\left(u_{\lceil(m+n)/2\rceil+i}\right)=8+n+i,1\le i\le\lfloor(m-n)/2\rfloor.$$

Case 3): $m<n$.

Assign the label to $u_i,v_i\ (1\le i\le n)$ as in case 1). Define

$$f(v_{m+i})=-8-m-i,1\le i\le\lceil(n-m)/2\rceil$$

and

$$f(v_{\lceil(m+n)/2\rceil+i})=5+m+i,1\le i\le\lfloor(n-m)/2\rfloor.$$

Then G is a pair sum graph. □

Theorem 4.4: The tree G with vertex set

$$V(G)=(B_{m,n})\cup\{w_i:1\le i\le 5\}$$

and edge set

$$E(G)=E(B_{m,n})\cup\{uw_1,w_1v,vw_2,w_2w_3,vw_4,w_4w_5\}/\{uv\}.$$

Then G is a pair sum tree.

Proof: Define

$$f:V(G)\to\{\pm1,\pm2,\cdots,\pm(m+n+7)\}$$

by

$$f(u)=-1,\ f(v)=1,\ f(w_1)=-4,\ f(w_2)=2,$$
$$f(w_3)=3,\ f(w_4)=-3,\ f(w_5)=5.$$

Case 1): $m=n$

$$f(u_i)=-5-i,1\le i\le n$$

and

$$f(v_i)=5+i,1\le i\le n$$

Case 2): $m>n$

Label the vertices u_i and v_i as in case 1) for $1\le i\le n.$ Define

$$f(u_{n+i})=-5-n-i,1\le i\le\lceil(m-n)/2\rceil$$

and

$$f\left(u_{\lceil(m+n)/2\rceil+i}\right)=6+n+i,1\le i\le\lceil(m-n)/2\rceil.$$

Case 3): $m<n$

Assign the label to $u_i,v_i\ (1\le i\le n)$ as in case 1). Define

$$f(v_{m+i})=5+m+i,1\le i\le\lceil(n-m)/2\rceil$$

and

$$f\left(v_{\lceil(m+n)/2\rceil+i}\right)=-7-m-i,1\le i\le\lceil(n-m)/2\rceil.$$

Then G is a pair sum graph. □

Theorem 4.5: Let G be the tree with

$$V(G)=V(B_{m,n})\cup\{w_i:1\le i\le 5\}$$

and

$$E(G)=E(B_{m,n})\cup\{w_1w_2,w_2u,uw_3,w_3v,vw_4,w_4w_5\}/\{uv\}.$$

Then G is a pair sum tree.

Proof: Define a function

$$f:V(G)\to\{\pm1,\pm2,\cdots,\pm(m+n+7)\}$$

by

$$f(u)=-4,\ f(v)=2,\ f(w_1)=-6,\ f(w_2)=-1,$$
$$f(w_3)=1,\ f(w_4)=3,\ f(w_5)=4.$$

Case 1): $m=n$.

$$f(u_i)=-6-i, 1\le i\le n$$

and

$$f(v_i)=8+i, 1\le i\le n.$$

Case 2): $m<n$.

Assign the label to $u_i, v_i\ (1\le i\le m)$ as in case 1). Define

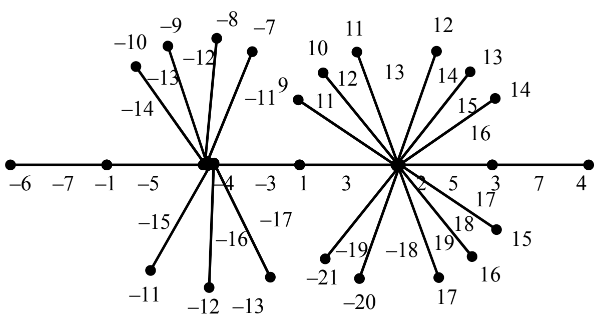
$$f(v_{m+i})=8+m+i, 1\le i\le\lceil(n-m)/2\rceil$$

and

$$f\left(v_{\lceil(n-m)/2\rceil+i}\right)=-12-m-i, 1\le i\le\lfloor(n-m)/2\rfloor.$$

Then G is a pair sum graph.

Illustration 5: A pair sum labeling of the tree in theorem 4.5 with $m=6$, $n=11$ is given below:



Theorem 4.6: Let G be the tree with

$$V(G)=V(B_{m,n})\cup\{w_i:1\le i\le 4\}$$

and

$$E(G)=E(B_{m,n})\cup\{uw_1, w_1v, vw_2, w_2w_3, w_3w_4\}/\{uv\}.$$

Then G is a pair sum tree.

Proof: Define a function

$$f:V(G)\to\{\pm1,\pm2,\cdots,\pm(m+n+6)\}$$

by

$$f(u)=-1,\ f(v)=1,\ f(w_1)=-4,\ f(w_2)=2,$$
$$f(w_3)=3,\ f(w_4)=4.$$

Case 1): $m=n$.

$$f(u_i)=-6-i, 1\le i\le n$$

and

$$f(v_i)=6+i, 1\le i\le n.$$

Case 2): $m<n$.

Assign the label to $u_i, v_i\ (1\le i\le m)$ as in case 1). Define

$$f(v_{m+i})=-7-m-i, 1\le i\le\lceil(n-m)/2\rceil$$

and

$$f\left(v_{\lceil(m+n)/2\rceil+i}\right)=6+m+i, 1\le i\le\lfloor(m-n)/2\rfloor.$$

Case 3): $m>n$.

Assign the label to $u_i, v_i\ (1\le i\le n)$ as in case 1). Define

$$f(u_{n+i})=-5-n-i, 1\le i\le\lceil(m-n)/2\rceil$$

and

$$f\left(u_{\lceil(m+n)/2\rceil+i}\right)=8+n+i, 1\le i\le\lfloor(m-n)/2\rfloor.$$

Then G is a pair sum graph. □

Theorem 4.7: Let G be the tree with

$$V(G)=V(B_{m,n})\cup\{w_i:1\le i\le 4\}$$

and

$$E(G)=E(B_{m,n})\cup\{vw_1, w_1w_2, w_2w_3, w_3w_4\}.$$

Then G is a pair sum tree.

Proof: Define a function

$$f:V(G)\to\{\pm1,\pm2,\cdots,\pm(m+n+6)\}$$

by

$$f(u)=1,\ f(v)=2,\ f(w_1)=3,\ f(w_2)=-1,$$
$$f(w_3)=-2,\ f(w_4)=-3.$$

Case 1): $m=n$.

$$f(u_i)=-6-i, 1\le i\le m$$

and

$$f(v_{1+i})=3+i, 1\le i\le m-1.$$

Case 2): $m<n$.

Assign the label to $u_i, v_i\ (1\le i\le m)$ as in case 1). Define

$$f(v_{m+i})=2+m+i, 1\le i\le\lceil(n-m)/2\rceil$$

and

$$f\left(u_{\lceil(m+n)/2\rceil+i}\right)=-7-m-i, 1\le i\le\lfloor(n-m)/2\rfloor.$$

Case 3): $m>n$.

Assign the label to $u_i, v_i\left(1 \le i \le n\right)$ as in case 1). Define

$$f\left(u_{n+i}\right) = -6 - n - i, 1 \le i \le \left\lceil \left(m-n\right)/2 \right\rceil$$

and

$$f\left(u_{\lceil (m+n)/2 \rceil + i}\right) = 5 + n + i, 1 \le i \le \left\lfloor \left(m-n\right)/2 \right\rfloor.$$

Then G is a pair sum graph. □

Theorem 4.8: The tree G with

$$V\left(G\right) = V\left(B_{m,n}\right) \cup \left\{w_i : 1 \le i \le 5\right\}$$

and

$$E\left(G\right) = E\left(B_{m,n}\right) \cup \left\{vw_1, w_1w_2, w_2w_3, vw_4, w_4w_5\right\}.$$

Then G is a pair sum graph.

Proof: Define a map

$$f : V\left(G\right) \to \left\{\pm 1, \pm 2, \cdots, \pm\left(m+n+7\right)\right\}$$

by

$$f\left(u\right) = 1,\ f\left(v\right) = 2,\ f\left(w_1\right) = 3,\ f\left(w_2\right) = -4,$$
$$f\left(w_3\right) = -1,\ f\left(w_4\right) = -5,\ f\left(w_5\right) = 4.$$

Case 1): $m = n$.

$$f\left(u_i\right) = -7 - i, 1 \le i \le m$$

and

$$f\left(v_i\right) = 4 + i, 1 \le i \le m.$$

Case 2): $m < n$.

Assign the label to $u_i, v_i\left(1 \le i \le m\right)$ as in case 1). Define

$$f\left(v_{m+i}\right) = 4 + m + i, 1 \le i \le \left\lceil \left(n-m\right)/2 \right\rceil$$

and

$$f\left(u_{\lceil (m+n)/2 \rceil + i}\right) = -8 - m - i, 1 \le i \le \left\lfloor \left(n-m\right)/2 \right\rfloor.$$

Case 3): $m > n$.

Assign the label to $u_i, v_i\left(1 \le i \le n\right)$ as in case 1). Define

$$f\left(u_{n+i}\right) = -7 - n - i, 1 \le i \le \left\lceil \left(m-n\right)/2 \right\rceil$$

and

$$f\left(u_{\lceil (m+n)/2 \rceil + i}\right) = 5 + n + i, 1 \le i \le \left\lfloor \left(m-n\right)/2 \right\rfloor.$$

Then G is a pair sum graph. □

Theorem 4.9: The tree G with

$$V\left(G\right) = V\left(B_{m,n}\right) \cup \left\{w_i : 1 \le i \le 5\right\}$$

and

$$E\left(G\right) = E\left(B_{m,n}\right) \cup \left\{w_1w_2, w_2u, vw_3, w_3w_4, w_4w_5\right\}.$$

Then G is a pair sum graph.

Proof: Define a map

$$f : V\left(G\right) \to \left\{\pm 1, \pm 2, \cdots, \pm\left(m+n+7\right)\right\}$$

by

$$f\left(u\right) = -4,\ f\left(v\right) = 1,\ f\left(w_1\right) = -6,\ f\left(w_2\right) = -1,$$
$$f\left(w_3\right) = 2,\ f\left(w_4\right) = 3,\ f\left(w_5\right) = 4.$$

Case 1): $m = n$.

$$f\left(u_1\right) = -5,\ f\left(u_{i+1}\right) = -6 - i, 1 \le i \le m-1,$$
$$f\left(v_1\right) = 8$$

and

$$f\left(u_{i+1}\right) = 9 + i, 1 \le i \le m-1.$$

Case 2): $m < n$.

Assign the label to $u_i, v_i\left(1 \le i \le m\right)$ as in case 1). Define

$$f\left(v_{m+i}\right) = 8 + m + i, 1 \le i \le \left\lceil \left(n-m\right)/2 \right\rceil$$

and

$$f\left(u_{\lceil (m+n)/2 \rceil + i}\right) = -10 - m - i, 1 \le i \le \left\lfloor \left(n-m\right)/2 \right\rfloor.$$

Case 3): $m > n$.

Assign the label to $u_i, v_i\left(1 \le i \le n\right)$ as in case 1). Define

$$f\left(u_{n+i}\right) = -5 - n - i, 1 \le i \le \left\lceil \left(m-n\right)/2 \right\rceil$$

and

$$f\left(u_{\lceil (m+n)/2 \rceil + i}\right) = 13 + n + i,\ 1 \le i \le \left\lfloor \left(m-n\right)/2 \right\rfloor.$$

Then G is a pair sum graph. □

Theorem 4.10: The tree G with

$$V\left(G\right) = V\left(B_{m,n}\right) \cup \left\{w_i : 1 \le i \le 6\right\}$$

and

$$E\left(G\right) = E\left(B_{m,n}\right) \cup \left\{w_1w_2, w_2u, vw_3, w_3w_4, vw_5, w_5w_6\right\}.$$

Then G is a pair sum graph.

Proof: Define a map

$$f : V\left(G\right) \to \left\{\pm 1, \pm 2, \cdots, \pm\left(m+n+8\right)\right\}$$

by

$$f\left(u\right) = -4,\ f\left(v\right) = 1,\ f\left(w_1\right) = -6,\ f\left(w_2\right) = -1,$$
$$f\left(w_3\right) = 2,\ f\left(w_4\right) = 3,\ f\left(w_5\right) = -3,\ f\left(w_6\right) = 5.$$

Case 1): $m = n$.

$$f\left(u_i\right) = -4 - i, 1 \le i \le m,\ f\left(v_i\right) = 6$$

and

$$f\left(v_{1+i}\right) = 7 + i, 1 \le i \le m - 1.$$

Case 2): $m < n$.

Assign the label to $u_i, v_i \left(1 \le i \le m\right)$ as in case 1). Define

$$f\left(v_{m+i}\right) = 6 + m + i, 1 \le i \le \left\lceil (n-m)/2 \right\rceil$$

and

$$f\left(u_{\lceil (m+n)/2 \rceil + i}\right) = -8 - m - i, 1 \le i \le \left\lfloor (n-m)/2 \right\rfloor.$$

Case 3): $m > n$.

Assign the label to $u_i, v_i \left(1 \le i \le n\right)$ as in case 1). Define

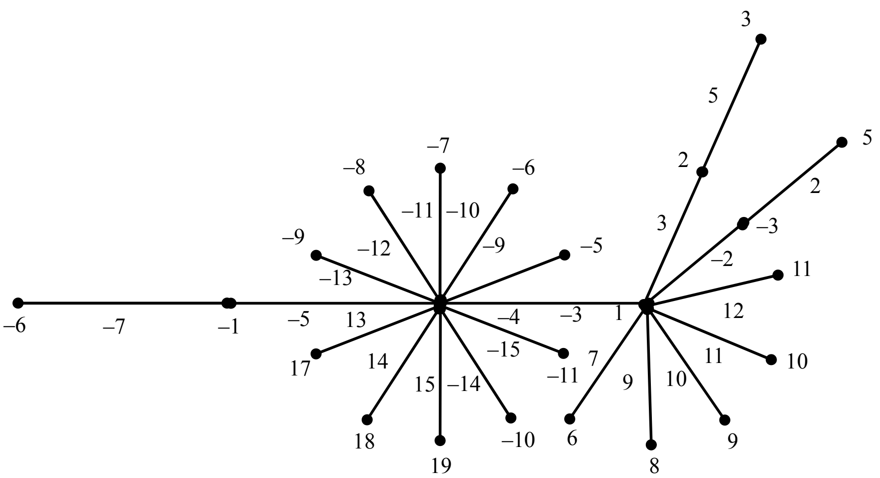
$$f\left(u_{n+i}\right) = 11 + n + i, 1 \le i \le \left\lceil (m-n)/2 \right\rceil$$

and

$$f\left(u_{\lceil (m+n)/2 \rceil + i}\right) = -4 - n - i, 1 \le i \le \left\lfloor (m-n)/2 \right\rfloor.$$

Then G is a pair sum graph. □

Illustration 6: A pair sum labeling of the tree in theorem 4.10 with $m = 10$, $n = 5$ is given below



Theorem 4.11: The tree G with

$$V(G) = V\left(B_{m,n}\right) \cup \left\{w_i : 1 \le i \le 4\right\}$$

$$E(G) = E\left(B_{m,n}\right) \cup \left\{w_1 w_2, w_2 u, u w_3, w_3 w_4, w_4 v\right\} / \left\{uv\right\}.$$

Then G is a pair sum graph.

Proof: Define a map

$$f : V(G) \to \left\{\pm 1, \pm 2, \cdots, \pm (m+n+6)\right\}$$

by

$$f(u) = 1,\ f(v) = -1,\ f(w_1) = 3,\ f(w_2) = 2,$$
$$f(w_3) = -4,\ f(w_4) = -2.$$

Case 1): $m = n$.

$$f\left(u_i\right) = 4 + i, 1 \le i \le m$$

and

$$f\left(v_i\right) = -5 - i, 1 \le i \le m.$$

Case 2): $m < n$.

Assign the label to $u_i, v_i \left(1 \le i \le m\right)$ as in case 1). Define

$$f\left(v_{m+i}\right) = 6 + m + i, 1 \le i \le \left\lceil (n-m)/2 \right\rceil$$

and

$$f\left(u_{\lceil (m+n)/2 \rceil + i}\right) = -5 - m - i, 1 \le i \le \left\lfloor (n-m)/2 \right\rfloor.$$

Case 3): $m > n$.

Assign the label to $u_i, v_i \left(1 \le i \le n\right)$ as in case 1). Define

$$f\left(u_{n+i}\right) = 4 + n + i, 1 \le i \le \left\lceil (m-n)/2 \right\rceil$$

and

$$f\left(u_{\lceil (m+n)/2 \rceil + i}\right) = -6 - n - i, 1 \le i \le \left\lfloor (m-n)/2 \right\rfloor.$$

Then G is a pair sum graph. □

Theorem 4.12: The tree G with

$$V(G) = V\left(B_{m,n}\right) \cup \left\{w_i : 1 \le i \le 3\right\}$$

$$E(G) = E\left(B_{m,n}\right) \cup \left\{u w_1, w_1 w_2, w_2 w_3, w_2 v\right\} / \left\{uv\right\}.$$

Then G is a pair sum graph.

Proof: Define a map

$$f : V(G) \to \left\{\pm 1, \pm 2, \cdots, \pm (m+n+5)\right\}$$

by

$$f(u) = -1,\ f(v) = 1,\ f(w_1) = 2,\ f(w_2) = -3,$$
$$f(w_3) = 5.$$

Case 1): $m = n$.

$$f(u_i) = -3 - i, 1 \le i \le m$$

and

$$f(v_i) = 3 + i, 1 \le i \le m.$$

Case 2): $m < n$.

Assign the label to $u_i, v_i (1 \le i \le m)$ as in case 1). Define

$$f(v_{m+i}) = 3 + m + i, 1 \le i \le \lceil (n-m)/2 \rceil$$

and

$$f\left(u_{\lceil (m+n)/2 \rceil + i}\right) = -5 - m - i, 1 \le i \le \lfloor (n-m)/2 \rfloor.$$

Case 3): $m > n$.

Assign the label to $u_i, v_i (1 \le i \le n)$ as in case 1). Define

$$f(u_{n+i}) = -3 - n - i, 1 \le i \le \lceil (m-n)/2 \rceil$$

and

$$f\left(u_{\lceil (m+n)/2 \rceil + i}\right) = 5 + n + i, 1 \le i \le \lfloor (m-n)/2 \rfloor.$$

Then G is a pair sum graph. □

Theorem 4.13: The tree G with

$$V(G) = V(B_{m,n}) \cup \{w_i : 1 \le i \le 3\}$$

and

$$E(G) = E(B_{m,n}) \cup \{uw_1, w_1w_2, w_2w_3, w_1v\} / \{uv\}.$$

Then G is a pair sum graph.

Proof: Define a map

$$f : V(G) \to \{\pm 1, \pm 2, \cdots, \pm (m+n+5)\}$$

by

$$f(u) = -1,\ f(v) = 3,\ f(w_1) = 2,\ f(w_2) = -3,$$
$$f(w_3) = -2.$$

Case 1): $m = n$.

$$f(u_i) = -5 - i, 1 \le i \le m,$$

$$f(v_i) = 3 + i, 1 \le i \le m.$$

Case 2): $m < n$.

Assign the label to $u_i, v_i (1 \le i \le m)$ as in case 1). Define

$$f(v_{m+i}) = 3 + m + i, 1 \le i \le \lceil (n-m)/2 \rceil$$

and

$$f\left(u_{\lceil (m+n)/2 \rceil + i}\right) = -9 - m - i, 1 \le i \le \lfloor (n-m)/2 \rfloor.$$

Case 3): $m > n$.

Assign the label to $u_i, v_i (1 \le i \le n)$ as in case 1). Define

$$f(u_{n+i}) = -5 - n - i, 1 \le i \le \lceil (m-n)/2 \rceil$$

and

$$f\left(u_{\lceil (m+n)/2 \rceil + i}\right) = 7 + n + i, 1 \le i \le \lfloor (m-n)/2 \rfloor.$$

Then G is a pair sum graph. □

Theorem 4.14: The tree G with

$$V(G) = V(B_{m,n}) \cup \{w_i : 1 \le i \le 3\}$$

and

$$E(G) = E(B_{m,n}) \cup \{uw_2, w_1w_2, w_2w_3, w_2v\} / \{uv\}.$$

Then G is a pair sum graph.

Proof: Define a map

$$f : V(G) \to \{\pm 1, \pm 2, \cdots, \pm (m+n+5)\}$$

by

$$f(u) = -1,\ f(v) = -3,\ f(w_1) = -5,\ f(w_2) = 2,$$
$$f(w_3) = 1.$$

Case 1): $m = n$.

$$f(u_i) = 4 + 2i, 1 \le i \le m$$

and

$$f(v_i) = -2i, 1 \le i \le m.$$

Case 2): $m > n$.

Assign the label to $u_i, v_i (1 \le i \le n)$ as in case 1). Define

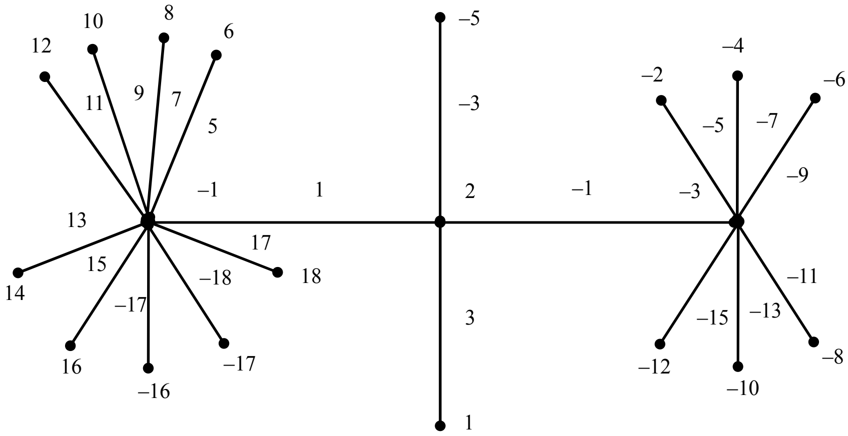
$$f(u_{n+i}) = -3 - 2n - i, 1 \le i \le \lceil (m-n)/2 \rceil$$

and

$$f\left(u_{\lceil (m+n)/2 \rceil + i}\right) = 5 + 2n + i, 1 \le i \le \lfloor (m-n)/2 \rfloor.$$

Then G is a pair sum graph. □

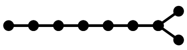
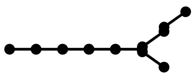
Illustration 7: A pair sum labeling of the tree in theorem 4.14 with $m = 9$, $n = 6$ is given below:

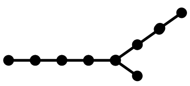
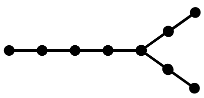


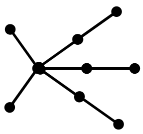
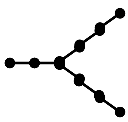
5. Trees of Order 9

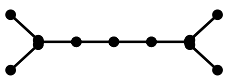
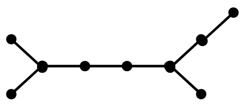
Here we prove that all trees of order ≤9 are pair sum.

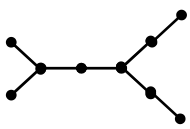
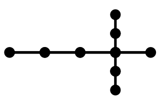
Theorem 5.1: The trees given below are pair sum.

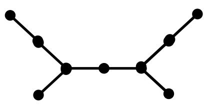
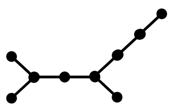
1) 
2) 

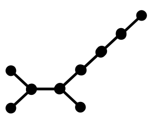
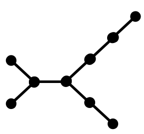
3) 
4) 

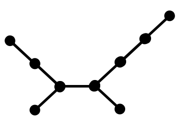
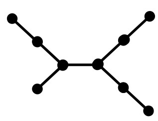
5) 
6) 

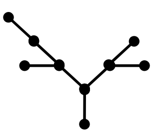
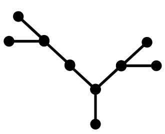
7) 
8) 

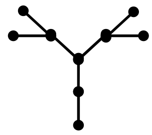
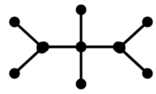
9) 
, 

10) 
11) 

12) 
13) 

14) 
15) 

16) 
17) 

18) 
19) 

Proof: Graphs in case 1) to case 5) are pair sum by theorem 3.1. and case 6) to case 19) graphs are pair sum by theorem 4.1 to 4.14. □

Remark 5.2: The remaining trees of order 9 are pair sum by theorems in [5].

Theorem 5.3: All trees of order 9 are pair sum.

Proof: Follow from theorem 5.1 and Remark 5.2.

Theorem 5.4: All trees of order ≤9 are pair sum.

Proof: Follow from theorems 2.3, 5.3. □

6. Acknowledgements

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7. References

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