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High Accuracy Arithmetic Average Discretization for Non-Linear Two Point Boundary Value Problems with a Source Function in Integral Form

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Abstract

In this article, we report the derivation of high accuracy finite difference method based on arithmetic average discretization for the solution of $u'' = F(x,u,u') + \int_0^1 K(x,s)\,\mathrm{d}s$, $0 < x < 1$, $0 < s < 1$ subject to natural boundary conditions on a non-uniform mesh. The proposed variable mesh approximation is directly applicable to the integro-differential equation with singular coefficients. We need not require any special discretization to obtain the solution near the singular point. The convergence analysis of a difference scheme for the diffusion convection equation is briefly discussed. The presented variable mesh strategy is applicable when the internal grid points of the solution space are both even and odd in number as compared to the method discussed by authors in their previous work in which the internal grid points are strictly odd in number. The advantage of using this new variable mesh strategy is highlighted computationally.

Keywords: Variable Mesh, Arithmetic Average Discretization, Non-Linear Integro-Differential Equation, Diffusion Equation, Simpson's $\frac{1}{3}$ Rd Rule, Singular Coefficients, Burgers' Equation, Maximum Absolute Errors

1. Introduction

We consider the non-linear differential equation with a source function in integral form:

$$u'' = F(x,u,u') + \int_0^1 K(x,s)\,\mathrm{d}s,\ 0 < x, s < 1. \qquad (1)$$

The two point boundary conditions associated with (1) are given by:

$$u(0) = \gamma_0,\ u(1) = \gamma_1 \qquad (2)$$

where γ_0, γ_1 are finite constants. We assume that $K(x, s)$ is a real valued function of both variables in the range $0 \le x, s \le 1$.

Let

$$I(x) = \int_0^1 K(x,s)\,\mathrm{d}s$$

and

$$F(x,u,u') + I(x) = G(x,u,u')$$

Then we may re-write (1) as

$$u'' = G(x,u,u'),\ 0 < x < 1 \qquad (3)$$

Keller [1] has given the conditions under which the differential Equation (3) together with the boundary conditions (2) has a unique solution. We assume that these conditions are satisfied in the problem that we are considering. In addition, we assume that $u(x) \in C^6[0,1]$ and $K(x,s) \in C^4[0,1]$.

Many physical problems from fluid mechanics, fluid dynamics, elasticity, magneto-hydrodynamics, plasma dynamics, oceanography, biological model, boundary layer theory, …etc are described mathematically by non-linear integro-differential equations. Davis and Rabinowitz [2], Philips [3], Linz [4], Lakshmikantham and

Rao [5], Atkinson [6], and Agarwal and O'Regan [7] have discussed various techniques for numerical integration and methods for approximate solution of integro-differential equations and their applications to various physical models. Most of the nonlinear differential equations cannot be solved analytically. So it is required to obtain efficient numerical methods. Jain *et al.* [8] have discussed variable mesh methods for the solution of two point nonlinear boundary value problems; however, their methods are not applicable to differential equations with singular coefficients. In recent years, Mohanty *et al.* ([9-12]) have discussed a family of third order variable mesh methods for the solution of two point non-linear boundary value problems and obtained convergent solution for singular problems. More recently, Mohanty and Dhall [13] have proposed a three-point third order variable mesh method for the solution of non-linear integro-differential Equation (1), which is applicable only when the internal grid points of the solution region are odd in number. In this paper, we propose an efficient third order variable mesh method based on arithmetic average discretization for the solution of non-linear integro-differential Equation (1), which is applicable when the internal grid points of the solution region are both odd and even in number. In next section, we give mathematical details of the method. In Section 3, we discuss the application of the proposed method to an integro-differential equation with singular coefficients and study the convergence analysis. In Section 4, we give numerical results to justify the utility of the proposed new strategy. Final remarks are given in Section 5.

2. Mathematical Details of the Discretization

We discretize the solution region [0,1] with the non-uniform mesh such that $0 = x_0 < x_1 \cdots < x_{N+1} = 1$. Let $h_{k+1} = x_{k+1} - x_k > 0$ be the variable mesh size in *x*-direction, where $k = 0(1)N+1$. Grid points are given by $x_i = x_0 + \sum_{k=1}^{i} h_k$, $i = 1(1)N+1$. The mesh ratio is $\sigma_k = (h_{k+1}/h_k) > 0$. When $\sigma_k = 1$, then it reduces to the constant mesh case. The off-step points are defined by $x_{k+\frac{1}{2}} = x_k + \frac{\sigma_k h_k}{2}$ and $x_{k-\frac{1}{2}} = x_k - \frac{h_k}{2}$ etc. Let the exact solution of *u*(*x*) at the grid point x_k be denoted by $U_k = u(x_k)$ and u_k be the approximate value of U_k.

Let us construct a numerical method for evaluating the integral $\int_0^1 \phi(x)\,\mathrm{d}x$, where $\phi(x)$ is a real-valued continuous function in [0,1].

Using the derivation technique discussed in [13], we obtain a fourth order accurate integral formula based on Simpson's $\frac{1}{3}$ rd rule (see Evans [14]).

$$\int_{x_k}^{x_{k+1}} \phi(x)\,\mathrm{d}x = \frac{h_{k+1}}{6}\left[\phi_k + 4\phi_{k+\frac{1}{2}} + \phi_{k+1}\right], \quad k = 0,1,2,\cdots N \tag{4}$$

where $\phi_k = \phi(x_k)$, $\phi_{k+1} = \phi(x_{k+1}) = \phi(x_k + h_{k+1})$, $\phi_{k+\frac{1}{2}} = \phi\left(x_{k+\frac{1}{2}}\right) = \phi\left(x_k + \frac{h_{k+1}}{2}\right)$, etc.

Then on the variable mesh the value of the integral

$$\int_0^1 \phi(x)\,\mathrm{d}x = \int_{x_0}^{x_1} \phi(x)\,\mathrm{d}x + \int_{x_1}^{x_2} \phi(x)\,\mathrm{d}x + \cdots + \int_{x_N}^{x_{N+1}} \phi(x)\,\mathrm{d}x = \sum_{k=0}^{N} \frac{h_{k+1}}{6}\left[\phi_k + 4\phi_{k+\frac{1}{2}} + \phi_{k+1}\right] \tag{5}$$

can be found by the repeated application of (4).

Now we discuss the third order numerical method based on arithmetic average discretization for the differential Equation (3).

At the grid point x_k, we denote

$$U_k'' = G(x_k, U_k, U_k') \equiv G_k \quad \text{(say)},$$

and

$$\alpha_k = \left(\frac{\partial G}{\partial u}\right)_{x_k}, \ \beta_k = \left(\frac{\partial G}{\partial u'}\right)_{x_k}.$$

Using Taylor expansion, from (3), we obtain

$$\left(U_{k+1} - (1+\sigma_k)U_k + \sigma_k U_{k-1}\right) = \frac{\sigma_k h_k^2}{3}\left[\sigma_k G_{k+\frac{1}{2}} + \frac{(1+\sigma_k)}{2} G_k + G_{k-\frac{1}{2}}\right] + O\left(h_k^5\right), \quad \sigma_k \neq 1 \tag{6}$$

We need the following approximations:

$$\bar{U}_{k+\frac{1}{2}} = \frac{1}{2}\left(U_{k+1} + U_k\right), \tag{7a}$$

$$\bar{U}_{k-\frac{1}{2}} = \frac{1}{2}\left(U_{k-1} + U_k\right), \tag{7b}$$

$$\bar{U}'_{k+\frac{1}{2}} = \frac{1}{\sigma_k h_k}\left(U_{k+1} - U_k\right), \tag{7c}$$

$$\bar{U}'_{k-\frac{1}{2}} = \frac{1}{h_k}\left(U_k - U_{k-1}\right), \tag{7d}$$

$$\bar{U}'_k = \frac{1}{\sigma_k(1+\sigma_k)h_k}\left[U_{k+1} - \left(1-\sigma_k^2\right)U_k - \sigma_k^2 U_{k-1}\right], \quad (7e)$$

and let

$$\bar{G}_{k\pm\frac{1}{2}} = G\left(x_{k\pm\frac{1}{2}}, \bar{U}_{k\pm\frac{1}{2}}, \bar{U}'_{k\pm\frac{1}{2}}\right). \quad (8)$$

It is then easy to see that

$$\bar{G}_{k+\frac{1}{2}} = G_{k+\frac{1}{2}} + \frac{\sigma_k^2 h_k^2}{24}\left(3U''_k\alpha_k + U'''_k\beta_k\right) + O\left(h_k^3\right), \quad \sigma_k \neq 1 \quad (9a)$$

$$\bar{G}_{k-\frac{1}{2}} = G_{k-\frac{1}{2}} + \frac{h_k^2}{24}\left(3U''_k\alpha_k + U'''_k\beta_k\right) + O\left(h_k^3\right), \quad \sigma_k \neq 1 \quad (9b)$$

From (9a) and (9b), it follows that

$$\bar{G}_{k+\frac{1}{2}} + \bar{G}_{k-\frac{1}{2}} = 2U''_k + \frac{h_k}{2}(\sigma_k - 1)U'''_k + \frac{h_k^2}{24}\left(1+\sigma_k^2\right)\left(3U''_k\alpha_k + U'''_k\beta_k + 3U_k^{IV}\right) + O\left(h_k^3\right), \sigma_k \neq 1 \quad (10a)$$

$$\bar{G}_{k+\frac{1}{2}} - \bar{G}_{k-\frac{1}{2}} = \frac{h_k}{2}(1+\sigma_k)U'''_k + \frac{h_k^2}{24}\left(\sigma_k^2 - 1\right)\left(3U''_k\alpha_k + U'''_k\beta_k + 3U_k^{IV}\right) + O\left(h_k^3\right), \sigma_k \neq 1 \quad (10b)$$

Now, let

$$\bar{\bar{U}}_k = U_k + ah_k^2\left(\bar{G}_{k+\frac{1}{2}} + \bar{G}_{k-\frac{1}{2}}\right), \quad (11a)$$

$$\bar{\bar{U}}'_k = \bar{U}'_k + bh_k\left(\bar{G}_{k+\frac{1}{2}} - \bar{G}_{k-\frac{1}{2}}\right), \quad (11b)$$

where "a" and "b" are parameters to be determined.

With the help of (10a) and (10b), from (11a) and (11b), we obtain

$$\bar{\bar{U}}_k = U_k + 2ah_k^2U''_k + O\left(h_k^3\right), \sigma_k \neq 1, \quad (12a)$$

$$\bar{\bar{U}}'_k = U'_k + \frac{h_k^2}{6}\left[\sigma_k + 3b(1+\sigma_k)\right]U'''_k + O\left(h_k^3\right), \quad \sigma_k \neq 1 \quad (12b)$$

Further, we define

$$\bar{\bar{G}}_k = G\left(x_k, \bar{\bar{U}}_k, \bar{\bar{U}}'_k\right) \quad (13)$$

and with the help of (12a) and (12b), it follows that

$$\bar{\bar{G}}_k = G_k + 2ah_k^2U''_k\alpha_k + \frac{h_k^2}{6}\left[\sigma_k + 3b(1+\sigma_k)\right]U'''_k\beta_k + O\left(h_k^3\right), \sigma_k \neq 1 \quad (14)$$

Then at each internal grid point x_k, the differential Equation (3) is discretized by

$$\left(U_{k+1} - (1+\sigma_k)U_k + \sigma_k U_{k-1}\right) = \frac{\sigma_k h_k^2}{3}\left[\sigma_k\bar{G}_{k+\frac{1}{2}} + \frac{(1+\sigma_k)}{2}\bar{\bar{G}}_k + \bar{G}_{k-\frac{1}{2}}\right] + \bar{T}_k, k = 1(1)N \quad (15)$$

where $\bar{T}_k = O\left(h_k^5\right)$, provided $\sigma_k \neq 1$.

Now with the help of the approximations (9a), (9b) and (14), from (6) and (15), we obtain the local truncation error

$$\bar{T}_k = \frac{-h_k^4}{72}\sigma_k(1+\sigma_k)\left[3\left(1-\sigma_k+\sigma_k^2+8a\right)\alpha_k U''_k + \left(1+\sigma_k+\sigma_k^2+6b(1+\sigma_k)\right)\beta_k U'''_k\right] + O\left(h_k^5\right), \sigma_k \neq 1 \quad (16)$$

The proposed numerical method (15) to be of $O\left(h_k^3\right)$, the coefficient of h_k^4 in (16) must be zero and we obtain the values of parameters

$$a = \frac{-\left(1-\sigma_k+\sigma_k^2\right)}{8}, b = \frac{-\left(1+\sigma_k+\sigma_k^2\right)}{6(1+\sigma_k)},$$

and the local truncation error given by (16) becomes $\bar{T}_k = O\left(h_k^5\right)$, $\sigma_k \neq 1$.

3. Application to Singular Problems

Consider the linear second order model integro-differential equation

$$\frac{d^2u}{dr^2} = A(r)\frac{du}{dr} + B(r)u + f(r) + \int_0^1 K(r,s)\,\mathrm{d}s, \quad 0 < r, s < 1 \quad (17)$$

where $A(r) = \frac{-\alpha}{r}, B(r) = \frac{\alpha}{r^2}$. For α = 1 or 2, the equation above represents cylindrical or spherical problem, respectively. Replacing the variable x by r and applying the formula (15) to the integro-differential equation (17), we obtain

$$-\left[\sigma_k U_{k-1}-(1+\sigma_k)U_k+U_{k+1}\right]+\frac{\sigma_k h_k^2}{3}\left[\sigma_k\left(A_{k+\frac{1}{2}}\bar{U}'_{k+\frac{1}{2}}+B_{k+\frac{1}{2}}\bar{U}_{k+\frac{1}{2}}+f_{k+\frac{1}{2}}+I_{k+\frac{1}{2}}\right)\right.$$
$$\left.+\left(A_{k-\frac{1}{2}}\bar{U}'_{k-\frac{1}{2}}+B_{k-\frac{1}{2}}\bar{U}_{k-\frac{1}{2}}+f_{k-\frac{1}{2}}+I_{k-\frac{1}{2}}\right)+\left(\frac{1+\sigma_k}{2}\right)\left(A_k\bar{\bar{U}}'_k+B_k\bar{\bar{U}}_k+f_k+I_k\right)\right]+T_k=0,\ k=1(1)N \tag{18}$$

where

$$U_k=u(r_k),\ A_k=A(r_k),\ B_k=B(r_k),\ f_k=f(r_k),$$

$$I_k=I(r_k)=\int_0^1 K(r_k,s)\,\mathrm{d}s=\left\{\int_{r_0}^{r_1}+\int_{r_1}^{r_2}+\cdots+\int_{r_N}^{r_{N+1}}\right\}K(r_k,s)\,\mathrm{d}s,$$

$$\bar{\bar{U}}_k=U_k-\frac{(1-\sigma_k+\sigma_k^2)h_k^2}{8}\left[A_{k+\frac{1}{2}}\bar{U}'_{k+\frac{1}{2}}+B_{k+\frac{1}{2}}\bar{U}_{k+\frac{1}{2}}+f_{k+\frac{1}{2}}+I_{k+\frac{1}{2}}+A_{k-\frac{1}{2}}\bar{U}'_{k-\frac{1}{2}}+B_{k-\frac{1}{2}}\bar{U}_{k-\frac{1}{2}}+f_{k-\frac{1}{2}}+I_{k-\frac{1}{2}}\right],$$

$$\bar{\bar{U}}'_k=\bar{U}'_k-\frac{(1+\sigma_k+\sigma_k^2)h_k}{6(1+\sigma_k)}\left[A_{k+\frac{1}{2}}\bar{U}'_{k+\frac{1}{2}}+B_{k+\frac{1}{2}}\bar{U}_{k+\frac{1}{2}}+f_{k+\frac{1}{2}}+I_{k+\frac{1}{2}}-A_{k-\frac{1}{2}}\bar{U}'_{k-\frac{1}{2}}-B_{k-\frac{1}{2}}\bar{U}_{k-\frac{1}{2}}-f_{k-\frac{1}{2}}-I_{k-\frac{1}{2}}\right],$$

and $T_k=O(h_k^5)$.

Note that the scheme (18) is directly applicable to singular problem (17) and do not require any fictitious points outside the solution region to compute the scheme. The scheme is also applicable when the internal grid points of the solution region are both even and odd in number as compared to the scheme discussed by Mohanty and Dhall [13] in which the internal grid points are strictly odd in number.

For convergence of the scheme (18), we use the following approximations:

$$I_{k+\frac{1}{2}}=I_k+\frac{\sigma_k h_k}{2}I'_k+\frac{\sigma_k^2 h_k^2}{8}I''_k+O(h_k^3), \tag{19a}$$

$$I_{k-\frac{1}{2}}=I_k-\frac{h_k}{2}I'_k+\frac{h_k^2}{8}I''_k-O(h_k^3), \tag{19b}$$

$$A_{k+\frac{1}{2}}=A_k+\frac{\sigma_k h_k}{2}A'_k+\frac{\sigma_k^2 h_k^2}{8}A''_k+O(h_k^3), \tag{19c}$$

$$A_{k-\frac{1}{2}}=A_k-\frac{h_k}{2}A'_k+\frac{h_k^2}{8}A''_k-O(h_k^3). \tag{19d}$$

Similarly, we can define the approximations for $B_{k\pm\frac{1}{2}}$ and $f_{k\pm\frac{1}{2}}$, where

$$A'_k=\frac{d}{dr}A(r_k),\ B'_k=\frac{d}{dr}B(r_k),$$
$$f'_k=\frac{d}{dr}f(r_k),\ K'(r,s)=\frac{\partial K}{\partial r},\cdots\text{etc}$$

and

$$I'_k=\frac{d}{dr}I(r_k)=\int_0^1 K'(r_k,s)\,\mathrm{d}s=\left\{\int_{r_0}^{r_1}+\int_{r_1}^{r_2}+\cdots+\int_{r_N}^{r_{N+1}}\right\}K'(r_k,s)\,\mathrm{d}s,$$

$$I''_k=\frac{d^2}{dr^2}I(r_k)=\int_0^1 K''(r_k,s)\,\mathrm{d}s=\left\{\int_{r_0}^{r_1}+\int_{r_1}^{r_2}+\cdots+\int_{r_N}^{r_{N+1}}\right\}K''(r_k,s)\,\mathrm{d}s.$$

Using the approximations (19) in (18), neglecting high order terms and simplifying we get the modified scheme in compact form

$$\left[-\sigma_k+sub_k\right]U_{k-1}+\left[1+\sigma_k+diag_k\right]U_k+\left[-1+sup_k\right]U_{k+1}+\phi_k+T_k=0 \tag{20}$$

where

$$sub_k=\frac{-\sigma_k^2 h_k}{6}A_k-\frac{\sigma_k h_k}{3}\left[A_k-\frac{h_k}{2}A'_k+\frac{h_k^2}{8}A''_k\right]+\frac{\sigma_k h_k^2}{6}\left[B_k-\frac{h_k}{2}B'_k\right]-\frac{\sigma_k(1+\sigma_k+\sigma_k^2)A_k h_k^2}{72}\left[2A_k-h_k(A'_k+B_k)\right]+\frac{\sigma_k(1+\sigma_k)(1-\sigma_k+\sigma_k^2)A_k B_k h_k^3}{48},$$

$$diag_k = \frac{-\left(1-\sigma_k^2\right)}{6} h_k A_k - \frac{\sigma_k\left(1+\sigma_k\right)h_k^2}{6}\left[A_k' - \frac{\left(1-\sigma_k\right)h_k}{4}A_k''\right] + \frac{\sigma_k\left(1+\sigma_k\right)h_k^2}{3}\left[B_k + \frac{\left(\sigma_k-1\right)h_k B_k'}{4} + \frac{\left(1-\sigma_k+\sigma_k^2\right)h_k^2 B_k''}{8}\right]$$
$$+\frac{\left(1+\sigma_k\right)\left(1+\sigma_k+\sigma_k^2\right)A_k h_k{}^2}{36}\left[A_k - \frac{\sigma_k B_k' h_k^2}{2}\right] + \frac{\left(1+\sigma_k\right)\left(1-\sigma_k+\sigma_k^2\right)B_k h_k^3}{48}\left[\left(1-\sigma_k\right)A_k - 2\sigma_k B_k h_k\right],$$

$$sup_k = \frac{h_k}{6}A_k + \frac{\sigma_k h_k}{3}\left[A_k + \frac{\sigma_k h_k}{2}A_k' + \frac{\sigma_k^2 h_k^2}{8}A_k''\right] + \frac{\sigma_k^2 h_k^2}{6}\left[B_k + \frac{\sigma_k h_k}{2}B_k'\right]$$
$$-\frac{\left(1+\sigma_k+\sigma_k^2\right)A_k h_k^2}{72}\left[2A_k + \sigma_k\left(A_k' + B_k\right)h_k\right] - \frac{\left(1+\sigma_k\right)\left(1-\sigma_k+\sigma_k^2\right)A_k B_k h_k^3}{48},$$

$$\phi_k = \frac{\sigma_k\left(1+\sigma_k\right)h_k^2}{6}\left[\left(3 - \frac{B_k\left(1-\sigma_k+\sigma_k^2\right)h_k^2}{4}\right)\left(f_k + I_k\right)\right.$$
$$\left.+\left(\sigma_k - 1 - \frac{\left(1+\sigma_k+\sigma_k^2\right)A_k h_k}{12}\right)h_k\left(f_k' + I_k'\right) + \frac{\left(1-\sigma_k+\sigma_k^2\right)}{4}h_k^2\left(f_k'' + I_k''\right)\right],$$

and $T_k = O\left(h_k^5\right)$.

Incorporating the boundary values $U_0 = \gamma_0,\ U_{N+1} = \gamma_1$, the difference Equation (20) in matrix form can be written as

$$\left(\boldsymbol{D}+\boldsymbol{P}\right)\boldsymbol{U} + \boldsymbol{\phi} + \boldsymbol{T}\left(h_k\right) = \boldsymbol{0} \tag{21}$$

where $\boldsymbol{D} = \left[-\sigma_k, 1+\sigma_k, -1\right]$ and $\boldsymbol{P}$ = [sub_k, $diag_k$, sup_k] are tri-diagonal matrices of order N and

$$\boldsymbol{\phi} = \left[\phi_1 + \left(sub_1 - \sigma_1\right)\gamma_0, \phi_2, \cdots, \phi_{N-1}, \phi_N + \left(sup_N - 1\right)\gamma_1\right]^T,$$

$\boldsymbol{U} = \left[U_1, U_2, \cdots, U_N\right]^T$, $\boldsymbol{T}\left(h_k\right) = \left[T_1, T_2, \cdots, T_N\right]^T$ and $\boldsymbol{0} = \left[0, 0, \cdots, 0\right]^T$ are vectors.

Let $\boldsymbol{u} = \left[u_1, u_2, \cdots, u_N\right]^T \cong \boldsymbol{U}$ which satisfies

$$\left(\boldsymbol{D}+\boldsymbol{P}\right)\boldsymbol{u} + \boldsymbol{\phi} = \boldsymbol{0} \tag{22}$$

Let $e_k = u_k - U_k$ be the discretization error (in the absence of round of errors) at the grid point r_k and $\boldsymbol{E} = \boldsymbol{u} - \boldsymbol{U} = \left[e_1, e_2, \cdots, e_N\right]^T$ be the error vector.

Subtracting (21) from (22), we obtain the error equation

$$\left(\boldsymbol{D}+\boldsymbol{P}\right)\boldsymbol{E} = \boldsymbol{T}\left(h_k\right) \tag{23}$$

Let $\left|A_k\right| \le G_1,\ \left|A_k'\right| \le G_2,\ \left|A_k''\right| \le G_3,\ \left|B_k\right| \le H_1,\ \left|B_k'\right| \le H_2,\ \left|B_k''\right| \le H_3$, where G_1, G_2, G_3, H_1, H_2 and H_3 are some positive constants. If $p_{i,j}$ be the $(i, j)^{\text{th}}$-element of $\boldsymbol{P}$, then

$$\left|p_{k,k+1}\right| \le \frac{h_k}{6}\left[\left(1+2\sigma_k\right)G_1 + \sigma_k^2 h_k\left(G_2 + H_1\right) + \frac{\left(1+\sigma_k+\sigma_k^2\right)h_k}{6}G_1^2\right] + O\left(h_k^3\right),\ k = 1(1)N-1, \tag{24a}$$

$$\left|p_{k,k-1}\right| \le \frac{\sigma_k h_k}{6}\left[\left(1+2\sigma_k\right)G_1 + h_k\left(G_2 + H_1\right) + \frac{\left(1+\sigma_k+\sigma_k^2\right)h_k}{6}G_1^2\right] + O\left(h_k^3\right),\ k = 2(1)N, \tag{24b}$$

Thus for sufficiently small h_k, the matrix ($\boldsymbol{D}$ + $\boldsymbol{P}$) is irreducible (see Varga [15] and Young [16]).

Let S_k be the sum of elements of the k^{th}-row of ($\boldsymbol{D}$ + $\boldsymbol{P}$), then

$$S_k = \sigma_k\left[1 + \frac{h_k}{6}\left(\left(2+\sigma_k\right)A_k - h_k A_k'\right) + \frac{h_k^2}{6}\left(\left(2+3\sigma_k\right)B_k + \frac{\left(1+\sigma_k+\sigma_k^2\right)}{6}A_k^2\right)\right] + O\left(h_k^3\right),\ k = 1 \tag{25a}$$

$$S_k = 1 - \frac{h_k}{6}\left(\left(1+2\sigma_k\right)A_k + \sigma_k^2 h_k A_k'\right) + \frac{h_k^2}{6}\left(\sigma_k\left(3+2\sigma_k\right)B_k + \frac{\left(1+\sigma_k+\sigma_k^2\right)}{6}A_k^2\right)O\left(h_k^3\right),\ k = N, \tag{25b}$$

$$S_k = \sigma_k\left(1+\sigma_k\right)\frac{h_k^2}{2}B_k + O\left(h_k^3\right), k = 2(1)N-1. \quad (25c)$$

Let $G_{1*} = \min_{1\le k\le N}|A_k|$, $G_1^* = \max_{1\le k\le N}|A_k|$, $H_{1*} = \min_{1\le k\le N}|B_k|$, $H_1^* = \max_{1\le k\le N}|B_k|$, then $0 < G_{1*} \le G_1 \le G_1^*$ and $0 < H_{1*} \le H_1 \le H_1^*$.

It is straightforward to show that for sufficiently small h_k, $(\boldsymbol{D}+\boldsymbol{P})$ is Monotone (see Varga [15] and Young [16]). Hence $(\boldsymbol{D}+\boldsymbol{P})^{-1}$ exists and $(\boldsymbol{D}+\boldsymbol{P})^{-1} \ge \boldsymbol{0}$.

From error Equation (23), we have

$$\|\boldsymbol{E}\| \le \left\|\left(\boldsymbol{D}+\boldsymbol{P}\right)^{-1}\right\| \cdot \left\|\boldsymbol{T}\left(h_k\right)\right\| \quad (26)$$

Thus for sufficiently small h_k, it is easy to show that

$$S_k > \sigma_k\left(2+3\sigma_k\right)\frac{h_k^2}{6}H_{1*}, k=1, \quad (27a)$$

$$S_k > \sigma_k\left(3+2\sigma_k\right)\frac{h_k^2}{6}H_{1*}, k=N, \quad (27b)$$

$$S_k \ge \sigma_k\left(1+\sigma_k\right)\frac{h_k^2}{2}H_{1*}, k=2(1)N-1. \quad (27c)$$

Since

$$\left(\boldsymbol{D}+\boldsymbol{P}\right)_{i,k}^{-1} \ge 0 \text{ and } \sum_{k=1}^{N}\left(\boldsymbol{D}+\boldsymbol{P}\right)_{i,k}^{-1}.\ S_k = 1,\ i=1(1)N,$$

hence

$$\left(\boldsymbol{D}+\boldsymbol{P}\right)_{i,k}^{-1} \le \frac{1}{S_k} < \frac{6}{\sigma_k\left(2+3\sigma_k\right)H_{1*}h_k^2}, k=1 \quad (28a)$$

$$\left(\boldsymbol{D}+\boldsymbol{P}\right)_{i,k}^{-1} \le \frac{1}{S_k} < \frac{6}{\sigma_k\left(3+2\sigma_k\right)H_{1*}h_k^2}, k=N \quad (28b)$$

Further,

$$\sum_{k=2}^{N-1}\left(\boldsymbol{D}+\boldsymbol{P}\right)_{i,k}^{-1} \le \frac{1}{\min_{2\le k\le N-1} S_k} \le \frac{2}{\sigma_k\left(1+\sigma_k\right)H_{1*}h_k^2}, \quad i=1(1)N \quad (29)$$

For any matrix $\boldsymbol{M}$, we define $\|\boldsymbol{M}\| = \max_{1\le i\le N}\sum_{k=1}^{N}|m_{i,k}|$, where $m_{i,k}$ is the $(i, k)^{\text{th}}$-element of $\boldsymbol{M}$ and $\|\boldsymbol{T}(h_k)\| = \max_{1\le i\le N}|T_i|$.

With the help of (28a), (28b), (29), from (26), we obtain

$$\|\boldsymbol{E}\| \le \frac{6}{H_{1*}\sigma_k h_k^2}\left(\frac{1}{2+3\sigma_k}+\frac{1}{3\left(1+\sigma_k\right)}+\frac{1}{3+2\sigma_k}\right)\cdot O\left(h_k^5\right) = O\left(h_k^3\right) \quad (30)$$

This establishes the third order convergence of the method (20).

4. Numerical Results

In this section, we consider another new variable mesh method for the solution of non-linear integro-differential Equation (1) as

$$\begin{aligned}&\left(U_{k+1}-\left(1+\sigma_k\right)U_k+\sigma_k U_{k-1}\right)\\ &= \frac{\sigma_k h_k^2}{2}\left[\frac{\left(1+2\sigma_k\right)}{3}\bar{F}_{k+\frac{1}{2}}+\frac{\left(2+\sigma_k\right)}{3}\bar{F}_{k-\frac{1}{2}}\right]\\ &+\frac{\sigma_k h_k^2}{2}\left[\frac{\left(1+2\sigma_k\right)}{3}I_{k+\frac{1}{2}}+\frac{\left(2+\sigma_k\right)}{3}I_{k-\frac{1}{2}}\right]+O\left(h_k^4\right),\\ &k=1(1)N\end{aligned} \quad (31)$$

where

$$\bar{F}_{k+\frac{1}{2}} = F\left(x_{k+\frac{1}{2}}, \bar{U}_{k+\frac{1}{2}}, \bar{U}'_{k+\frac{1}{2}}\right),$$

$$\bar{F}_{k-\frac{1}{2}} = F\left(x_{k-\frac{1}{2}}, \bar{U}_{k-\frac{1}{2}}, \bar{U}'_{k-\frac{1}{2}}\right),$$

and

$$I_k = I\left(x_k\right) = \int_{s=0}^{1}K\left(x_k, s\right)\mathrm{d}s, k=1(1)N.$$

The approximations associated with $\bar{F}_{k\pm\frac{1}{2}}$ are already defined by (7a)-(7d). The order of accuracy of the method (31) is of $O\left(h_k^2\right)$. For evaluating the integral associated with (31), we replace the integral by the trapezoidal rule (see Evans [14]).

$$\int_0^1\phi(x)\mathrm{d}x = \int_{x_0}^{x_1}\phi(x)\mathrm{d}x+\int_{x_1}^{x_2}\phi(x)\mathrm{d}x+\cdots+\int_{x_N}^{x_{N+1}}\phi(x)\mathrm{d}x = \sum_{k=0}^{N}\frac{h_{k+1}}{2}\left[\phi_k+\phi_{k+1}\right] \quad (32)$$

In this section, we have solved two benchmark problems using the proposed method described by equation (15) and compared our results with those obtained by using the variable mesh method discussed by Mohanty and Dhall [13] only for the cases when internal grid points are odd in number. We have also computed our results using uniform mesh (when $\sigma_k = 1$) for all values of N. The boundary conditions may be obtained using the exact solution as a test procedure. The linear difference equation has been solved using a tri-diagonal solver, whereas non-linear difference equations have been solved using the Newton-Raphson method (see Kelly [17] and

Evans [18]). While using the Newton-Raphson method, the iterations were stopped when absolute error tolerance $\le 10^{-12}$ was achieved.

The unit interval [0,1] in the space-direction is divided into $(N + 1)$ points with

$$0 = x_0 < x_1 < \cdots x_N < x_{N+1} = 1,$$

where

$$h_k = x_k - x_{k-1}$$

and

$$\sigma_k = \left(h_{k+1}/h_k\right) > 0,\ k = 1, 2, \cdots, N.$$

We may write

$$\begin{aligned} 1 = x_{N+1} - x_0 &= \left(x_{N+1} - x_N\right) + \left(x_N - x_{N-1}\right) + \cdots + \left(x_1 - x_0\right) \\ &= h_{N+1} + h_N + \cdots + h_1 = \left(1 + \sigma_1 + \sigma_1\sigma_2 + \cdots + \sigma_1\sigma_2\cdots\sigma_N\right)h_1. \end{aligned} \tag{33}$$

For simplicity, we consider $\sigma_k = \sigma$ (a constant), $k = 1, 2, \cdots, N$, then from (33) we have

$$h_1 = \frac{1-\sigma}{1-\sigma^{N+1}}. \tag{34}$$

By prescribing the total number of mesh points to be $(N + 2)$, we can compute the value of h_1 from (34). This is the first mesh spacing on the left of the boundary and the remaining mesh is determined by $h_{k+1} = \sigma h_k$, $k = 1, 2, \cdots, N$. For variable mesh, we choose the values of $\sigma = 1.2$. All computations were carried out using double precision arithmetic.

Example 4.1 $$\varepsilon\frac{\mathrm{d}^2u}{\mathrm{d}r^2} + \frac{\alpha}{r}\frac{\mathrm{d}u}{\mathrm{d}r} - \frac{\alpha}{r^2}u = \alpha + 2\varepsilon + r\left(r - \frac{\alpha}{\sqrt{\varepsilon}} - 4\sqrt{\varepsilon}\right)\mathrm{e}^{\frac{-r}{\sqrt{\varepsilon}}} - \frac{\alpha + 2\varepsilon}{\sqrt{\varepsilon}}\int_{s=0}^{1} r\mathrm{e}^{\frac{-rs}{\sqrt{\varepsilon}}}\,\mathrm{d}s,\ 0 < r < 1,$$

(*Linear equation in polar coordinates*) (35)

The exact solution is given by $u(r) = r^2 e^{\frac{-r}{\sqrt{\varepsilon}}}$. The maximum absolute errors are tabulated in **Table 1** for various values of N.

Example 4.2 $$\varepsilon\left(\frac{\mathrm{d}^2u}{\mathrm{d}r^2} + \frac{\alpha}{r}\frac{\mathrm{d}u}{\mathrm{d}r} - \frac{\alpha}{r^2}u\right) = u\frac{\mathrm{d}u}{\mathrm{d}r} + (2+\alpha)\varepsilon + \frac{r}{\varepsilon^2}\left[r - (4+\alpha)\varepsilon\right]\mathrm{e}^{\frac{-r}{\varepsilon}}$$
$$+ \frac{r^3}{\varepsilon}(r - 2\varepsilon)\mathrm{e}^{\frac{-2r}{\varepsilon}} - (2+\alpha)\int_{s=0}^{1} r\mathrm{e}^{\frac{-rs}{\varepsilon}}\,\mathrm{d}s,\ 0 < r < 1,$$

(*Model Burger's equation in polar coordinates*) (36)

The exact solution is given by $u(r) = r^2\mathrm{e}^{\frac{-r}{\varepsilon}}$. The maximum absolute errors are tabulated in **Table 2** for various values of N.

5. Final Remarks

Using three variable mesh points, we have discussed a new numerical methods of accuracy of $O\left(h_k^3\right)$ based on arithmetic average discretizations for the solution of the non-linear integro-differential Equation (1). Mohanty and Dhall [13] have developed a third order variable mesh method based on Numerov type discretization, which is only applicable to the solution space having odd number of grid points. Although the proposed variable mesh method involve more algebra, but applicable to the solution space having both odd and even number of internal grid points. In addition, the proposed methods are directly applicable to singular problems and we do not require any fictitious points near the boundaries to incorporate the singular point. The numerical results indicate that the proposed method is computationally nearly equal to the method discussed in [13] and applicable to the solution space with all internal grid points. We have tabulated maximum absolute errors for different mesh sizes. Our mesh sizes are either in increasing or in decreasing order. So it is not possible to estimate the order of convergence of the proposed method. Order of convergence can be estimated for uniform mesh using the formula $\log\left(e_{h_1}/e_{h_2}\right)\big/\log\left(h_1/h_2\right)$, where e_{h_1} and e_{h_2} are the maximum absolute errors for two grid mesh widths h_1 and h_2, respectively. For ex: in **Table 2**, let us consider the case $\alpha = 1$, $\varepsilon = 0.01$, $N = 30$ and $N = 60$, *i.e.* $h = \frac{1}{30} = h_1$ (say) and $h = \frac{1}{60} = h_2$ (say) and the corresponding maximum absolute errors are 0.3116 (–8) and 0.1978 (–9), respectively. Using the above formula the

Table 1. The maximum absolute errors for Example 4.1.

N	$O(h_k^3)$ -proposed method (20)				$O(h_k^2)$ -proposed method (31)			
	$\alpha=1$		$\alpha=2$		$\alpha=1$		$\alpha=2$	
	$\varepsilon=0.1$	$\varepsilon=0.01$	$\varepsilon=0.1$	$\varepsilon=0.01$	$\varepsilon=0.1$	$\varepsilon=0.01$	$\varepsilon=0.1$	$\varepsilon=0.01$
20	0.2062E–4	0.3334E–4	0.2528E–4	0.2800E–3	0.1906E–3	0.4772E–3	0.2040E–3	0.2240E–2
25	0.1519E–4 *0.1416E–4	0.2511E–4 *0.2405E–4	0.1810E–4 *0.1723E–4	0.9455E–4 *0.9376E–4	0.1665E–3	0.1209E–3	0.1783E–3	0.8935E–3
30	0.1149E–4	0.1216E–4	0.1491E–4	0.4234E–4	0.1507E–3	0.2864E–4	0.1614E–3	0.3438E–3
35	0.1082E–4 *0.1022E–4	0.9128E–5 *0.9075E–5	0.1348E–4 *0.1289E–4	0.1661E–4 *0.1618E–4	0.1402E–3	0.2224E–4	0.1502E–3	0.8741E–4
40	0.1030E–4	0.8811E–5	0.1199E–4	0.1116E–4	0.1299E–3	0.1599E–4	0.1391E–3	0.5240E–4
45	0.8989E–5 *0.8910E–5	0.7612E–5 *0.7588E–5	0.9217E–5 *0.9148E–5	0.7880E–5 *0.7823E–5	0.1238E–3	0.1522E–4	0.1322E–3	0.3338E–4
50	0.8599E–5	0.6423E–5	0.8875E–5	0.6345E–5	0.1161E–3	0.1451E–4	0.1243E–3	0.1448E–4
55	0.8110E–5 *0.8066E–5	0.5666E–5 *0.5612E–5	0.8420E–5 *0.8366E–5	0.4529E–5 *0.4488E–5	0.1112E–3	0.1384E–4	0.1188E–3	0.1360E–4
60	0.7501E–5	0.4892E–5	0.7977E–5	0.2828E–5	0.1059E–3	0.1325E–4	0.1135E–3	0.1284E–4
65	0.7123E–5 *0.7070E–5	0.4088E–5 *0.4024E–5	0.7362E–5 *0.7316E–5	0.2524E–5 *0.2487E–5	0.1022E–3	0.1278E–4	0.1092E–3	0.1252E–4
70	0.6699E–5	0.3236E–5	0.6886E–5	0.2228E–5	0.9812E–4	0.1227E–4	0.1051E–3	0.1216E–4
75	0.6128E–5 *0.6084E–5	0.2422E–5 *0.2392E–5	0.6332E–5 *0.6304E–5	0.1907E–5 *0.1888E–5	0.9512E–4	0.1190E–4	0.9926E–4	0.1182E–4
80	0.5511E–5	0.1772E–5	0.5834E–5	0.1664E–5	0.9179E–4	0.1148E–4	0.9831E–4	0.1143E–4

*: Results obtained by using the method discussed in [13].

Table 2. The maximum absolute errors for Example 4.2.

N	$O(h_k^3)$ -proposed method (15)				$O(h_k^2)$ -proposed method (15) [for uniform mesh $\sigma_k=1$]			
	$\alpha=1$		$\alpha=2$		$\alpha=1$		$\alpha=2$	
	$\varepsilon=0.1$	$\varepsilon=0.01$	$\varepsilon=0.1$	$\varepsilon=0.01$	$\varepsilon=0.1$	$\varepsilon=0.01$	$\varepsilon=0.1$	$\varepsilon=0.01$
10	0.3582E–5	0.2318E–6	0.4054E–5	0.2022E–6	0.3318E–6	0.2421E–7	0.3844E–6	0.2100E–7
15	0.4972E–6 *0.4818E–6	0.8959E–7 *0.8872E–7	0.1422E–6 *0.1366E–6	0.8889E–7 *0.8801E–7	0.5756E–7	0.6250E–8	0.7224E–7	0.8181E–8
20	0.1104E–6	0.5934E–7	0.1951E–6	0.3283E–7	0.1929E–7	0.1412E–8	0.1616E–7	0.1012E–8
25	0.1525E–7 *0.1488E–7	0.2785E–7 *0.2740E–7	0.5790E–7 *0.5720E–7	0.6503E–8 *0.6472E–8	0.7782E–8	0.9215E–9	0.8269E–8	0.5440E–9
30	0.4865E–8	0.1987E–8	0.1354E–7	0.2173E–8	0.3116E–8	0.3812E–9	0.3880E–8	0.4004E–9
35	0.4133E–8 *0.4096E–8	0.4344E–9 *0.4312E–9	0.8218E–8 *0.8176E–8	0.7437E–9 *0.7414E–9	0.8868E–9	0.5381E–10	0.1614E–8	0.7818E–10
40	0.3373E–8	0.5659E–10	0.6534E–8	0.2147E–9	0.8617E–9	0.5733E–11	0.9210E–9	0.1817E–10
45	0.3008E–8 *0.3001E–8	0.4234E–10 *0.4228E–10	0.6166E–8 *0.6122E–8	0.8872E–10 *0.8845E–10	0.5417E–9	0.3230E–11	0.7106E–9	0.7117E–11
50	0.2743E–8	0.2606E–10	0.5657E–8	0.5971E–10	0.4413E–9	0.1006E–11	0.4818E–9	0.4343E–11
55	0.2174E–8 *0.2170E–8	0.1918E–10 *0.1911E–10	0.4828E–8 *0.4810E–8	0.4815E–10 *0.4804E–10	0.2830E–9	0.8716E–12	0.3636E–9	0.1156E–11
60	0.1676E–8	0.1210E–10	0.4138E–8	0.4104E–10	0.1978E–9	0.7677E–12	0.2442E–9	0.8821E–12

*: Results obtained by using the method discussed in [13].

order of the convergence of method can be estimated as 3.97 which is nearly equal to 4.0. Similarly in other cases, we found that the order of the convergence of the method for uniform mesh case is nearly equal to four.

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