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Generalization of Certain Subclasses of Multivalent Functions with Negative Coefficients Defined by Cho-Kwon-Srivastava Operator

Elsayed A. Elrifai, Hanan E. Darwish, Abdusalam R. Ahmed
Department of Mathematics, Faculty of Science, University of Mansoura, Mansoura, Egypt
E-mail: *Rifai@mans.edu.eg*, *Darwish333@yahoo.com*, *Abdusalam5056@yahoo.com*
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Abstract

Making use of the Cho-Kwon-Srivastava operator, we introduce and study a certain $SC_n(j, p, \lambda, \alpha, \delta)$ of p-valently analytic functions with negative coefficients. In this paper, we obtain coefficient estimates, distortion theorem, radii of close-to-convexity, starlikeness, convexity and modified Hadamard products of functions belonging to the class $SC_n(j, p, \lambda, \alpha, \delta)$. Finally, several applications investigate an integral operator, and certain fractional calculus operators also considered.

Keywords: Multivalent Functions, Cho-Kwon-Srivastava Operator, Modified-Hadamard Product, Fractional Calculus

1. Introduction

Let $T(j,p)$ denote the class of functions of the form:

$$f(z) = z^p - \sum_{k=j+p}^{\infty} a_k z^k \left(a_k \geq 0;\ p,\ j \in N = \{1,2,3,\cdots\}\right), \tag{1.1}$$

which are analytic and p-valent in the open unit disc $U = \{z : |z| < 1\}$. A function $f(z) \in T(j,p)$ is said to be p-valently starlike of order α if it satisfies the inequality :

$$\operatorname{Re}\left(\frac{zf'(z)}{f(z)}\right) > \alpha \left(z \in U; 0 \leq \alpha < p;\ p \in N\right). \tag{1.2}$$

We denote by $T_j^*(p,\alpha)$ the class of all p-valently starlike functions of order α. Also a function $f(z) \in T(j,p)$ is said to be p-valently convex of order α if it satisfies the inequality:

$$\operatorname{Re}\left\{1 + \frac{zf''(z)}{f'(z)}\right\} > \alpha \left(z \in U;\ 0 \leq \alpha < p;\ p \in N\right). \tag{1.3}$$

We denote by $C_j(p,\alpha)$ the class of all p-valently convex functions of order α. We note that (see for example Duren [1] and Goodman [2])

$$f(z) \in C_j(p,\alpha) \Leftrightarrow \frac{zf'(z)}{p} \in T_j^*(p,\alpha) \tag{1.4}$$
$$\left(0 \leq \alpha < p;\ p \in N\right).$$

The classes $T_j^*(p,\alpha)$ and $C_j(p,\alpha)$ are studied by Owa [3].

In [4] Wang *et al.* defined Cho-kwon-Srivastava operator which

$$\ell_{p,j}^{\lambda}(a,c) f(z) : T(j,p) \to T(j,p),$$

by

$$\ell_{p,j}^{\lambda}(a,c) f(z) = z^p - \sum_{k=j+p}^{\infty} \frac{(\lambda+p)_{k-p}(c)_{k-p}}{(k-p)!(a)_{k-p}} a_k z^k \tag{1.5}$$

for

$$\left(a, c \in R/Z_0^- = \{0,-1,-2,-3,\cdots\},\ z \in U,\ \lambda > -p\right)$$

and

$$(\lambda)_k = \begin{cases} 1 & ;\ k = 0 \\ \lambda(\lambda+1)\cdots(\lambda+k-1) & ;\ k \in N. \end{cases}$$

Clearly, $\ell_{p,1}^{\lambda}(a,c)$ is the well-known Cho-kwon-Srivastava operator (see [5]) where

$$\ell_{p,j}^{1}(p+1,1) = f(z),\ \ell_{p,j}^{1}(p,1) = \frac{zf'(z)}{p},$$

and

$$\ell_{p,1}^{\lambda}(a,a) = D^{\lambda+p-1}f(z)(\lambda > -p),$$

where $D^{\lambda+p-1}$ is the well-known Ruscheweyh derivative of $(\lambda + p - 1)$-th order.

With the help of the Cho-Kwon-Srivastava $\ell_{p,j}^{\lambda}(a,c)f(z)$, we say that a function $f(z)$ belonging to $T(j,p)$ is in the class $SC_n(j,p,\lambda,\alpha,\delta)$ if and only if

$$\operatorname{Re}\left\{\frac{z\left(\ell_{p,j}^{\lambda}(a,c)f(z)\right)' + \delta z^2\left(\ell_{p,j}^{\lambda}(a,c)f(z)\right)''}{(1-\delta)\ell_{p,j}^{\lambda}(a,c)f(z) + \delta z\left(\ell_{p,j}^{\lambda}(a,c)f(z)\right)'}\right\} > \alpha$$
$$(p \in N,\ j \in N_0) \tag{1.6}$$

We note that:

1) when $\delta = 0$, we have

$$\operatorname{Re}\left\{\frac{z\left(\ell_{p,j}^{\lambda}(a,c)f(z)\right)'}{\ell_{p,j}^{\lambda}(a,c)f(z)}\right\} > \alpha$$

which is the class of starlike of order α.

2) when $\delta = 0$, $a = p+1$, $\lambda = 1$, $c = 1$, we have the class

$$\operatorname{Re}\left\{\frac{zf'(z)}{f(z)}\right\} > \alpha;\ 0 \le \alpha < p$$

which is the class of starlike functions of order α studied by Owa [3] and Yamakawa [6]

3) when $\delta = 1$, we have

$$\operatorname{Re}\left\{1 + \frac{z\left(\ell_{p,j}^{\lambda}(a,c)f(z)\right)''}{\left(\ell_{p,j}^{\lambda}(a,c)f(z)\right)'}\right\} > \alpha;$$
$$0 \le \alpha \le p$$

which is the class of convex operator of order α.

4) when $\delta = 1$, $a = p+1$, $c = 1$, $\lambda = 1$, we have

$$\operatorname{Re}\left\{1 + \frac{zf''(z)}{f'(z)}\right\} > \alpha;\ (0 \le \alpha < p)$$

which is the class of convex functions of order α studied by Owa [3] and Yamakawa [6].

In our present paper, we shall make use of the familiar $J_{c,p}$ defined by (c.f. [7,8], see also [9])

$$\left(J_{c,p}f\right)(z) = \frac{c+p}{z^c}\int_0^z t^{c-1}f(t)\,\mathrm{d}t, \tag{1.7}$$
$$\left(f(z) \in T(j,p);\ c > -p,\ p \in N\right),$$

as well as the fractional calculus operator D_z^{μ} for which it is well known that (see, for details, [10,11]; see also Section 5 below)

$$D_z^{\mu}\left\{z^{\rho}\right\} = \frac{\Gamma(\rho+1)}{\Gamma(\rho+1-\mu)}z^{\rho-\mu}, \tag{1.8}$$
$$(\rho > -1;\ \mu \in R)$$

in terms of Gamma functions.

2. Coefficient Estimates

Theorem 1. Let the function $f(z)$ defined by (1.1). Then $f(z) \in SC_n(j,p,\lambda,\alpha,\delta)$ if and only if

$$\sum_{k=j+p}^{\infty}\frac{(\lambda+p)_{k-p}(c)_{k-p}}{(k-p)!(a)_{k-p}}(k-\alpha)\left[1+\delta(k-1)\right]a_k \le (p-\alpha)\left[1+\delta(p-1)\right] \tag{2.1}$$

$$\left(z \in U;\ 0 \le \alpha < p,\ 0 \le \delta \le 1,\ p,\ j \in N,\ n \in N_0\right).$$

Proof. Assume that the inequality (2.1) holds true. Then we have

$$(j+p-\alpha)\sum_{k=j+p}^{\infty}\frac{(\lambda+p)_{k-p}(c)_{k-p}}{(k-p)!(a)_{k-p}}\left[1+\delta(k-1)\right]a_k$$
$$\le \sum_{k=j+p}^{\infty}\frac{(\lambda+p)_{k-p}(c)_{k-p}}{(k-p)!(a)_{k-p}}(k-\alpha)\left[1+\delta(k-1)\right]a_k \le (p-\alpha)\left[1+\delta(p-1)\right]$$

that is, that

$$\sum_{k=j+p}^{\infty}\frac{(\lambda+p)_{k-p}(c)_{k-p}}{(k-p)!(a)_{k-p}}\left[1+\delta(k-1)\right]a_k \le \frac{(p-\alpha)\left[1+\delta(p-1)\right]}{j+p-\alpha}.$$

Since

$$\left|\left[1+\delta(p-1)\right]-\sum_{k=j+p}^{\infty}\frac{(\lambda+p)_{k-p}(c)_{k-p}}{(k-p)!(a)_{k-p}}\left[1+\delta(k-1)\right]a_k z^{k-p}\right|$$
$$\geq\left[1+\delta(p-1)\right]-\sum_{k=j+p}^{\infty}\frac{(\lambda+p)_{k-p}(c)_{k-p}}{(k-p)!(a)_{k-p}}\left[1+\delta(k-1)\right]a_k |z|^{k-p}$$
$$\geq\left[1+\delta(p-1)\right]-\sum_{k=j+p}^{\infty}\frac{(\lambda+p)_{k-p}(c)_{k-p}}{(k-p)!(a)_{k-p}}\left[1+\delta(k-1)\right]a_k \geq \frac{j\left[1+\delta(p-1)\right]}{j+p-\alpha}>0.$$

Then we find that

$$\left|\frac{z\left(\ell_{p,j}^{\lambda}(a,c)f(z)\right)'+\delta z^2\left(\ell_{p,j}^{\lambda}(a,c)f(z)\right)''}{(1-\delta)\ell_{p,j}^{\lambda}(a,c)f(z)+\delta z\left(\ell_{p,j}^{\lambda}(a,c)f(z)\right)'}-p\right|$$
$$\leq\frac{\sum_{k=j+p}^{\infty}\frac{(\lambda+p)_{k-p}(c)_{k-p}}{(k-p)!(a)_{k-p}}(k-p)\left[1+\delta(k-1)\right]a_k|z|^{k-p}}{\left[1+\delta(k-1)\right]-\sum_{k=j+p}^{\infty}\frac{(\lambda+p)_{k-p}(c)_{k-p}}{(k-p)!(a)_{k-p}}\left[1+\delta(k-1)\right]a_k|z|^{k-p}} \tag{2.2}$$
$$\leq\frac{\sum_{k=j+p}^{\infty}\frac{(\lambda+p)_{k-p}(c)_{k-p}}{(k-p)!(a)_{k-p}}(k-p)\left[1+\delta(k-1)\right]a_k}{\left[1+\delta(p-1)\right]-\sum_{k=j+p}^{\infty}\frac{(\lambda+p)_{k-p}(c)_{k-p}}{(k-p)!(a)_{k-p}}\left[1+\delta(k-1)\right]a_k}\leq p-\alpha.$$

This shows that the values of the function

$$\phi(z)=\frac{z\left(\ell_{p,j}^{\lambda}(a,c)f(z)\right)'+\delta z^2\left(\ell_{p,j}^{\lambda}(a,c)f(z)\right)''}{(1-\delta)\ell_{p,j}^{\lambda}(a,c)f(z)+\delta z\left(\ell_{p,j}^{\lambda}(a,c)f(z)\right)'} \tag{2.3}$$

lie in a circle which is centered at $w=p$ and whose radius is $(p-\alpha)$. Hence $f(z)$ satisfies the condition (1.6).

Conversely, assume that the function $f(z)$ is in the class $SC_n(j,p,\lambda,\alpha,\delta)$. Then we have

$$\operatorname{Re}\left\{\frac{z\left(\ell_{p,j}^{\lambda}(a,c)f(z)\right)'+\delta z^2\left(\ell_{p,j}^{\lambda}(a,c)f(z)\right)''}{(1-\delta)\ell_{p,j}^{\lambda}(a,c)f(z)+\delta z\left(\ell_{p,j}^{\lambda}(a,c)f(z)\right)'}\right\}$$
$$=\operatorname{Re}\left\{\frac{p\left[1+\delta(p-1)\right]-\sum_{k=j+p}^{\infty}\frac{(\lambda+p)_{k-p}(c)_{k-p}}{(k-p)!(a)_{k-p}}k\left[1+\delta(k-1)\right]a_k z^{k-p}}{\left[1+\delta(p-1)\right]-\sum_{k=j+p}^{\infty}\frac{(\lambda+p)_{k-p}(c)_{k-p}}{(k-p)!(a)_{k-p}}\left[1+\delta(k-1)\right]a_k z^{k-p}}\right\}>\alpha \tag{2.4}$$

for some $\alpha(0\leq\alpha<p)$, some $\delta(0\leq\delta\leq1)$, p, $j\in N$, $n\in N_0$, and $z\in U$. Choose values of z on the real axis so that $\phi(z)$ given by (2.3) is real. Upon clearing the denominator in (2.4) and letting $z\to1^-$ through real values, we can see that

$$p\left[1+\delta(p-1)\right]-\sum_{k=j+p}^{\infty}\frac{(\lambda+p)_{k-p}(c)_{k-p}}{(k-p)!(a)_{k-p}}k\left[1+\delta(k-1)\right]a_k$$
$$\geq\alpha\left\{\left[1+\delta(p-1)\right]-\sum_{k=j+p}^{\infty}\frac{(\lambda+p)_{k-p}(c)_{k-p}}{(k-p)!(a)_{k-p}}\left[1+\delta(k-1)\right]a_k\right\}. \tag{2.5}$$

Thus we have the inequality (2.1).

Corollary 1. Let the function $f(z)$ defined by (1.1) be in the class $SC_n(j,p,\lambda,\alpha,\delta)$. Then

$$a_k \le \frac{(p-\alpha)[1+\delta(p-1)]}{\frac{(\lambda+p)_{k-p}(c)_{k-p}}{(k-p)!(a)_{k-p}}(k-\alpha)[1+\delta(k-1)]} \qquad (2.6)$$

$$(k \ge j+p,\ p,\ j \in N).$$

The result is sharp for the function $f(z)$ given by

$$f(z) = z^p - \frac{(p-\alpha)[1+\delta(p-1)]}{\frac{(\lambda+p)_{k-p}(c)_{k-p}}{(k-p)!(a)_{k-p}}(k-\alpha)[1+\delta(k-1)]} z^k$$

$$(k \ge j+p,\ p,\ j \in N,\ n \in N_0). \qquad (2.7)$$

3. Distortion Theorem

Theorem 2. If a function $f(z)$ defined by (1.1) is in the class $SC_n(j,p,\lambda,\alpha,\delta)$ then

$$\left\{\frac{p!}{(p-m)!} - \frac{(p-\alpha)[1+\delta(p-1)](j+p)!}{\frac{(\lambda+p)_j(c)_j}{j!(a)_j}(j+p-\alpha)[1+\delta(j+p-1)](j+p-m)!}|z|^j\right\}|z|^{p-m}$$
$$\le \left|f^m(z)\right| \le \left\{\frac{p!}{(p-m)!} + \frac{(p-\alpha)[1+\delta(p-1)](j+p)!}{\frac{(\lambda+p)_j(c)_j}{j!(a)_j}(j+p-\alpha)[1+\delta(j+p-1)](j+p-m)!}|z|^j\right\}|z|^{p-m} \qquad (3.1)$$

$$(z \in U;\ 0 \le \alpha < p;\ 0 \le \lambda \le 1,\ p,\ j \in N,\ n \in N_0).$$

The result is sharp for the function $f(z)$ given by

$$f(z) = z^p - \frac{(p-\alpha)[1+\delta(p-1)]}{\frac{(\lambda+p)_j(c)_j}{j!(a)_j}(j+p-\alpha)[1+\delta(j+p-1)]} z^{j+p}.\ (p,\ j \in N,\ n \in N_0). \qquad (3.2)$$

Proof. In view of Theorem 1, we have

$$\frac{\left(\frac{(\lambda+p)_j(c)_j}{j!(a)_j}\right)(j+p-\alpha)[1+\delta(j+p-1)]}{(p-\alpha)[1+\delta(p-1)](j+p)!}\sum_{k=j+p}^{\infty} k!a_k \le \sum_{k=j+p}^{\infty}\frac{\frac{(\lambda+p)_{k-p}(c)_{k-p}}{(k-p)!(a)_{k-p}}(k-\alpha)[1+\delta(k-1)]}{(p-\alpha)[1+\delta(p-1)]}a_k \le 1$$

which readily yields

$$\sum_{k=j+p}^{\infty} k!a_k \le \frac{(p-\alpha)[1+\delta(p-1)](j+p)!}{\frac{(\lambda+p)_j(c)_j}{j!(a)_j}(j+p-\alpha)[1+\delta(j+p-1)]}. \qquad (3.3)$$

Now, by differentiating both sides of (1.1) m times, we obtain

$$f^{(m)}(z) = \frac{p!}{(p-m)!}z^{p-m} - \sum_{k=j+p}^{\infty}\frac{k!}{(k-m)!}a_k z^{k-m} \qquad (3.4)$$

$$(k \ge j+p,\ j,\ p \in N).$$

Theorem 2, follows from (3.3) and (3.4).

Finally, it is easy to see that the bounds in (3.1) are attained for the function $f(z)$ given by (3.2).

4. Radii of Close-to-Convexity, Starlikeness and Convexity

Theorem 3. Let the function $f(z)$ defined by (1.1) be in the class $SC_n(j,p,\lambda,\alpha,\delta)$ then

1) $f(z)$ is *p*-valently close-to-convex of order

$\varphi(0 \le \varphi < p)$ in $|z| < r_1$, where

$$r_1 = \inf_k \left\{ \frac{\frac{(\lambda+p)_{k-p}(c)_{k-p}}{(k-p)!(a)_{k-p}}(k-\alpha)[1+\delta(k-1)]}{(p-\alpha)[1+\delta(p-1)]} \left(\frac{p-\varphi}{k}\right) \right\}^{\frac{1}{k-p}}, \ (k \ge j+p,\ p,\ j \in N,\ n \in N_0), \tag{4.1}$$

2) $f(z)$ is p-valently starlike of order $\phi(0 \le \varphi < p)$ in $|z| < r_2$, where

$$r_2 = \inf_k \left\{ \frac{\frac{(\lambda+p)_{k-p}(c)_{k-p}}{(k-p)!(a)_{k-p}}(k-\alpha)[1+\delta(k-1)]}{(p-\alpha)[1+\delta(p-1)]} \left(\frac{p-\varphi}{k-\varphi}\right) \right\}^{\frac{1}{k-p}}, \ (k \ge j+p\ p,\ j \in N) \tag{4.2}$$

3) $f(z)$ is p-valently convex of order $\varphi(0 \le \varphi < p)$ in $|z| < r_3$, where

$$r_3 = \inf_k \left\{ \frac{\frac{(\lambda+p)_{k-p}(c)_{k-p}}{(k-p)!(a)_{k-p}}(k-\alpha)[1+\delta(k-1)]}{(p-\alpha)[1+\delta(p-1)]} \cdot \frac{p(p-\varphi)}{k(k-\varphi)} \right\}^{\frac{1}{k-p}}, \ (k \ge j+p,\ j,\ p \in N). \tag{4.3}$$

Each of these results is sharp for the function $f(z)$ given by (2.7).

Proof. It is sufficient to show that

$$\left|\frac{f'(z)}{z^{p-1}} - \varphi\right| \le p - \varphi \quad \left(|z| < r_1;\ 0 \le \varphi < p,\ p \in N\right), \tag{4.4}$$

$$\left|\frac{zf'(z)}{f(z)} - p\right| \le p - \varphi \quad \left(|z| < r_2;\ 0 \le \varphi < p,\ p \in N\right), \tag{4.5}$$

and

$$\left|1 + \frac{zf''(z)}{f'(z)} - p\right| \le p - \varphi \quad \left(|z| < r_3;\ 0 \le \varphi < p,\ p \in N\right), \tag{4.6}$$

for a function $f(z) \in SC_n(j,p,\lambda,\alpha,\delta)$ where r_1, r_2 and r_3 are defined by (4.1) - (4.3) respectively. The details involved are fairly straightforward and may omitited.

5. Modified Hadamard Products

For the functions $f_i(z)(i=1,2)$ defined by

$$f_i(z) = z^p - \sum_{k=j+p}^{\infty} a_{k,i} z^k \ \left(a_{k,i} \ge 0;\ i = 1,2\right), \tag{5.1}$$

we denote by $(f_1 * f_2)(z)$ the modified Hadamard product (or convolution) of the functions $f_1(z)$ and $f_2(z)$, defined by

$$(f_1 * f_2)(z) = z^p - \sum_{k=j+p}^{\infty} a_{k,1} a_{k,2} z^k. \tag{5.2}$$

Theorem 4. Let the functions $f_i(z)(i=1,2)$ defined by (5.1) be in the class $SC_n(j,p,\lambda,\alpha,\delta)$ then $(f_1 * f_2)(z) \in SC_n(j,p,\lambda,\gamma,\delta)$, where

$$\gamma = p - \frac{j(p-\alpha)^2[1+\delta(p-1)]}{\frac{(\lambda+p)_j(c)_j}{j!(a)_j}(j+p-\alpha)^2[1+\delta(j+p-1)] - (p-\alpha)^2[1+\delta(p-1)]}. \tag{5.3}$$

The result is sharp for the functions $f_i(z)(i=1,2)$ given by

$$f_i(z) = z^p - \frac{(p-\alpha)[1+\delta(p-1)]}{\frac{(\lambda+p)_j(c)_j}{j!(a)_j}(j+p-\alpha)[1+\delta(j+p-1)]} z^{j+p}, \ (p,\ j \in N,\ i = 1,2). \tag{5.4}$$

Proof. Employing the technique used earlier by Schild and Silverman [12], we need to find the largest γ such that

$$\sum_{k=j+p}^{\infty} \frac{\frac{(\lambda+p)_{k-p}(c)_{k-p}}{(k-p)!(a)_{k-p}}(k-\gamma)\left[1+\delta(k-1)\right]}{(p-\gamma)\left[1+\delta(p-1)\right]} a_{k,1}a_{k,2} \le 1. \tag{5.5}$$

Since $f_i(z) \in SC_n(j,p,\lambda,\alpha,\delta)(i=1,2)$, we readily see that

$$\sum_{k=j+p}^{\infty} \frac{\frac{(\lambda+p)_{k-p}(c)_{k-p}}{(k-p)!(a)_{k-p}}(k-\alpha)\left[1+\delta(k-1)\right]}{(p-\alpha)\left[1+\delta(p-1)\right]} a_{k,i} \le 1 \quad (i=1,2). \tag{5.6}$$

Therefore, by the Cauchy-Schwarz inequality, we obtain

$$\sum_{k=j+p}^{\infty} \frac{\frac{(\lambda+p)_{k-p}(c)_{k-p}}{(k-p)!(a)_{k-p}}(k-\alpha)\left[1+\delta(k-1)\right]}{(p-\alpha)\left[1+\delta(p-1)\right]} \sqrt{a_{k,1}a_{k,2}} \le 1. \tag{5.7}$$

Thus we only need to show that

$$\frac{(k-\gamma)}{(p-\gamma)} a_{k,1}\cdot a_{k,2} \le \frac{(k-\alpha)}{(p-\alpha)}\sqrt{a_{k,1}\cdot a_{k,2}} \quad (k \ge j+p,\ p,\ j\in N), \tag{5.8}$$

or, equivalently, that

$$\sqrt{a_{k,1}\cdot a_{k,2}} \le \frac{(p-\gamma)(k-\alpha)}{(p-\alpha)(k-\gamma)} \quad (k \ge j+p,\ p,\ j\in N), \tag{5.9}$$

Hence, in light of the inequality (5.7), it is sufficient to prove that

$$\frac{(p-\alpha)\left[1+\delta(p-1)\right]}{\frac{(\lambda+p)_{k-p}(c)_{k-p}}{(k-p)!(a)_{k-p}}(k-\alpha)\left[1+\delta(k-1)\right]} \le \frac{(p-\gamma)(k-\alpha)}{(p-\alpha)(k-\gamma)},\ (k \ge j+p,\ p,\ j\in N), \tag{5.10}$$

It follows from (5.10) that

$$\gamma \le p - \frac{(k-p)(p-\alpha)^2\left[1+\delta(p-1)\right]}{\frac{(\lambda+p)_{k-p}(c)_{k-p}}{(k-p)!(a)_{k-p}}(k-\alpha)^2\left[1+\delta(k-1)\right]-(p-\alpha)^2\left[1+\delta(p-1)\right]},\ (k \ge j+p,\ j,\ p\in N). \tag{5.11}$$

Now, defining the function $G(k)$ by

$$G(k) = p - \frac{(k-p)(p-\alpha)^2\left[1+\delta(p-1)\right]}{\frac{(\lambda+p)_{k-p}(c)_{k-p}}{(k-p)!(a)_{k-p}}(k-\alpha)^2\left[1+\delta(k-1)\right]-(p-\alpha)^2\left[1+\delta(p-1)\right]},\ (k \ge j+p,\ j,\ p\in N). \tag{5.12}$$

We see that $G(k)$ is an increasing function of *k*. Therefore, we conclude that

$$\gamma \le G(j+p) = p - \frac{j(p-\alpha)^2\left[1+\delta(p-1)\right]}{\frac{(\lambda+p)_j(c)_j}{j!(a)_j}(j+p-\alpha)^2\left[1+\delta(j+p-\alpha)\right]-(p-\alpha)^2\left[1+\delta(p-1)\right]}, \tag{5.13}$$

which evidently completes the proof of Theorem 4.

Remark: Putting 1) $a=p+1$, $\lambda=1$, $c=1$, $\delta=0$ and 2) $a=p+1$, $\lambda=1$, $c=1$, $\delta=1$ in Theorem 4, we obtain

Corollary 2. Let the functions $f_i(z)(i=1,2)$ defined by (5.1) be in the class $T_j^*(p,\alpha)$. Then $(f_1 * f_2)(z) \in T_j^*(p,\gamma)$, where

$$\gamma = p - \frac{j(p-\alpha)^2}{(j+p-\alpha)^2-(p-\alpha)^2}. \tag{5.14}$$

The result is sharp.

Corollary 3. Let the functions $f_i(z)(i=1,2)$ defined by (5.1) be in the class $C_j(p,\alpha)$. Then $(f_1 * f_2)(z) \in C_j(p,\gamma)$, where

$$\gamma = p - \frac{j(p-\alpha)^2 p}{(j+p-\alpha)^2(1+j+p-\alpha)-(p-\alpha)^2 p}. \tag{5.15}$$

The result is sharp.

Using arguments similar to those in the proof of

Theorem 4, we obtain the following result.

Theorem 5. Let the function $f_1(z)$ defined by (5.1) be in the class $SC_n(j,p,\lambda,\alpha,\delta)$ Suppose also that the function $f_2(z)$ defined by (5.1) be in the class $SC_n(j,p,\lambda,\tau,\delta)$, Then $(f_1 * f_2)(z) \in SC_n(j,p,\lambda,\zeta,\delta)$, where

$$\zeta = p - \frac{j(p-\alpha)(p-\tau)[1+\delta(p-1)]}{\frac{(\lambda+p)_{k-p}(c)_{k-p}}{(k-p)!(a)_{k-p}}(j+p-\alpha)(j+p-\tau)[1+\delta(j+p-1)] - \Omega}, \tag{5.16}$$

and

$$\Omega = (p-\alpha)(p-\tau)[1+\delta(p-1)]. \tag{5.17}$$

The result is the best possible for the functions

$$f_1(z) = z^p - \frac{(p-\alpha)[1+\delta(p-1)]}{\frac{(\lambda+p)_j(c)_j}{j!(a)_j}(j+p-\alpha)[1+\delta(j+p-1)]} z^{j+p}, \ (p, j \in N) \tag{5.18}$$

and

$$f_2(z) = z^p \frac{(p-\tau)[1+\delta(p-1)]}{\frac{(\lambda+p)_j(c)_j}{j!(a)_j}(j+p-\tau)[1+\delta(j+p-1)]} z^{j+p}, \ (p, j \in N). \tag{5.19}$$

Theorem 6. Let the functions $f_i(z)(i=1,2,\cdots,m)$ defined by (5.1) be in the class $SC_n(j,p,\lambda,\alpha,\delta)$. Then the function

$$h(z) = z^p - \sum_{k=j+p}^{\infty}\left(\sum_{i=1}^{\infty} a_{k,i}^2\right) z^k, \tag{5.20}$$

belongs to the class $SC_n(j,p,\lambda,\zeta,\delta)$ where

$$\zeta = p - \frac{jm(p-\alpha)^2[1+\delta(p-1)]}{\frac{(\lambda+p)_j(c)_j}{j!(a)_j}(j+p-\alpha)^2[1+\delta(j+p-1)] - m(p-\alpha)^2[1+\delta(p-1)]}. \tag{5.21}$$

The result is sharp for the functions $f_i(z)(i=1,2,\cdots,m)$ given by (5.4).

Proof. Noting that

$$\sum_{k=j+p}^{\infty}\left\{\frac{\frac{(\lambda+p)_{k-p}(c)_{k-p}}{(k-p)!(a)_{k-p}}(k-\alpha)[1+\delta(k-1)]}{(p-\alpha)[1+\delta(p-1)]}\right\}^2 a_{k,i}^2 \le \left\{\sum_{k=j+p}^{\infty}\frac{\frac{(\lambda+p)_{k-p}(c)_{k-p}}{(k-p)!(a)_{k-p}}(k-\alpha)[1+\delta(k-1)]}{(p-\alpha)[1+\delta(p-1)]} a_{k,i}\right\}^2 \le 1, \tag{5.22}$$

$$\left(f_i(z) \in SC_n(j,p,\lambda,\alpha,\delta)(i=1,2,\cdots,m)\right)$$

we have

$$\sum_{k=j+p}^{\infty}\frac{1}{m}\left\{\frac{\frac{(\lambda+p)_{k-p}(c)_{k-p}}{(k-p)!(a)_{k-p}}(k-\alpha)[1+\delta(k-1)]}{(p-\alpha)[1+\delta(p-1)]}\right\}^2\left(\sum_{i=1}^{m} a_{k,i}^2\right) \le 1. \tag{5.23}$$

Therefore, we have find largest ζ such that

$$\frac{(k-\zeta)}{(p-\zeta)} \leq \frac{\frac{(\lambda+p)_{k-p}(c)_{k-p}}{(k-p)!(a)_{k-p}}(k-\alpha)^2[1+\delta(k-1)]}{m(p-\alpha)^2[1+\delta(p-1)]} \quad (k \geq j+p,\ p,\ j \in N), \tag{5.24}$$

that is, that

$$\zeta \leq p - \frac{m(k-p)(p-\alpha)^2[1+\delta(p-1)]}{\frac{(\lambda+p)_{k-p}(c)_{k-p}}{(k-p)!(a)_{k-p}}(k-\alpha)^2[1+\delta(k-1)] - m(p-\alpha)^2[1+\delta(p-1)]}, \quad (k \geq j+p,\ p,\ j \in N). \tag{5.25}$$

Now, defining the function $\psi(k)$ by

$$\psi(k) = p - \frac{m(k-p)(p-\alpha)^2[1+\delta(p-1)]}{\frac{(\lambda+p)_{k-p}(c)_{k-p}}{(k-p)!(a)_{k-p}}(k-\alpha)^2[1+\delta(k-1)] - m(p-\alpha)^2[1+\delta(p-1)]}, \quad (k \geq j+p,\ p,\ j \in N) \tag{5.26}$$

we observe that $\psi(k)$ is an increasing function of *k*. We thus conclude that

$$\zeta \leq \psi(j+p) = p - \frac{mj(p-\alpha)^2[1+\delta(p-1)]}{\frac{(\lambda+p)_j(c)_j}{j!(a)_j}(j+p-\alpha)^2[1+\delta(j+p-1)] - m(p-\alpha)^2[1+\delta(p-1)]}, \tag{5.27}$$

which completes the proof of Theorem 6.

6. Applications of Fractional Calculus

Various operators of fractional calculus (that is, fractional integral and fractional derivatives) have been studied in (cf., e.g., [9,10,13-15]; see also the various references cited therein).

For our present investigation, we recall the following definitions.

Definition 1. The fractional integral of order μ is defined, for a function $f(z)$, by

$$D_z^{-\mu} f(z) = \frac{1}{\Gamma(\mu)} \int_0^z \frac{f(\zeta)}{(z-\zeta)^{1-\mu}} \mathrm{d}\zeta \quad (\mu > 0), \tag{6.1}$$

where the function $f(z)$ is analytic in a simply-connected domain of the complex z-plane containing the origin and the multiplicity of $(z-\zeta)^{\mu-1}$ is removed by requiring $\log(z-\zeta)$ to be real when $(z-\zeta) > 0$.

Definition 2. The fractional derivative of order μ is defined, for a function $f(z)$, by

$$D_z^{\mu} f(z) = \frac{1}{\Gamma(1-\mu)} \int_0^z \frac{f(\zeta)}{(z-\zeta)^{\mu}} \mathrm{d}\zeta \quad (0 \leq \mu < 1), \tag{6.2}$$

where the function $f(z)$ is constrained, and the multiplicity of $(z-\zeta)^{-\mu}$ is removed, as in Definition 1.

Definition 3. Under the hypotheses of Definition 2, the fractional derivative of order $n+\mu$ is defined, for a function $f(z)$, by

$$D_z^{n+\mu} f(z) = \frac{d^n}{dz^n}\left\{D_z^{\mu} f(z)\right\} \quad (0 \leq \mu < 1, n \in N_0). \tag{6.3}$$

In this section, we shall investigate the growth and distortion properties of functions in the class $SC_n(j, p, \lambda, \alpha, \delta)$ involving the operators $J_{c,p}$ and D_z^{μ}. In order to derive our results, we need the following Lemma given by Chen *et al.* [14].

Lemma 1 (see [14]). Let the function $f(z)$ defined by (1.1). Then

$$D_z^{\mu}\left\{J_{c,p} f(z)\right\} = \frac{\Gamma(p+1)}{\Gamma(p+1-\mu)} z^{p-\mu} - \sum_{k=j+p}^{\infty} \frac{(c+p)\Gamma(k+1)}{(c+k)\Gamma(k+1-\mu)} a_k z^{k-\mu}, \quad (\mu \in R; c > -p,\ p,\ j \in N) \tag{6.4}$$

and

$$J_{c,p}\left(D_z^{\mu}\left\{f(z)\right\}\right) = \frac{(c+p)\Gamma(p+1)}{(c+p-\mu)\Gamma(p+1-\mu)} z^{p-\mu} - \sum_{k=j+p}^{\infty} \frac{(c+p)\Gamma(k+1)}{(c+k-\mu)\Gamma(k+1-\mu)} a_k z^{k-\mu}, \quad (\mu \in R; c > -p,\ p,\ j \in N) \tag{6.5}$$

provided that no zeros appear in the denominators in (6.4) and (6.5).

Theorem 7. Let the function $f(z)$ defined by (1.1) be in the class $SC_n(j,p,\lambda,\alpha,\delta)$. Then

$$\left|D_z^{-\mu}\left\{\left(J_{c,p}f\right)(z)\right\}\right| \ge \left\{\frac{\Gamma(p+1)}{\Gamma(p+1-\mu)} - \frac{(c+p)\Gamma(j+p+1)(p-\alpha)\left[1+\delta(p-1)\right]}{(c+j+p)\Gamma(j+p+1+\mu)\frac{(\lambda+p)_j(c)_j}{j!(a)_j}(j+p-\alpha)\left[1+\delta(j+p-1)\right]}|z|^j\right\}|z|^{p+\mu},$$

$$\left(z\in U;\ 0\le\alpha<p;\ 0\le\delta\le 1;\ \mu>0;\ c>-p,\ p,\ j\in N\right) \tag{6.6}$$

and

$$\left|D_z^{-\mu}\left\{\left(J_{c,p}f\right)(z)\right\}\right| \le \left\{\frac{\Gamma(p+1)}{\Gamma(p+1+\mu)} + \frac{(c+p)\Gamma(j+p+1)(p-\alpha)\left[1+\delta(p-1)\right]}{(c+j+p)\Gamma(j+p+1+\mu)\frac{(\lambda+p)_j(c)_j}{j!(a)_j}(j+p-\alpha)\left[1+\delta(j+p-1)\right]}|z|^j\right\}|z|^{p+\mu},$$

$$\left(z\in U;\ 0\le\alpha<p;\ 0\le\delta\le 1;\ \mu>0;\ c>-p,\ p,\ j\in N\right). \tag{6.7}$$

Each of the assertion (6.6) and (6.7) is sharp.

Proof. In view of Theorem 1, we have

$$\frac{\frac{(\lambda+p)_j(c)_j}{j!(a)_j}(j+p-\alpha)\left[1+\delta(j+p-1)\right]}{(p-\alpha)\left[1+\delta(p-1)\right]}\sum_{k=j+p}^{\infty}a_k \le \sum_{k=j+p}^{\infty}\frac{\frac{(\lambda+p)_{k-p}(c)_{k-p}}{(k-p)!(a)_{k-p}}(j+p-\alpha)\left[1+\delta(k-1)\right]}{(p-\alpha)\left[1+\delta(p-1)\right]}a_k \le 1, \tag{6.8}$$

which readily yields

$$\sum_{k=j+p}^{\infty}a_k \le \frac{(p-\alpha)\left[1+\delta(p-1)\right]}{\frac{(\lambda+p)_j(c)_j}{j!(a)_j}(j+p-\alpha)\left[1+\delta(j+p-1)\right]}. \tag{6.9}$$

Consider the function $f(z)$ defined in U by

$$F(z)=\frac{\Gamma(p+1+\mu)}{\Gamma(p+1)}z^{-\mu}D_z^{-\mu}\left\{\left(J_{c,p}f\right)(z)\right\} = z^p-\sum_{k=j+p}^{\infty}\frac{(c+p)\Gamma(k+1)\Gamma(p+1+\mu)}{(c+k)\Gamma(k+1+\mu)\Gamma(p+1)}a_kz^k$$

$$=z^p-\sum_{k=j+p}^{\infty}\phi(k)a_kz^k\quad(z\in U),$$

where

$$\phi(k)=\frac{(c+p)\Gamma(k+1)\Gamma(p+1+\mu)}{(c+k)\Gamma(k+1+\mu)\Gamma(p+1)}\ \left(k\ge j+p,\ p,\ j\in N,\ \mu>0\right). \tag{6.10}$$

Since $\phi(k)$ is a decreasing function of k when $\mu>0$, we get

$$0<\phi(k)\le\phi(j+p)=\frac{(c+p)\Gamma(j+p+1)\Gamma(p+1+\mu)}{(c+j+p)\Gamma(j+p+1+\mu)\Gamma(p+1)}\ \left(c>-p,\ p,\ j\in N,\ \mu>0\right). \tag{6.11}$$

Thus, by using (6.9) and (6.11), we deduce that

$$|F(z)|\ge|z|^p-\phi(j+p)|z|^{j+p}\sum_{k=j+p}^{\infty}a_k$$

$$\ge|z|^p-\frac{(c+p)\Gamma(j+p+1)\Gamma(p+1+\mu)(p-\alpha)\left[1+\delta(p-1)\right]}{(c+j+p)\Gamma(j+p+1+\mu)\Gamma(p+1)\frac{(\lambda+p)_j(c)_j}{j!(a)_j}(j+p-\alpha)\left[1+\delta(j+p-1)\right]}|z|^{j+p},$$

and

$$\begin{aligned}\left|F(z)\right| &\le |z|^p + \phi(j+p)|z|^{j+p}\sum_{k=j+p}^{\infty} a_k \\ &\le |z|^p + \frac{(c+p)\Gamma(j+p+1)\Gamma(p+1+\mu)(p-\alpha)\left[1+\delta(p-1)\right]}{(c+j+p)\Gamma(j+p+1+\mu)\Gamma(p+1)\frac{(\lambda+p)_j (c)_j}{j!(a)_j}(j+p-\alpha)\left[1+\delta(j+p-1)\right]}|z|^{j+p},\end{aligned}$$

which yield the inequalities (6.6) and (6.7) of Theorem 7. The equalities in (6.6) and (6.7) are attained for the function $f(z)$ given by

$$D_z^{-\mu}\left\{\left(J_{c,p}f\right)(z)\right\} = \left\{\frac{\Gamma(p+1)}{\Gamma(p+1+\mu)} - \frac{(c+p)(p-\alpha)\Gamma(j+p+1)\left[1+\delta(p-1)\right]}{(c+j+p)\Gamma(j+p+1+\mu)\frac{(\lambda+p)_j (c)_j}{j!(a)_j}(j+p-\alpha)\left[1+\delta(j+p-1)\right]}|z|^j\right\} z^{p+\mu} \tag{6.12}$$

or, equivalently, by

$$\left(J_{c,p}f\right)(z) = z^p - \frac{(c+p)(p-\alpha)\left[1+\delta(p-1)\right]}{(c+j+p)\Gamma(j+p+1+\mu)\frac{(\lambda+p)_j (c)_j}{j!(a)_j}(j+p-\alpha)\left[1+\delta(j+p-1)\right]} z^{j+p}. \tag{6.13}$$

Thus we complete the proof of Theorem 7.

Using arguments similar to those in the proof of Theorem 7, we obtain the following result.

Theorem 8. Let the function $f(z)$ defined by (1.1) be in the class $SC_n(j,p,\lambda,\alpha,\delta)$. Then

$$\left|D_z^{\mu}\left\{\left(J_{c,p}f\right)(z)\right\}\right| \ge \left\{\frac{\Gamma(p+1)}{\Gamma(p+1+\mu)} - \frac{(c+p)\Gamma(j+p+1)(p-\alpha)\left[1+\delta(p-1)\right]}{(c+j+p)\Gamma(j+p+1-\mu)\frac{(\lambda+p)_j (c)_j}{j!(a)_j}(j+p-\alpha)\left[1+\delta(j+p-1)\right]}|z|^j\right\}|z|^{p-\mu},$$
$$\left(z\in U;\ 0\le\alpha<p,\ 0\le\delta\le 1,\ 0\le\mu<1,\ c>-p,\ j,\ p\in N\right), \tag{6.14}$$

and

$$\left|D_z^{\mu}\left\{\left(J_{c,p}f\right)(z)\right\}\right| \le \left\{\frac{\Gamma(p+1)}{\Gamma(p+1-\mu)} + \frac{(c+p)\Gamma(j+p+1)(p-\alpha)\left[1+\delta(p-1)\right]}{(c+j+p)\Gamma(j+p+1-\mu)\frac{(\lambda+p)_j (c)_j}{j!(a)_j}(j+p-\alpha)\left[1+\delta(j+p-1)\right]}|z|^j\right\}|z|^{p-\mu},$$
$$\left(z\in U;\ 0\le\alpha<p,\ 0\le\delta\le 1,\ 0\le\mu<1,\ j,\ p\in N\right). \tag{6.15}$$

Each of the assertions (6.14) and (6.15) is sharp.

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