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The Maximum Size of an Edge Cut and Graph Homomorphisms

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Abstract

For a graph *G*, let $b(G)=\max\{|D|: D \text{ is an edge cut of } G\}$. For graphs *G* and *H*, a map $\varphi: V(G)\to V(H)$ is a graph homomorphism if for each $e=uv\in E(G)$, $\varphi(u)\varphi(v)\in E(H)$. In 1979, Erdös proved by probabilistic methods that for $p\ge 2$ with

$$f(p)=\begin{cases}\frac{1}{2}+\frac{1}{2p-2} & \text{if } p\equiv 0 \pmod 2,\\ \frac{1}{2}+\frac{1}{2p} & \text{if } p\equiv 1 \pmod 2,\end{cases}$$

if there is a graph homomorphism from *G* onto K_p, then $b(G)\ge f(p)|E(G)|$. In this paper, we obtained the best possible lower bounds of $b(G)$ for graphs *G* with a graph homomorphism onto a Kneser graph or a circulant graph and we characterized the graphs *G* reaching the lower bounds when *G* is an edge maximal graph with a graph homomorphism onto a complete graph, or onto an odd cycle.

Keywords: Maximum Edge Cuts, Graph Homomorphisms

1. Introduction

In this paper, the graphs we consider are finite, simple and connected. Undefined notation and terminology will follow those in [1]. Let *G* be a graph. For disjoint nonempty subsets $X,Y\subseteq V(G)$, let $[X,Y]$ denote the set of edges of *G* with one end in *X* and the other end in *Y*. For a subset $S\subseteq V(G)$, let $\bar{S}=V(G)\setminus S$. An edge cut of *G* is an edge subset of the form $[S,\bar{S}]$ for some nonempty proper subset $S\subseteq V(G)$. Define $b(G)=\max\{|D|: D \text{ is an edge cut of } G\}$.

For graphs *G* and *H*, a map $\varphi: V(G)\to V(H)$ is a graph homomorphism if for each $e=uv\in E(G)$, $\varphi(u)\varphi(v)\in E(H)$. If there is a graph homomorphism from *G* to *H*, then *G* is called *H*-colorable. Suppose that *G* is *H*-colorable. Then every graph homomorphism $\varphi: V(G)\to V(H)$ also induces a map $\varphi_e: E(G)\to E(H)$. If for any graph homomorphism $\varphi: V(G)\to V(H)$, and for any e_1, $e_2\in E(H)$, we always have $|\varphi_e^{-1}(e_1)|=|\varphi_e^{-1}(e_2)|$, then *G* is called edge-uniformly *H*-colorable.

Throughout this paper, we use K_p to denote the complete graph with *p* vertices and C_{2p+1} to denote an odd cycle with $V(C_{2p+1})=\{v_i, i\in Z_{2p+1}\}$ and $N(v_i)=\{v_{i-1},v_{i+1}\}$.

Theorem 1.1 (Erdös [2]) Let $p\ge 2$ be an integer and

$$f(p)=\begin{cases}\frac{1}{2}+\frac{1}{2p-2} & \text{if } p\equiv 0 \pmod 2,\\ \frac{1}{2}+\frac{1}{2p} & \text{if } p\equiv 1 \pmod 2.\end{cases}$$

If there is a graph homomorphism from *G* onto K_p, then $b(G)\ge f(p)|E(G)|$.

Let *p* and *q* be two positive integers with $p\ge 2q$. The Kneser graph $K_{p:q}$ is the graph whose vertices represent the q-subsets of $\{0,1,\cdots,p-1\}$, where two vertices are connected if and only if they correspond to disjoint subsets.

Theorem 1.2 Let p, q be two positive integers with $p \ge 2q$. If there is a graph homomorphism from G onto $K_{p:q}$, then $b(G) \ge \frac{2q}{p}|E(G)|$.

Let p and q be two positive integers with $p \ge 2q$. The circulant graph $K_{p/q}$ is the graph with vertex set $V(G)=\{0,1,2,\cdots,p-1\}$ and the neighbors of vertex v are $\{v+q, v+q+1,\cdots,v+p-q\}$.

Theorem 1.3 Let p, q be two positive integers with $p \ge 2q$. If there is a graph homomorphism from G onto $K_{p/q}$, then $b(G) \ge \frac{2q}{p}|E(G)|$.

Theorem 1.4 Let p, q be two positive integers with $p \ge 2q$ and let

$$f(p,q)=\begin{cases}\dfrac{p^2-4q^2+4q}{2p(p-2q+1)} & \text{if } p \text{ is even,}\\[2mm] \dfrac{p^2-4q^2+4q-1}{2p(p-2q+1)} & \text{if } p \text{ is odd.}\end{cases}$$

If G is an edge-uniformly $K_{p/q}$-colorable, then $b(G) \ge f(p,q)|E(G)|$.

A graph G is edge-maximal H-colorable if G is H-colorable but for any graph G' such that G' contains G as a spanning subgraph with $|E(G')|>|E(G)|$, G is not H-colorable.

In Section 2, we prove an associate Theorem which will be used in the proofs of other theorems. In Subsection 3.1, we give an alternative proof of Theorem 1.1 and characterize the graphs G reaching the lower bound in Theorem 1.1 when G is edge-maximal K_p-colorable. In Subsection 3.2, we show the validity and sharpness of Theorem 1.2. In Subsection 3.3, we show the validity of Theorem 1.3 and Theorem 1.4 and characterize the graphs G reaching the lower bound in Theorem 1.4 when G is edge-maximal $K_{p/q}$-colorable. In Subsection 3.5, we show a best possible lower bound for $b(G)$ when G has a graph homomorphism onto an odd cycle C_{2p+1} and characterize the graphs reaching the lower bound among all edge-maximal C_{2p+1}-colorable graphs. There are a lot of researches about graph homomorphism can be found in [3-7].

2. An Associate Theorem

In this section, we shall prove an associate Theorem which will be used in the proofs of other theorems. Throughout this section, we assume that G and H are two graphs and φ is an onto graph homomorphism from G to H. Note that we can also view φ as a map from $E(G)$ to $E(H)$ such that for each $e=uv\in E(G)$, $\varphi(e)=\varphi(u)\varphi(v)\in E(H)$.

Lemma 2.1 Let g be a function between sets X and Y. Then

1) For any $S\subseteq X$, $g^{-1}(\bar{S})=\overline{g^{-1}(S)}$.

2) For any sets $C\subseteq Y$ and $D\subseteq Y$, $C\subseteq D$ if and only if $g^{-1}(C)\subseteq g^{-1}(D)$.

Lemma 2.2 If $D=[S,\bar{S}]$ is an edge cut of H, then $\varphi^{-1}(D)=\left[\varphi^{-1}(S),\overline{\varphi^{-1}(S)}\right]$ is an edge cut of G. Consequently, $\varphi^{-1}(D)=\left[\varphi^{-1}(S),\varphi^{-1}(\bar{S})\right]$.

Proof. For each $x\in\varphi^{-1}(S)$, $y\in\overline{\varphi^{-1}(S)}$ with $xy\in E(G)$, by the definition of inverse image, $\varphi(x)\in S$, and $\varphi(y)\notin S$. Hence, $\varphi(xy)=\varphi(x)\varphi(y)\in D$, and so $xy\in\varphi^{-1}(D)$. It follows that $\varphi^{-1}(D)=\left[\varphi^{-1}(S),\overline{\varphi^{-1}(S)}\right]$. Conversely, for each $xy\in\varphi^{-1}(D)$, $\varphi(x)\varphi(y)=\varphi(xy)\in D=[S,\bar{S}]$. We may assume that $\varphi(x)\in S$ and $\varphi(y)\in\bar{S}$. Then $x\in\varphi^{-1}(S)$ and $y\notin\varphi^{-1}(S)$. This proves $\varphi^{-1}(D)=\left[\varphi^{-1}(S),\overline{\varphi^{-1}(S)}\right]$. Therefore, $\varphi^{-1}(D)=\left[\varphi^{-1}(S),\overline{\varphi^{-1}(S)}\right]$ is an edge cut of G. By Lemma 2.1 1), $\varphi^{-1}(D)=\left[\varphi^{-1}(S),\varphi^{-1}(\bar{S})\right]$.

Lemma 2.3 Suppose that $\mathcal{F}=\{D_1,D_2,\cdots,D_k\}$ is an edge cut cover of H. Then $\mathcal{F}'=\{\varphi^{-1}(D_1),\varphi^{-1}(D_2),\cdots,\varphi^{-1}(D_k)\}$ is an edge cut cover of G. Moreover, if an edge e of H lies in exactly l' members of $\mathcal{F}$, then every edge in $\varphi^{-1}(e)$ lies in exactly l' members of $\mathcal{F}'$.

Proof. By the definition of graph homomorphism, for each $e=xy\in E(G)$, $\varphi(xy)=\varphi(x)\varphi(y)\in E(H)$. Since $\mathcal{F}$ is an edge cut cover of H, then $\varphi(xy)=\varphi(x)\varphi(y)\in D_i$ for some $D_i\in\mathcal{F}$. It follows that $xy\in\varphi^{-1}(D_i)$, and so $\mathcal{F}'$ is an edge cut cover of G. By Lemma 2.1 2), $e\in D_i$ if and only if $\varphi^{-1}(e)\subseteq\varphi^{-1}(D_i)$, and so if an edge e of H lies in exactly l' members of $\mathcal{F}$, then every edge of $\varphi^{-1}(e)$ lies in exactly l' members of $\mathcal{F}'$.

Let k, l be two positive integers. A k-edge cut at least l-cover of a graph H is a collection $\mathcal{F}=\{D_1,D_2,\cdots,D_k\}$ of k edge cuts of H such that every edge of H lies in at least l members of $\mathcal{F}$. A k-edge cut l-cover of a graph H is a collection $\mathcal{F}=\{D_1,D_2,\cdots,D_k\}$ of k edge cuts of H such that every edge of H lies in exactly l members of $\mathcal{F}$. A k-edge cut average l-cover of a graph H is a collection $\mathcal{F}=\{D_1,D_2,\cdots,D_k\}$ of edge cuts of H such that $\bigcup_{i=1}^k D_i=E(H)$ and $\sum_{i=1}^k|D_i|=l|E(H)|$.

Lemma 2.4 Suppose that φ is an onto graph homomorphism from G to H and $\mathcal{F}=\{D_1,D_2,\cdots,D_k\}$ is a k edge cut cover of H such that $\sum_{i=1}^k|\varphi^{-1}(D_i)|\ge l|E(G)|$.

Then $b(G) \ge \frac{l}{k}|E(G)|$.

Proof. By Lemma 2,

$$\mathcal{F}' = \{\varphi^{-1}(D_1), \varphi^{-1}(D_2), \cdots, \varphi^{-1}(D_k)\}$$

is an edge cut cover of G and for any edge e of H, if e lies in exactly l'-members of $\mathcal{F}$, then every edge of $\varphi^{-1}(e)$ lies in exactly l' members of $\mathcal{F}'$. Therefore, $b(G) \ge \frac{l}{k}|E(G)|$ follows from

$$l|E(G)| \le \sum_{i=1}^{k} |\varphi^{-1}(D_i)| \le kb(G).$$

Theorem 2.5 If one of the following is true, then $b(G) \ge \frac{l}{k}|E(G)|$.

1) There is a graph homomorphism from G onto H and H has a k-edge cut l-cover.

2) There is a graph homomorphism from G onto H and H has a k-edge cut at least l-cover.

3) G is edge-uniformly H-colorable and H has a k-edge cut average l-cover.

3. Main Results

3.1. Graphs with a Graph Homomorphism onto a Complete Graph

In this section, we present an alternative proof for Theorem 1.1 and determine all graphs G reaching the lower bound in Theorem 1.1 when G is edge-maximal K_p-colorable.

Lemma 3.1 Let $p \ge 2$ be an integer and let $= \binom{p}{\lfloor \frac{p}{2} \rfloor}$, $l = \frac{k \lfloor \frac{p}{2} \rfloor \lceil \frac{p}{2} \rceil}{\binom{p}{2}}$. Then the graph K_p has a k-edge cut l-cover.

Proof. Let $\mathcal{F} = \left\{ \left[X, V(K_p \setminus X)\right] : X \text{ is a } \lfloor \frac{p}{2} \rfloor\text{-subset of } V(K_p) \right\}$. Then $\mathcal{F}$ is an edge cut cover of K_p with $|\mathcal{F}| = \binom{p}{\lfloor \frac{p}{2} \rfloor} = k$. Since every $\left[X, V(K_p \setminus X)\right] \in \mathcal{F}$ has size $\left|\left[X, V(K_p \setminus X)\right]\right| = \lfloor \frac{p}{2} \rfloor \lceil \frac{p}{2} \rceil$, and since $|E(K_p)| = \binom{p}{2}$, every edge of K_p must be in exactly $\frac{k \lfloor \frac{p}{2} \rfloor \lceil \frac{p}{2} \rceil}{\binom{p}{2}} = l$ members of $\mathcal{F}$.

Thus, Theorem 1.3 follows from Theorem 2.5 1) and Lemma 3.1.

The lower bound of Theorem 1.1 is best possible, in the sense that there exists a family of graphs such that the lower bound of Theorem 1.1 is reached.

Let G and H be two graphs. The composition of G and H, denoted by $G[H]$, is the graph obtained from G by replacing each vertex of $v_i \in V(G)$ by H_i, a copy of H, and joining every vertex in H_k to every vertex in H_l if $v_k v_l \in E(G)$.

Theorem 3.2 Suppose that there is an onto graph homomorphism from G to K_p and G is edge maximal K_p-colorable.

Then each of the following holds.

1) $b(G) \ge f(p)|E(G)|$, where equality holds only if G is edge-uniformly K_p-colorable.

2) Among all edge-maximal K_p-colorable graph G, $b(G) = f(p)|E(G)|$ if and only if $G \cong K_p[\overline{K}_s]$.

Proof. 1) By Theorem 1.1, $b(G) \ge f(p)|E(G)|$. Suppose that $b(G) \ge f(p)|E(G)|$. Let $\varphi : V(G) \to V(K_p)$ be an arbitrarily given graph homomorphism and let $V(K_p) = \{v_1, v_2, \cdots, v_p\}$ be labelled such that if $V_i = \varphi^{-1}(v_i)$, then $|V_1| \le |V_2| \le \cdots \le |V_p|$. For any subset $X \subseteq V(K_p)$, let $Y = V(K_p) \setminus X$ and let φ also denote the induced map $\varphi : E(G) \to E(K_p)$. Then $\varphi^{-1}(X) = \bigcup_{v_i \in X} \varphi^{-1}(v_i) = \bigcup_{v_i \in X} V_i$ and $\varphi^{-1}(Y) = V(G) \setminus \varphi^{-1}(X)$. Since G is an edge-maximal K_p-colorable, $\varphi^{-1}\left([X, Y]_{K_p}\right) = \left[\varphi^{-1}(X), \varphi^{-1}(Y)\right]_{V(G)}$ is a complete bipartite graph. There are $k = \binom{p}{\lfloor \frac{p}{2} \rfloor}$ partitions of $V(K_p)$ into two parts $(X_i, Y_i), i = 1, 2, \cdots, k$, such that $0 \le |Y_i| - |X_i| \le 1$. Set $D_i = [X_i, Y_i]$. Label them so that $|\varphi^{-1}(D_1)| \le |\varphi^{-1}(D_2)| \le \cdots \le |\varphi^{-1}(D_k)|$. By Lemma 2.2 and Lemma 2.3 and the assumption that $b(G) = f(p)|E(G)|$, $\frac{l}{k}|E(G)| = b(G) \ge |\varphi^{-1}(D_k)|$ and so $k \cdot \frac{l}{k}|E(G)| \ge k|\varphi^{-1}(D_k)| \ge \sum_{i=1}^{k} |\varphi^{-1}(D_i)| = l|E(G)|$, where $l = \frac{k\left\lfloor \frac{p}{2} \right\rfloor \left\lceil \frac{p}{2} \right\rceil}{\binom{p}{2}}$. Therefore all the inequalities are equalities and so $\left|\varphi^{-1}(D_1)\right| = \left|\varphi^{-1}(D_2)\right| = \cdots = \left|\varphi^{-1}(D_k)\right|$.

Let $X' = \left\{v_1, v_{i_1}, v_{i_2}, \cdots, v_{i_{\lfloor \frac{p}{2} \rfloor -1}}\right\}$ with $v_p \notin X'$, $Y' = V(K_p) \setminus X'$ and $D' = [X', Y']$. Let $X'' = X' \cup \{v_p\} \setminus \{v_1\}$, $Y'' = V(K_p) \setminus X''$ and $D'' = [X'', Y'']$. Then $\left|\varphi^{-1}(D')\right| = \left|\varphi^{-1}(D'')\right|$. Since

$$\begin{aligned}\left|\varphi^{-1}(D')\right| &= \left|\varphi^{-1}(X')\right|\left|\varphi^{-1}(Y')\right| \\ &= \left(|V_1| + |V_{i_1}| + \cdots + \left|V_{i_{\lfloor \frac{p}{2} \rfloor -1}}\right|\right) \\ &\quad \left(\left|V(K_p)\right| - \left(|V_1| + |V_{i_1}| + \cdots + \left|V_{i_{\lfloor \frac{p}{2} \rfloor -1}}\right|\right)\right)\end{aligned}$$

and

$$\begin{aligned}\left|\varphi^{-1}(D'')\right| &= \left|\varphi^{-1}(X'')\right|\left|\varphi^{-1}(Y'')\right| \\ &= \left(|V_p| + |V_{i_1}| + \cdots + \left|V_{i_{\lfloor \frac{p}{2} \rfloor -1}}\right|\right) \\ &\quad \left(\left|V(K_p)\right| - \left(|V_p| + |V_{i_1}| + \cdots + \left|V_{i_{\lfloor \frac{p}{2} \rfloor -1}}\right|\right)\right),\end{aligned}$$

it follows from $\left|\varphi^{-1}(D')\right| = \left|\varphi^{-1}(D'')\right|$ that $|V_1| = |V_p|$, and so $|V_1| = |V_2| = \cdots = |V_p| = s$ for some positive integer s, which implies G is edge uniformly K_p-colorable.

2) Suppose that G is an edge-maximal K_p-colorable graph such that $b(G) = f(p)|E(G)|$. By the proof of 1), there is some positive integer s such that $G \cong K_p[\overline{K}_s]$

Conversely, suppose that $G \cong K_p[\overline{K}_s]$. By Theorem 1.1, $b(K_p(s)) \le f(p)\left|K_p[\overline{K}_s]\right|$. It remains to show that for any partition (X, Y) of $V(K_p[\overline{K}_s])$, $\|[X,Y]\| \le f(p)\left|K_p[\overline{K}_s]\right|$.

Let $V_1, V_2, \cdots, V_p$ denote the partition of $V(K_p[\overline{K}_s])$ such that for $i = 1, 2, \cdots, p$, V_i is an independent set of $K_p[\overline{K}_s]$. Let $[X,Y]$ be an edge cut of $K_p(s)$ such that 1) $\|[X,Y]\|$ is maximized and subject to 1), 2) $\left|\{i : |X_i||Y_i| \ne 0\}\right|$ is minimized, where $X_i = X \cap V_i$ and $Y_i = Y \cap V_i$. Then we have the following claims:

Claim 1. For each i, $1 \le i \le p$, $|X_i||Y_i| = 0$.

Otherwise, there is i such that $|X_i||Y_i| \ne 0$. Let $m = |X| - |X_i|$ and $t = |Y| - |Y_i|$, If $m = t$, let $X' = X \cup Y_i$ and $Y' = Y \setminus Y_i$ Then $[X', Y']$ is an edge cut of $K_p[\overline{K}_s]$ such that $\|[X,Y]\| = \|[X',Y']\|$ and $\left|\{i : |X_i'||Y_i'| \ne 0\}\right| = \left|\{i : |X_i||Y_i| \ne 0\}\right| - 1$, contrary to the choice of $[X,Y]$. Then $m \ne t$. Without less of generality, we may assume $m < t$. Let $X' = X \cup Y_i$ and $Y' = Y \setminus Y_i$. Then $[X'', Y'']$ is an edge cut of $K_p[\overline{K}_s]$ such that $\|[X'',Y'']\| = \|[X,Y]\| + (t-m)|Y_i| > \|[X,Y]\|$, contrary to the choice of $[X,Y]$.

Claim 2. $\left|\left|\{i : V_i \subseteq X\}\right| - \left|\{i : V_i \subseteq Y\}\right|\right| \le 1$.

Otherwise, $\left|\left|\{i : V_i \subseteq X\}\right| - \left|\{i : V_i \subseteq Y\}\right|\right| \ge 2$. Choose $V_i \subseteq X$ and set $X' = X \setminus V_i$ and $Y' = Y \cup V_i$. Then $[X', Y']$ is an edge cut such that

$$\begin{aligned}\|[X',Y']\| &= \|[X,Y]\| + \left(\left|\{i : V_i \subseteq X\}\right| - \left|\{i : V_i \subseteq Y\}\right| - 1\right)m^2 \\ &> \|[X,Y]\|,\end{aligned}$$

contrary to the choice of $[X,Y]$.

By Claim 1 and Claim 2, we have $\|[X,Y]\| = \frac{p^2 s^2}{4}$ when p is even and $\|[X,Y]\| = \frac{(p^2-1)s^2}{4}$ when p is odd, that is, $\|[X,Y]\| = f(p)\left|E(K_p[\overline{K}_s])\right|$.

3.2. Graphs with a Graph Homomorphism onto a Kneser Graph

In this section, we shall show the validity and sharpness of Theorem 1.2.

Theorem 3.3 Let p and q be two positive integers with $p \ge 2q$. Then $K_{p:q}$ has a p-edge cut $2q$-cover.

Proof. For $i = 0, 1, \cdots, p-1$, let $V_i = \{X \subseteq \{0,1,2,\cdots,p-1\} : |X| = q \text{ and } i \in X\}$ and $D_i = [V_i, \overline{V_i}]$. Then $|V_i| = \binom{p-1}{q-1}$. By the definition of $K_{p:q}$, V_i is an independent sets of $K_{p:q}$ and D_i is an edge cut of size $\binom{p-1}{q-1} \times \binom{p-q}{q}$. Then $\mathcal{F} = \{D_0, D_1, \cdots, D_{p-1}\}$ is an edge cut cover of $K_{p:q}$. Since $K_{p:q}$ has $\binom{p}{q}$ vertices and each of them is of degree $\binom{p-q}{q}$, $\left|E(K_{p:q})\right| = \frac{\binom{p}{q}\binom{p-q}{q}}{2}$.

Since each edge of $K_{p:q}$ is contained in

$$\frac{\binom{p-1}{q-1}\times\binom{p-q}{q}\times p}{\frac{\binom{p}{q}\times\binom{p-q}{q}}{2}}=2q$$ edges cuts, $K_{p:q}$ has a p-edge cut $2q$-cover.

Proof of Theorem 1.2 Theorem 1.2 follows from Theorem 2.5 and Theorem 3.3.

Theorem 3.4 (Poljak and Tuza, Theorem 2 in [8]). If $p \le 3q$, then the maximum edge cut of $K_{p:q}$ is induced by the maximum independent set of $K_{p:q}$ and is of size $\binom{p-1}{q-1}\times\binom{p-q}{q}$.

Theorem 3.5 The bound in Theorem 1.2 is best possible.

Proof. By Theorem 3.4, the edge cuts we choose in the proof of Theorem 4.1 are the maximum edge cuts of $K_{p:q}$ when $p \le 3q$. Therefore, the bound in Theorem 1.2 is reached by $K_{p:q}$ when $p \le 3q$.

3.3. Graphs with a Graph Homomorphism onto a Circulant Graph

In this section, we verify the validity and sharpness of Theorem 1.3 and Theorem 1.4.

Theorem 3.6 Let p and q be two positive integers with $p \ge 2q$. Then the following hold.

1) $K_{p/q}$ has a p-edge cut at least $2q$-cover.

2) $K_{p/q}$ has a p-edge cut average $\frac{2\left(\left\lfloor\frac{p}{2}\right\rfloor\left\lceil\frac{p}{2}\right\rceil-q(q-1)\right)}{p-1-2(q-1)}$-cover.

Proof. For $i=0,1,\cdots,p-1$, let $V_i=\left\{i,i+1,\cdots,\left\lfloor\frac{p}{2}\right\rfloor-1\right\}$ and let $D_i=\left[V_i,\overline{V_i}\right]$. Then $\mathcal{F}=\{D_0,D_1,\cdots,D_{p-1}\}$ is an edge cut cover of $K_{p/q}$ with $|D_i|=\left\lfloor\frac{p}{2}\right\rfloor\left\lceil\frac{p}{2}\right\rceil-q(q-1)$.

1) Notice that for any edge $e=uv\in K_{p/q}$, e lies in at least $2q$ members of $\mathcal{F}=\{D_0,D_1,\cdots,D_{p-1}\}$, then $K_{p/q}$ has a p-edge cut at least $2q$-cover.

2) Since $\left|E\left(K_{\frac{p}{q}}\right)\right|=\frac{p(p-1-2(q-1))}{2}$,

$$\sum_{i=0}^{p}|D_i|=\left(\left\lfloor\frac{p}{2}\right\rfloor\left\lceil\frac{p}{2}\right\rceil-q(q-1)\right)p=\frac{2\left(\left\lfloor\frac{p}{2}\right\rfloor\left\lceil\frac{p}{2}\right\rceil-q(q-1)\right)}{p-1-2(q-1)}\left|E\left(K_{p/q}\right)\right|$$

and so $K_{p/q}$ has a p-edge cut average $\frac{2\left(\left\lfloor\frac{p}{2}\right\rfloor\left\lceil\frac{p}{2}\right\rceil-q(q-1)\right)}{p-1-2(q-1)}$-cover.

Proof of Theorem 1.3 Theorem 1.3 follows from Theorem 2.5 and Theorem 3.6 1).

Proof of Theorem 1.4 Theorem 1.4 follows from Theorem 2.5 and Theorem 3.6 2).

Theorem 3.7 Let G be an edge-maximal graph such that G is edge uniformly $K_{p/q}$-colorable. Then $G\cong K_{p/q}\left[\overline{K_s}\right]$ and $b(G)=f(p,q)|E(G)|$.

Proof. Suppose that G is an edge-maximal graph such that G is edge-uniformly $K_{p/q}$-colorable. Then, $G\cong K_{p/q}\left[\overline{K_s}\right]$ for some positive integer s. By Theorem 1.4, we have $b\left(K_{p/q}\left[\overline{K_s}\right]\right)\ge f(p,q)\left|E\left(K_{p/q}\left[\overline{K_s}\right]\right)\right|$.

Now we prove that $b\left(K_{p/q}\left[\overline{K_s}\right]\right)=f(p,q)\left|E\left(K_{p/q}\left[\overline{K_s}\right]\right)\right|$.

For $i\in\{0,1,\cdots,p-1\}=V\left(K_{p/q}\right)$, let $V_i=h^{-1}(i)$, where $h:V(G)\to V\left(K_{p/q}\right)$ is an onto graph homomorphism from G to $K_{p/q}$ such that for any edge xy of G, $q\le|h(x)-h(y)|\le p-q$. By the definition of graph homomorphism, V_i is an independent set. Let $[X,Y]$ be an edge cut of $K_{p/q}\left[\overline{K_s}\right]$ such that 1) $\|[X,Y]\|$ is maximized and subject to 1), 2) $\left|\{i:|X_i||Y_i|\ne 0\}\right|$ is minimized, where $X_i=X\cap V_i$ and $Y_i=Y\cap V_i$.

Claim 1. For each i, $|X_i||Y_i|=0$.

Proof. Otherwise, there is i such that $|X_i||Y_i|\ne 0$. Since for each $v_i\in V_i$, either $|N(v_i)\cap X|\ge|N(v_i)\cap Y|$ or $|N(v_i)\cap X|\le|N(v_i)\cap Y|$. If the former is true, then $\|[X\setminus X_i,\ Y\cup X_i]\|\ge\|[X,Y]\|$, contradicting the choice of $[X,Y]$; if the latter is true, then $\|[X\cup Y_i,Y\setminus Y_i]\|\ge\|[X,Y]\|$, contradicting the choice of $[X,Y]$.

Assume that $X=\bigcup_{i=j_1}^{j_m}V_i$ and let $\{j_1,j_2,\cdots,j_m\}=J$. Without loss of generality, we can assume that $|J|\le\left\lfloor\frac{p}{2}\right\rfloor$.

Let C_p be the cycle with vertex set $\{0,1,2,\cdots,p-1\}=V\left(K_{p/q}\right)$, where i is adjacent to j if and only if $|i-j|=1(mod\ p)$. Let $dist_{C_p}(J)$ be the length of the shortest path of C_p which contains all the elements in J. Then $dist_C(J)\ge m-1$.

Claim 1. $dist_C(J)=m-1$.

Proof. Suppose, to the contrary, that $dist_{C_p}(J)>m-1$. Let P be a path of C_p which contains all the elements in J and assume that j_1, j_m are the endpoints of P. Then there is $k \in V(P)\setminus J$. Let $X' = \left(X \setminus \{V_{j_m}\}\right) \cup \{V_m\}$ and $J' = J\setminus\{j_m\}\cup\{m\}$. Then $dist_{C_p}(J') < dist_{C_p}(J)$ and $\left\|\left[X', \bar{X}'\right]\right\| > \left\|\left[X, \bar{X}\right]\right\|$, a contradiction.

Claim 2. $m = \left\lfloor \frac{p}{2} \right\rfloor$.

Proof. Suppose, to the contrary, that $m < \left\lfloor \frac{p}{2} \right\rfloor$. Let P be a path of C_p which contains all the elements in J and assume that j_1, j_m are the endpoints of P. Let $j_{m+1} \in V(C_p)\setminus V(P)$ be such that $j_m j_{m+1} \in E(C_p)$. Let $X' = X \cup \{V_{m+1}\}$ and $J' = J \cup \{j_{m+1}\}$. Then $dist_C(J') = dist_C(J) + 1 \le \left\lfloor \frac{p}{2} \right\rfloor$ and $\left\|\left[X', \bar{X}'\right]\right\| > \left\|\left[X, \bar{X}\right]\right\|$, a contradiction.

By the above discussion, we can calculate that $b\left(K_{p/q}\left[\overline{K}_s\right]\right) = f(p,q)\left|E\left(K_{p/q}(s)\left[\overline{K}_s\right]\right)\right|$.

3.4. Graphs with a Graph Homomorphism onto an Odd Cycle

In this section, we will show a best possible lower bound for $b(G)$ when G has a graph homomorphism onto an odd cycle C_{2p+1} and characterize the graphs reaching the lower bound among all edge-maximal C_{2p+1}-colorable graphs.

Theorem 3.8 Suppose that there is an onto graph homomorphism from G to C_{2p+1}. Then each of the following holds.

1) $b(G) \ge \frac{2p}{2p+1}|E(G)|$, where equality holds only if G is edge-uniformly C_{2p+1}-colorable.

2) Among all edge-maximal C_{2p+1}-colorable graph G, $b(G) = \frac{2p}{2p+1}|E(G)|$ if and only if $G \cong C_{2p+1}\left[\overline{K}_s\right]$.

Proof. 1) Notice that $C_{2p+1} \cong K_{(2p+1)/(2p)}$, $b(G) \ge \frac{2p}{2p+1}|E(G)|$ follows from Theorem 1.3. Now suppose that $b(G) = \frac{2p}{2p+1}|E(G)|$. Let $\varphi : V(G) \to V(C_{2p+1})$ be an arbitrarily given graph homomorphism and for each $i \in Z_{2p+1}$, let $V_i = \varphi^{-1}(v_i)$. Set $W_i = [V_i, V_{i+1}]$, for each $i \in Z_{2p+1}$. Since φ is a graph homomorphism, we have $E(G) = \bigcup_{i\in Z_{2p+1}} W_i$. It follows that $|E(G)| = \sum_{i=0}^{2p}|W_i|$. We may assume that $|W_0| = \min\{|W_i| : i \in Z_{2p+1}\}$. Then $E(G) - W_0$ is an edge-cut of G, and so by the assumption on G, we have $\frac{2p}{2p+1}|E(G)| = b(G) \ge |E(G)| - |W_0| = \sum_{i=1}^{2p}|W_i|$. It follows that $2p\left(\sum_{i=0}^{2p}|W_i|\right) = 2p|E(G)| \ge (2p+1)\sum_{i=1}^{2p}|W_i|$. Hence $2p|W_0| \ge |W_1| + |W_1| + \cdots + |W_{2p}|$. By the choice of X_0, we must have $|W_0| = |W_1| = \cdots = |W_{2p}|$. Let $m = |W_0|$. Then for any edge $e \in E(C_{2p+1})$, $|\varphi^{-1}(e)| = m$. Since φ is arbitrarily, then G is edge-uniformly C_{2p+1}-colorable.

2) Suppose that G is an edge-maximal C_{2p+1}-colorable graph with $b(G) = \frac{2p}{2p+1}|E(G)|$. Let W_i be defined as in 1). Since G is an edge-maximal C_{2p+1}-colorable graph, the subgraph induces by W_i is a complete bipartite graph. Since $b(G) = \frac{2p}{2p+1}|E(G)|$, by 1), G is edge-uniformly C_{2p+1}-colorable, which means, $|W_0| = |W_1| = \cdots = |W_{2p}|$ Therefore, $|V_0| = |V_1| = \cdots = |V_{2p}| = s$ for some positive integer s and so $G \cong C_{2p+1}\left[\overline{K}_s\right]$.

Conversely, suppose $G \cong C_{2p+1}\left[\overline{K}_s\right]$. Note that $\left|E\left(C_{2p+1}\left[\overline{K}_s\right]\right)\right| = (2p+1)\mathrm{m}^2$. Then $b\left(C_{2p+1}\left[\overline{K}_s\right]\right) \ge 2\,\mathrm{pm}^2$ and so it suffices to show that for any subset $\varnothing \ne X \subseteq V\left(C_{2p+1}\left[\overline{K}_s\right]\right)$ and $Y = V\left(C_{2p+1}\left[\overline{K}_s\right]\right)\setminus X$, the edge cut $[X,Y]$ of $C_{2p+1}\left[\overline{K}_s\right]$ satisfies $\|[X,Y]\| \le 2\,\mathrm{pm}^2$.

Let $[X,Y]$ be an edge cut of $C_{2p+1}\left[\overline{K}_s\right]$ such that 1) $\|[X,Y]\|$ is maximized and subject to 1), 2) $\left|\{i : |X_i||Y_i| \ne 0\}\right|$ is minimized, where $X_i = X \cap V_i$ and $Y_i = Y \cap V_i$. Since $\left[\bigcup_{i\in I_1} V_i, \bigcup_{i\in I_2} V_i\right]$, where $I_1 = \{i \in Z_{2p+1} : i \text{ is odd}\}$ and $I_2 = \{i \in Z_{2p+1} : i \text{ is even}\}$, is an edge cut with cardinality 2 pm^2, then $\|[X,Y]\| \ge \left\|\left[\bigcup_{i\in I_1} V_i, \bigcup_{i\in I_2} V_i\right]\right\| \ge 2\,\mathrm{pm}^2$. Then we have the following claims:

Claim 1. For each i, $|X_i||Y_i| = 0$.

Proof. Otherwise, there is i such that $|X_i||Y_i| \ne 0$. Let $s = |X_{i+1}| + |X_{i-1}|$ and $t = |Y_{i+1}| + |Y_{i-1}|$. If $s = t$, let $X' = X \cup Y_i$ and $Y' = Y \setminus Y_i$. Then $[X', Y']$ is an edge cut of G_m such that $\|[X,Y]\| = \|[X',Y']\|$ and

$ \left|\{i:|X_i'||Y_i'|\neq 0\}\right| = \left|\{i:|X_i'||Y'\neq 0_i|\right|-1$, contrary to the choice of $[X,Y]$. Then $s\neq t$. Without loss of generality, we may assume $s<t$. Let $X''=X\cup Y_i$ and $Y''=Y\setminus Y_i$. Then $[X'',Y'']$ is an edge cut of G_m such that $\|[X'',Y'']\|=\|[X,Y]\|+(t-s)|Y_i|>\|[X,Y]\|$, contrary to the choice of $[X,Y]$.

Claim 2. There is exactly one pair of consecutive number $\{i,i+1\}$ in $\{0,1,\cdots,2p\}$ (we consider 0 and $2p$ as a pair of consecutive numbers, too) such that $|X_i|=|X_{i+1}|$.

Proof. By Claim 1 and the structure of $C_{2p+1}\left[\overline{K}_s\right]$, there exist some pair of consecutive numbers $\{i,i+1\}$ in $\{0,1,\cdots,2p\}$ such that $|X_i|=|X_{i+1}|$ Since $\|[X,Y]\|\geq 2\ \mathrm{pm}^2$, then there is at most one pair of consecutive number $\{i,i+1\}$ in $\{0,1,\cdots,2p\}$ such that $|X_i|=|X_{i+1}|$ and so Claim 2 follows.

By Claim 1 and Claim 2, $\|[X,Y]\|\geq 2\ \mathrm{pm}^2$.

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