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Positive Solutions to the Nonhomogenous p-Laplacian Problem with Nonlinearity Asymptotic to u^{p-1} at Infinity in $\mathbb{R}^N$

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Abstract

In this paper, we study the following problem

$$\begin{cases} -\Delta_p u + V(x)|u|^{p-2}u = K(x)f(u) + h(x) & \text{in } \mathbb{R}^N, \\ u \in W^{1,p}(\mathbb{R}^N), \quad u > 0 & \text{in } \mathbb{R}^N, \end{cases} \qquad (*)$$

where $1 < p < N$, the potential $V(x)$ is a positive bounded function, $h \in L^{p'}(\mathbb{R}^N)$, $\frac{1}{p'} + \frac{1}{p} = 1,\ 1 < p < N,\ h \ge 0,$ $h \not\equiv 0$, $f(s)$ is nonlinearity asymptotical to s^{p-1} at infinity, that is, $f(s) \sim O(s^{p-1})$ as $s \to +\infty$. The aim of this paper is to discuss how to use the Mountain Pass theorem to show the existence of positive solutions of the present problem. Under appropriate assumptions on V, K, h and f, we prove that problem (*) has at least two positive solutions even if the nonlinearity $f(s)$ does not satisfy the Ambrosetti-Rabinowitz type condition:

$$0 \le F(u) \le \int_0^u f(s)\,\mathrm{d}s \le \frac{1}{p+\theta} f(u)u,\ u > 0,\ \theta > 0.$$

Keywords: Positive Solutions, p-Laplacian, Nonlinearity Asymptotic, Mountain Pass Theorem

1. Introduction and Preliminaries

In this paper, we study the following problem

$$\begin{cases} -\Delta_p u + V(x)|u|^{p-2}u = K(x)f(u) + h(x) & \text{in } \mathbb{R}^N, \\ u \in W^{1,p}(\mathbb{R}^N), \quad u > 0 & \text{in } \mathbb{R}^N, \end{cases} \qquad (1.1)$$

where

$$\Delta_p u = \operatorname{div}\left(|\nabla u|^{p-2}\nabla u\right),\ h \in L^{p'}(\mathbb{R}^N),$$
$$\frac{1}{p'} + \frac{1}{p} = 1,\ 1 < p < N,\ h \ge 0,\ h \ne 0,$$

and the function V, K and f satisfy the following conditions:

(V_1) $V: \mathbb{R}^N \to \mathbb{R}$ is continuous and there exist $a, \alpha, A, B > 0$ such that

$$\frac{a}{1+|x|^\alpha} + \beta \le V(x) \le A.$$

(F_1) $f(t) \in C(\mathbb{R}, \mathbb{R}^+)$, $f(t) \equiv 0$ if $t \le 0$.

(F_2) $\lim\limits_{t \to 0^+} \frac{f(t)}{t^{p-1}} = 0$.

(F_3) There exists $q \in (p, p^* - 1)$,such that $\lim\limits_{n \to \infty} \frac{f(t)}{t^q} = 0$, where $p^* = \frac{Np}{N-p}$.

(F_4) $\lim\limits_{t \to +\infty} \frac{f(t)}{t^{p-1}} = l \in (1, +\infty)$.

(K_1) K is a positive continuous bounded function and there exists $R_0 > 0$ such that

$$\sup\left\{\frac{f(s)}{s^{p-1}} : s > 0\right\} < \inf\left\{\frac{V(x)}{K(x)} : |x| \ge R_0\right\}.$$

Throughout this paper, we define the following Weighted Sobolev space

$$W = \left\{u \in \mathcal{D}^{1,p}(\mathbb{R}^N) : \int_{\mathbb{R}^N}\left[|\nabla u|^p + V(x)u^p\right]\mathrm{d}x < +\infty\right\}$$

Clearly, $W^{1,p}(\mathbb{R}^N) \subset W$. W is a Hilbert space with its scalar product and norm are given by

$$(u,v) = \int_{\mathbb{R}^N} \left[|\nabla u|^{p-2} \nabla u \nabla v + V(x) u^{p-1} v \right] dx$$
$$\text{and } \|u\|^p = \int_{\mathbb{R}^N} \left[|\nabla u|^p + V(x) u^p \right] dx,$$

because of (V_1) it is equivalent to the standard $W^{1,p}(\mathbb{R}^N)$ norm. So, We associate with (1.1) the functional $I : W \to \mathbb{R}$ defined by

$$I(u) = \frac{1}{p} \int_{\mathbb{R}^N} \left[|\nabla u|^p + V(x) u^p \right] dx - \int_{\mathbb{R}^N} K(x) F(u) dx - \int_{\mathbb{R}^N} h(x) u dx, \quad (1.2)$$

where $F(u) = \int_0^u f(t) dt$. By (V_1) and (K_1) there exists $C_0 > 0$ such that

$$K(x) \le C_0 V(x), \text{ for all } x \in \mathbb{R}^N. \quad (1.3)$$

Thus, I is well defined on W and $I \in C^1(W, \mathbb{R})$ with

$$\langle I'(u), v \rangle = \int_{\mathbb{R}^N} \left[|\nabla u|^{p-2} \nabla u \nabla v + V(x) u^{p-1} v \right] dx - \int_{\mathbb{R}^N} K(x) f(u) v dx - \int_{\mathbb{R}^N} h(x) v dx,$$

for all $v \in W$. We also use the notation:

$|u|_t = \left(\int_{\mathbb{R}^N} |u|^t dx \right)^{\frac{1}{t}}$ for all $t \in (1, +\infty)$.

Under the conditions (F_1) and (F_2), we are able to prove I has a Mountain Pass geometry. Namely setting

$$\Gamma = \left\{ \gamma \in C([0,1], W), \gamma(0) = 0, \text{ and } I(\gamma(1)) < 0 \right\},$$

we have $\Gamma \ne \varnothing$ and $c = \inf_{\gamma \in \Gamma} \max_{t \in [0,1]} I(\gamma(t)) > 0.$

The value $c \in \mathbb{R}$ is called the Mountain Pass level for I. Ekeland's variational principle implies that there exists a Cerami sequence at c, namely a sequence $\{u_n\} \subset W$ such that

$$I(u_n) \to c \text{ and } \|I'(u_n)\|_{W^*} (1 + \|u_n\|) \to 0 \text{ as } n \to +\infty, \quad (1.4)$$

where W^* denotes the dual space of W. At this point, to get an existence result, it clearly suffices to show that $\{u_n\}$ is bounded and then that $\{u_n\}$ has a strongly convergent subsequence whose limit is a non-trivial critical point of I. These two steps consist the heart of the proofs of Theorems 1.1 below.

For problems like (1.1) as $p = 2$, in most works, the following superlinear condition of $f(t)$, the so-called Ambrosetti-Rabinowitz type condition is assumed

$$0 \le F(u) \le \int_0^u f(s) ds \le \frac{1}{p+\theta} f(u) u, \ u > 0, \ \theta > 0. \quad (1.5)$$

Our equation does not satisfy (1.5) under assumption of (F_4). The difficulty to prove that $\{u_n\}$ is bounded is linked to the fact that we are considering an nonlinearity asymptotically problem.

There are a few works on asymptotically linear problems on unbounded domains. The first result is due to Stuart and Zhou [1]. They study a problem of the type of

$$-\Delta u + V(x) u = f(u), \ x \in \mathbb{R}^N, \quad (1.6)$$

assuming that it has a radial symmetry. Thanks to this assumption, the problem is somehow set in $\mathbb{R}$ and possesses a stronger compactness. Moreover in [2], a problem of the form

$$-\Delta u + K(x) u = f(x,u), \ x \in \mathbb{R}^N$$

is studied, where $K > 0$ is a constant and $f(x,s)$ is asymptotically linear in s and periodic in $x \in \mathbb{R}^N$. Subsequently, taking advantages of some techniques introduced in [3], an extended study of radially symmetric problems on $\mathbb{R}^N$ was done in [4]. Jeanjean *et al.* in [5] discussed (1.6) under some different conditions of $V(x)$ and $f(u)$, it gives results that (1.6) has a positive solution. Recently, under the assumptions (V_1) as $\beta = 0$ with $0 < \alpha < 2$ and (K_2): $K : \mathbb{R}^N \to \mathbb{R}$ is smooth and there exist $k, \beta > 0$ such that $0 < K(x) \le \dfrac{k}{1 + |x|^\beta}$, Ambrosetti *et al.* in [6] proved that problem

$$\begin{cases} -\Delta u + V(x) u = K(x) f(u) & \text{in} \mathbb{R}^N, \\ u \in H^1(\mathbb{R}^N), \ u > 0 & \text{in } \mathbb{R}^N, \ N \ge 3, \end{cases} \quad (1.7)$$

has a bound state for $f(u) = u^p$ with $\sigma < p < \dfrac{N+2}{N-2}$ and

$$\sigma = \begin{cases} \dfrac{N+2}{N-2} - \dfrac{4\beta}{\alpha(N-2)}, & 0 < \beta < \alpha, \\ 1, & \beta \ge \alpha. \end{cases} \quad (1.8)$$

Moreover, it is also proved in [6] that, if $f(u) = u^p$ in (1.7), then the restriction of $\sigma < p < \dfrac{N+2}{N-2}$ is necessary to get a ground state (*i.e.* a least energy solution) of (1.7). Liu *et al.* in [7] showed that (1.7) has a bound state and ground state solution if f is asymptotically linear at infinity and other assumptions of V, K and f.

Similar to [8,9] considered the problem

$$\begin{cases} -\Delta u + u = K(x) f(u) + h(x) & \text{in } \mathbb{R}^N, \\ u \in H^1(\mathbb{R}^N), \quad u > 0, \quad N \ge 3, \end{cases} \quad (1.9)$$

with $\lim_{t\to\infty}\frac{f(t)}{t}=l\le+\infty$. It studied the problem by the case $l<+\infty$ and $l=+\infty$ to obtain the multiple positive solutions in these two cases.

Our result is motivated by some work on the existence of positive solutions for asymptotically linear Schrodinger equations as well as by some ideas used for bounded domain problems. Positive solutions of nonlinear elliptic problems on a bounded domain have been much studied (see, for example [3,10,11,12]). But to our best knowledge, it seems that there few results about (1.1) which is a p-laplacian equation with nonlinearity asymptotic to u^{p-1} at infinity in $\mathbb{R}^N$. In this paper, we shall extend the results of [9] to the more general case. As is known, to seek a weak solution of (1.1) is equivalent to find a nonzero critical point of I in W, so by the Ekeland's variational principle [13], we can get a weak solution u_0 for $h\in L^{p'}(\mathbb{R}^N)$ suitably small easily. Moreover, u_0 is the local minimizer of I and $I(u_0)<0$. However, under our assumptions it seems difficult to get a second solution (different from u_0) of (1.1) by applying the Mountain Pass theorem. Since we lose the (AR) condition, we must overcome the difficulty of the lack of a priori bound in *W* for Palais-Smale sequences. On the other hand, once a (PS) sequence is bounded in W, it also has some difficulties to show this sequence converges to a different solution from u_0. When $l=\infty$, it seems difficult to get the boundness result of $\{u_n\}\subset W$, so we only discuss the case $l<\infty$ and obtain Theorem 1.1:

Theorem 1.1. Suppose that $h\in L^{p'}(\mathbb{R}^N)$, $\frac{1}{p'}+\frac{1}{p}=1$, $1<p<N$, $h\ge0$, $h\not\equiv0$. Let $(V_1),(F_1)-(F_4)$, (K_1) be hold and $l>\mu^*$ with

$$\mu^*=\inf\left\{\int_{\mathbb{R}^N}\left[|\nabla u|^p+V(x)u^p\right]\mathrm{d}x:\ u\in W,\ \int_{\mathbb{R}^N}K(x)u^p\mathrm{d}x=1\right\}. \tag{1.10}$$

Then there exists $d>0$ such that problem (1.1) has at least two positive solutions $u_0,\ u_1\in W$ satisfying $I(u_0)<0$ and $I(u_1)>0$ if $|h|_{p'}<d$.

2. Existence of Minimum Positive Solution

In this section, we prove the existence of minimum positive solution for

$$\begin{cases}-\Delta_p u+V(x)|u|^{p-2}u=K(x)f(u)+h(x) & \text{in } \mathbb{R}^N,\\ u\in W^{1,p}(\mathbb{R}^N), \qquad u>0 & \text{in } \mathbb{R}^N,\end{cases} \tag{2.1}$$

by Ekeland's variational method. To this end, we need some lemmas.

Lemma 2.1. Assume that $(V_1),(F_1)-(F_4)$, (K_1) with $l<\infty$ hold and $0<\alpha<p$. Let $h\in L^{p'}(\mathbb{R}^N)$, $\frac{1}{p'}+\frac{1}{p}=1$ and $\{u_n\}\subset W$ be a bounded (PS) sequence of I. Then $\{u_n\}$ has a strongly convergent subsequence in *W*.

Proof. It is sufficient to prove that for any $\varepsilon>0$, there exist $R(\varepsilon)>R_0$ (R_0 is given by (K_1)) and $n(\varepsilon)>0$ such that for all $R\ge R(\varepsilon)$ and $n\ge n(\varepsilon)$

$$\int_{\{x:|x|\ge R\}}\left[|\nabla u|^p+V(x)u_n^p\right]\mathrm{d}x\le\varepsilon. \tag{2.2}$$

For R_0 given by (K_1), define

$$C_1(R_0,\alpha,\beta,a):=\sup\left\{\frac{1+(2R)^\alpha}{\left[\beta\left(1+(2R)^\alpha\right)+a\right]R^p}:R\ge R_0\right\},$$
$$=\frac{1+(2R_0)^\alpha}{\left[\beta\left(1+(2R)^\alpha\right)+a\right]R_0^{\,p}}, \tag{2.3}$$

$$C_2(R_0,\alpha,\beta,a):=\sup\left\{\frac{1+(2R)^\alpha}{\left[\beta\left(1+(2R)^\alpha\right)+a\right]R^\alpha}:R\ge R_0\right\},$$
$$=\frac{1+(2R_0)^\alpha}{\left[\beta\left(1+(2R)^\alpha\right)+a\right]R_0^{\,\alpha}}, \tag{2.4}$$

where $\alpha,\ \beta$ and a are given by (V_1). Then, by (V_1), (2.3) and (2.4), we have, for all $R>R_0$,

$$1/R^p\le C_1(R_0,\alpha,\beta,a)V(x),\quad \text{for all } |x|\le 2R. \tag{2.5}$$

and

$$1/R^\alpha\le C_2(R_0,\alpha,\beta,a)V(x),\quad \text{for all } |x|\le 2R. \tag{2.6}$$

Let $\xi_R(x):\mathbb{R}^N\to[0,1]$ be a smooth function such that

$$\xi_R(x)=\begin{cases}0, & 0\le|x|\le R,\\ 1, & |x|\ge 2R,\end{cases} \tag{2.7}$$

and, for some constant $C_0>0$ (independent of R),

$$|\nabla\xi_R(x)|\le\frac{C_0}{R},\quad \text{for all } x\in\mathbb{R}^N. \tag{2.8}$$

Then, by (2.5), for all $n\in\mathbb{N}$ and $R\ge R_0$, we have

$$\begin{aligned}\int_{\mathbb{R}^N}|\nabla(u_n\xi_R)|^p\,\mathrm{d}x&=\int_{\mathbb{R}^N}\left(|\nabla u_n\cdot\xi_R|+|u_n\cdot\nabla\xi_R|\right)^p\mathrm{d}x\\ &\le 2^{p-1}\int_{\mathbb{R}^N}\left[|\nabla u_n\cdot\xi_R|^p+|u_n\cdot\nabla\xi_R|^p\right]\mathrm{d}x\\ &\le 2^{p-1}\left(\int_{\mathbb{R}^N}|\nabla u_n|^p\,\mathrm{d}x+\int_{\{x:\,R\le|x|\le2R\}}\left(\frac{C_0}{R}\right)^p|u_n|^p\,\mathrm{d}x\right)\\ &\le 2^{p-1}\left[1+C_1(R_0,\alpha,\beta,a)\cdot C_0^p\right]\|u_n\|^p.\end{aligned} \tag{2.9}$$

This implies that

$$\|u_n\xi_R\| \le \left[2^{p-1}+1+2^{p-1}C_0^pC_1(R_0,\alpha,\beta,a)\right]^{1/p}\|u_n\|, \quad (2.10)$$

for all $n\in\mathbb{N}$ and $R\ge R_0$. By $0<\alpha<p$, for any $\varepsilon>0$, there exists $R(\varepsilon)\ge R_0$ such that

$$R^{\alpha-p} \le \frac{\varepsilon/C(\varepsilon)}{C_0^pC_2(R_0,\alpha,\beta,a)}, \quad \text{for all } R\ge R(\varepsilon), \quad (2.11)$$

where $C(\varepsilon)=\left(\varepsilon\frac{p}{p-1}\right)^{-(p-1)}p^{-1}$. By (1.4), $\|I'(u_n)\|_{W^*}\|u_n\|\to 0$ as $n\to\infty$, so for any $\varepsilon>0$, there exists $n(\varepsilon)>0$ such that

$$\|I'(u_n)\|_w\|u_n\| \le \frac{\varepsilon}{\left[2^{p-1}+1+2^{p-1}C_0^pC_1(R_0,\alpha,\beta,a)\right]^{1/p}}$$

for all $n\ge n(\varepsilon)$. (2.12)

Hence, it follows from (2.10) and (2.12) that

$$\left|\langle I'(u_n),u_n\xi_R\rangle\right| \le \|I'(u_n)\|_{W^*}\|u_n\xi_R\| \le \varepsilon, \quad (2.13)$$

for all $n\ge n(\varepsilon)$ and $R\ge R_0$. Note that

$$\begin{aligned}\langle I'(u_n),u_n\xi_R\rangle &= \int_{\mathbb{R}^N}\left(|\nabla u_n|^{p-1}\nabla(u_n\xi_R)+V(x)u_n^p\xi_R\right)\mathrm{d}x\\ &-\int_{\mathbb{R}^N}\left(K(x)f(u_n)u_n\xi_R+h(x)u_n\xi_R\right)\mathrm{d}x\\ &=\int_{\mathbb{R}^N}\left(|\nabla u_n|^p\xi_R+V(x)u_n^p\xi_R\right)\mathrm{d}x+\int_{\mathbb{R}^N}|\nabla u_n|^{p-1}u_n\nabla\xi_R\mathrm{d}x\\ &-\int_{\mathbb{R}^N}\left(K(x)f(u_n)u_n\xi_R+h(x)u_n\xi_R\right)\mathrm{d}x.\end{aligned} \quad (2.14)$$

For $R\ge R(\varepsilon)$, using (2.6) and (2.11), we have,

$$\frac{C_0^pC_2(R_0,\alpha,\beta,a)}{R^p} \le \frac{\varepsilon/C(\varepsilon)}{R^\alpha} \le \frac{\varepsilon}{C(\varepsilon)}C_2(R_0,\alpha,\beta,a)V(x), \quad \text{for all } |x|\le 2R,$$

that is,

$$\frac{C_0^p}{R^p} \le \frac{\varepsilon}{C(\varepsilon)}V(x), \text{ for all } |x|\le 2R, \quad (2.15)$$

Therefore, from (2.8) and (2.15), we get, for all $n\in\mathbb{N}$ and $R\ge R(\varepsilon)$,

$$\begin{aligned}&\int_{\mathbb{R}^N}\left||\nabla u_n|^{p-1}u_n\nabla\xi_R\right|\mathrm{d}x\\ &\le\varepsilon\int_{\mathbb{R}^N}|\nabla u_n|^p\mathrm{d}x+C(\varepsilon)\int_{\mathbb{R}^N}u_n^p|\nabla\xi_R|^p\mathrm{d}x\\ &\le\varepsilon\int_{\mathbb{R}^N}|\nabla u_n|^p\mathrm{d}x+C(\varepsilon)\int_{\mathbb{R}^N}\frac{C_0^p}{R^p}|u_n|^p\mathrm{d}x \quad (2.16)\\ &\le\varepsilon\int_{\mathbb{R}^N}|\nabla u_n|^p\mathrm{d}x+C(\varepsilon)\frac{\varepsilon}{C(\varepsilon)}\int_{\mathbb{R}^N}V(x)|u_n|^p\mathrm{d}x\\ &\le\varepsilon\|u_n\|^p.\end{aligned}$$

By (F_1), (K_1) and (2.7), there exists $\eta\in(0,1)$ such that, for all $n\in\mathbb{N}$ and $R\ge R_0$

$$\int_{\mathbb{R}^N}\left|K(x)f(u_n)u_n\xi_R\right|\mathrm{d}x \le \eta\int_{\mathbb{R}^N}V(x)u_n^p\xi_R\mathrm{d}x. \quad (2.17)$$

Since $h\in L^{p'}(\mathbb{R}^N)$, and $\|u_n\|\le C$ for some constant $C>0$, it follows from (2.7) there exists $R(\varepsilon)\ge R_0$ such that

$$\int_{\mathbb{R}^N}h(x)u_n\xi_R dx \le |h(x)\xi_R|_{p'}|u_n|_p \le \varepsilon, \text{for } R\ge R(\varepsilon). \quad (2.18)$$

Combining (2.13), (2.14) and (2.16)-(2.18), for all $n\ge n(\varepsilon)$ and $R\ge R(\varepsilon)$, we see that

$$\begin{aligned}\varepsilon &\ge \left|\langle I'(u_n),u_n\xi_R\rangle\right| = \int_{\mathbb{R}^N}\left[|\nabla u_n|^p\xi_R+V(x)u_n^p\xi_R\right]\mathrm{d}x\\ &+\int_{\mathbb{R}^N}|\nabla u_n|^{p-1}u_n\nabla\xi_R\mathrm{d}x\\ &-\int_{\mathbb{R}^N}K(x)f(x)u_n\xi_R\mathrm{d}x-\int_{\mathbb{R}^N}h(x)u_n\xi_R\mathrm{d}x\\ &\ge\int_{\mathbb{R}^N}\left[|\nabla u_n|^p\xi_R+V(x)u_n^p\xi_R\right]\mathrm{d}x-\varepsilon\|u_n\|^p\\ &-\eta\int_{\mathbb{R}^N}V(x)u_n^p\xi_R\mathrm{d}x-\varepsilon.\end{aligned}$$

That is,

$$\begin{aligned}&\int_{\mathbb{R}^N}|\nabla u_n|^p\xi_R\mathrm{d}x+(1-\eta)\int_{\mathbb{R}^N}V(x)u_n^p\xi_R\mathrm{d}x\\ &\le 2\varepsilon+\varepsilon\|u_n\|\le C_3\varepsilon.\end{aligned} \quad (2.19)$$

From $\eta\in(0,1)$ and (2.7), it is easy to see that (2.19) implies (2.2).

In the following, we give a property of variational functional I defined by (1.1):

Lemma 2.2. If $(V1)$, $(F1)$-$(F3)$, (K_1) hold, $h(x)\in L^{p'}(\mathbb{R}^N)$ and $K(x)\in L^\infty(\mathbb{R}^N)$. Then there exist $\rho,\alpha,d>0$ such that $I(u)_{\|u\|=\rho}\ge\alpha>0$ for $|h|_{p'}<d$.

Proof. It follows from (F_1)-(F_3) that for any $\varepsilon>0$, there exist $q\in(p,p^*-1)$ and $C(\varepsilon,q)>0$ such that for all $s>0$,

$$F(s)\le\frac{1}{p}\varepsilon s^p+C(\varepsilon,q)s^{q+1}. \quad (2.20)$$

By the Sobolev embedding and (1.3), we have

$$\begin{aligned}I(u)&=\frac{1}{p}\int_{\mathbb{R}^N}\left[|\nabla u|^p+V(x)u^p\right]\mathrm{d}x\\ &-\int_{\mathbb{R}^N}K(x)F(u)\mathrm{d}x-\int_{\mathbb{R}^N}h(x)u\mathrm{d}x.\\ &\ge\frac{1}{p}\|u\|^p-\int_{\mathbb{R}^N}K(x)\left(\frac{1}{p}\varepsilon u^p+C(\varepsilon,q)u^{q+1}\right)\mathrm{d}x-C|h|_{p'}\|u\|\\ &\ge\frac{1}{p}\|u\|^p-\varepsilon\frac{1}{p}\int_{\mathbb{R}^N}C_0V(x)u^p\mathrm{d}x\\ &-C_0C(\varepsilon,q)\int_{\mathbb{R}^N}V(x)u^{q+1}\mathrm{d}x-C|h|_{p'}\|u\|\\ &\ge\frac{1}{p}\|u\|^p-C_4\varepsilon\|u\|^p-C_5(\varepsilon,q)\|u\|^{q+1}-C|h|_{p'}\|u\|\\ &=\|u\|\left[\left(\frac{1}{p}-C_4\varepsilon\right)\|u\|^{p-1}-C_5(\varepsilon,q)\|u\|^q-C|h|_{p'}\right].\end{aligned} \quad (2.21)$$

Taking $\varepsilon = \dfrac{1}{C_4 \cdot 2p}$ and setting

$g(t) = \dfrac{1}{2p} t^{p-1} - C_5(\varepsilon, q) t^q$ for $t \ge 0$, we see that there exists $\rho > 0$ such that $\max_{t\ge0} g(t) = g(\rho) = d$. Then it follows from (2.21) that there exists $\alpha > 0$ such that $I(u)_{\|u\|=\rho} \ge \alpha > 0$ for $|h|_{p'} < d$.

Theorem 2.1. Assume that (V_1), (F_1)-(F_4), (K_1) hold, $h(x) \in L^{p'}(\mathbb{R}^N)$, $h(x) \ge 0$, $h(x) \not\equiv 0$. Let $B_\rho = \{u \in W : \|u\| < \rho\}$, if $|h|_{p'} < d$, d, ρ is given by Lemma 2.2, then there exists $u_0 \in W^{1,p}(\mathbb{R}^N)$ such that

$$I(u_0) = \inf\{I(u) : u \in \bar{B}_\rho\} < 0,$$

and u_0 is a positive solution of problem (1.1).

Proof. Since $h(x) \in L^{p'}(\mathbb{R}^N)$, $h(x) \ge 0$ and $h(x) \not\equiv 0$, we can choose a function $\varphi \in W$ such that

$$\int_{\mathbb{R}^N} h(x)\varphi(x)\mathrm{d}x > 0. \tag{2.22}$$

For $t > 0$, we have

$$I(t\varphi) = \frac{t^p}{p}\int_{\mathbb{R}^N}\left[|\nabla\varphi|^p + V(x)\varphi^p\right]\mathrm{d}x - \int_{\mathbb{R}^N} K(x)F(t\varphi)\mathrm{d}x - t\int_{\mathbb{R}^N} h(x)\varphi(x)\mathrm{d}x$$
$$\le \frac{t^p}{p}\|\varphi\|^p - t\int_{\mathbb{R}^N} h(x)\varphi(x)\mathrm{d}x < 0,$$

for $t > 0$ small enough. Hence $c_0 := \inf\{I(u) : u \in \bar{B}_\rho\} < 0$. By the Ekeland's variational principle, there exists $\{u_n\} \subset \bar{B}_\rho$ such that

1) $c_0 < I(u_n) < c_0 + \dfrac{1}{n}$,

2) $I(w) \ge I(u_n) - \dfrac{1}{n}\|w - u_n\|$ forall $w \in \bar{B}_\rho$.

Then by a standard procedure, see for example [14], we can show that $\{u_n\}$ is a bounded (PS) sequence of I. Hence Lemma 2.1 implies that there exists $u_0 \in W^{1,p}(\mathbb{R}^N)$ such that $I'(u_0) = 0$ and $I(u_0) = c_0 < 0$.

3. Existence of Second Solution

Next we prove that problem (1.1) has a Mountain Pass type solution. For this purpose, we use a variant version of Mountain Pass theorem ([13] Chapter IV), which allows us to find a so-called Cerami type (PS) sequence. The properties of this kind of (PS) sequence are very helpful in showing its boundedness. The following lemma shows that I defied in (1.1) has the so-called Mountain Pass geometry.

Lemma 3.1. Let $(V_1),(F_1)-(F_4)$, (K_1) be hold and $l > \mu^*$ with μ^* given by (1.10). Then there exists $v \in W$ with $\|v\| > \rho$, ρ is given by Lemma 2.2, such that $I(v) < 0$.

Proof. By the definition of μ^* and $l > \mu^*$, we can choose a nonnegative function $\varphi \in W$ such that $\varphi \ge 0$

$$\int_{\mathbb{R}^N} K(x)\varphi^p\mathrm{d}x = 1 \quad \text{and}$$
$$\mu^* \le \int_{\mathbb{R}^N}\left[|\nabla\varphi|^p + V(x)\varphi^p\right]\mathrm{d}x < l.$$

Therefore, by (F_4) and Fatou's lemma, we deduce that

$$\lim_{t\to+\infty}\frac{I(t\varphi)}{t^p} = \frac{1}{p}\|\varphi\|^p - \lim_{t\to+\infty}\int_{\mathbb{R}^N} K(x)\frac{F(t\varphi)}{t^p}\mathrm{d}x - \lim_{t\to+\infty}\frac{1}{t^p}\int_{\mathbb{R}^N} h(x)t\varphi\mathrm{d}x$$
$$\le \frac{1}{p}\|\varphi\|^p - \frac{1}{p}l < 0.$$

So the lemma is proved by taking $v = t_0\varphi$ with $t_0 > 1$ large enough.

From Lemma 2.2 and Lemma 3.1, there is a sequence $\{u_n\} \subset W$ such that

$$\lim_{n\to+\infty} I(u_n) = c \text{ and } \lim_{n\to+\infty}\|I'(u_n)\|_{W^*}(1 + \|u_n\|) = 0. \tag{3.1}$$

For this sequence $\{u_n\}$, let $w_n = \dfrac{u_n}{\|u_n\|}$.

Clearly, w_n is bounded in W and there is a $w \in W$ such that, up to a subsequence,

$$\begin{aligned} &w_n \to w && \text{weakly in } W,\\ &w_n \to w && \text{a.e. in } \mathbb{R}^N,\\ &w_n \to w && \text{strongly in } L_{loc}^q(\mathbb{R}^N) \text{ for } p \le q < p^*. \end{aligned} \tag{3.2}$$

For the above w, we have the following lemma.

Lemma 3.2. Let $(V_1),(F_1)-(F_4)$, (K_1) hold, $\alpha \in (0, p]$, $l > \mu^*$ for μ^* given by (1.10). If $\|u_n\| \to +\infty$ as $n \to +\infty$, then w given by (3.2) is a nontrivial nonnegative solution of

$$-\Delta_p u(x) + V(x)|u(x)|^{p-2}u(x) = lK(x)u^{p-1},\ u \in W. \tag{3.3}$$

Proof. The proof of this lemma is similar to that of ([7] Lemma 2.4). For the sake of completeness, we give a simple proof here.

Step 1. We claim that w is nontrivial, that is $w \not\equiv 0$. By contradiction, if $w \equiv 0$, we claim that

$$\limsup_{n\to+\infty}\int_{\mathbb{R}^N} K(x)\frac{f(u_n)}{u_n^{p-1}}w_n^p\mathrm{d}x < 1. \tag{3.4}$$

If (3.4) is true, then it leads to a contradiction immediately. Indeed, since $\|u_n\| \to +\infty$, it follows from (3.1) that

$$\left\langle I'(u_n),u_n\right\rangle \big/ \|u_n\|^p = o(1),$$

that is

$$o(1) = \|w_n\|^p - \int_{\mathbb{R}^N} K(x)\frac{f(u_n)}{u_n^{p-1}} w_n^p \mathrm{d}x = 1 - \int_{\mathbb{R}^N} K(x)\frac{f(u_n)}{u_n^{p-1}} w_n^p \mathrm{d}x,$$

where, and in what follows, $o(1)$ denotes a quantity which goes to zero as $n \to +\infty$. Clearly, this contradicts with (3.4). Hence $w \not\equiv 0$ and Step 1 is proved. So we only need to prove (3.4) holds.

In fact, by (K_1), there is a constant $\eta \in (0,1)$ such that

$$\sup\left\{\frac{f(s)}{s^{p-1}} : s > 0\right\} < \eta \inf\left\{\frac{V(x)}{K(x)} : |x| \ge R_0\right\}. \quad (3.5)$$

This yields, for all $n \in \mathbb{N}$,

$$\int_{\{x:|x|\ge R_0\}} K(x)\frac{f(u_n)}{u_n^{p-1}}|w_n|^p \mathrm{d}x \le \eta \int_{\{x:|x|\ge R_0\}} V(x)|w_n|^p \mathrm{d}x \le \eta < 1. \quad (3.6)$$

On the other hand, since the embedding $W^{1,p}(B_{R_0}) \hookrightarrow L^p(B_{R_0})$ is compact, $w_n \to w$ strongly in $L^p(B_{R_0})$. Passing to a subsequence, there exists $g \in L^p(B_{R_0})$ such that, for all $n \in \mathbb{N}$,

$$|w_n| \le g(x) \quad \text{a.e. in } B_{R_0}.$$

By $(F_1),(F_4)$, there exists $C_6 > 0$ such that

$$\frac{f(t)}{t^{p-1}} \le C_6 \quad \text{for all } t \in \mathbb{R}. \quad (3.7)$$

Then, for all $n \in \mathbb{N}$,

$$0 \le K(x)\frac{f(u_n)}{u_n^{p-1}} w_n^p \le C_6 K(x)|w_n|^p \le C_6 |K|_\infty |g|^p \le C \quad \text{a.e. in } B_{R_0}. \quad (3.8)$$

Noting that $w_n \to w \equiv 0$ a.e. in $\mathbb{R}^N$, we get

$$K(x)\frac{f(u_n)}{u_n^{p-1}} w_n^p \to 0 \quad \text{a.e. in } B_{R_0}. \quad (3.9)$$

It follows from (3.8), (3.9) and the dominated convergence theorem that

$$\lim_{n\to+\infty} \int_{\{x:|x|<R_0\}} K(x)\frac{f(u_n)}{u_n^p} w_n^p \mathrm{d}x = 0. \quad (3.10)$$

Hence, (3.4) is deduced from (3.6) and (3.10).

Step 2. we show that W is nonnegative, that is, $w \ge 0$.

Let $w_n^-(x) = -\min\{0, w_n(x)\}$, $w_n(x)$ is also bounded in W. If $\|u_n\| \to \infty$, then

$$\frac{\left\langle I'(u_n), w_n^-(x)\right\rangle}{\|u_n\|^{p-1}} = o(1),$$

that is,

$$-\|w_n^-\|^p = \int_{\mathbb{R}^N} K(x)\frac{f(u_n)}{\|u_n\|^{p-1}}|w_n^-|^p \mathrm{d}x + o(1). \quad (3.11)$$

By (F_1), $f(t) \equiv 0$ for all $t \le 0$. It follows from (3.11) that $\lim_{n\to\infty}\|w_n^-\|^p = o(1)$. Thus $w^- = 0$ a.e. in $x \in \mathbb{R}^N$ and $w \ge 0$.

Step 3. We prove w solves (3.3).

By Lemma 3.1, it is sufficient to prove that for any $\varphi \in C_0^\infty(\mathbb{R}^N)$,

$$\int_{\mathbb{R}^N}\left[|\nabla w|^{p-2}\nabla w \nabla\varphi + V(x)|w|^{p-1}\varphi\right]\mathrm{d}x = \int_{\mathbb{R}^N} lK(x) w^{p-1}\varphi \mathrm{d}x. \quad (3.12)$$

Using (3.1) and $\|u_n\| \to +\infty$, we have

$$\frac{\left\langle I'(u_n),\varphi\right\rangle}{\|u_n\|^{p-1}} = o(1) \quad \text{for any } \varphi \in C_0^\infty(\mathbb{R}^N),$$

that is,

$$\int_{\mathbb{R}^N}\left[|\nabla w_n|^{p-2}\nabla w \nabla\varphi + V(x) w_n^{p-1}\varphi\right]\mathrm{d}x = \int_{\mathbb{R}^N} K(x)\frac{f(u_n)}{u_n^{p-1}} w_n^{p-1}\varphi \mathrm{d}x + o(1). \quad (3.13)$$

Since $w_n \to w$ weakly in W as $n \to \infty$, we see that

$$\int_{\mathbb{R}^N}\left[|\nabla w|^{p-2}\nabla w \nabla\varphi + V(x) w^{p-1}\varphi\right]\mathrm{d}x = \int_{\mathbb{R}^N} K(x)\frac{f(u_n)}{u_n^{p-1}} w_n^{p-1}\varphi \mathrm{d}x + o(1). \quad (3.14)$$

So, Step 3 is complete provided that

$$\int_{\mathbb{R}^N} K(x)\frac{f(u_n)}{u_n^{p-1}} w_n^{p-1}\varphi \mathrm{d}x \to \int_{\mathbb{R}^N} lK(x) w^{p-1}\varphi \mathrm{d}x. \quad (3.15)$$

In fact, by (3.7) and (1.3) we have

$$\int_{\mathbb{R}^N}\left|K^{\frac{p-1}{p}}(x)\frac{f(u_n)}{u_n^{p-1}} w_n^{p-1}\right|^{\frac{p}{p-1}}\mathrm{d}x \le C_7\int_{\mathbb{R}^N} V(x) w_n^{p-1}\mathrm{d}x \le C_8\|w_n\|^p \le C_9. \quad (3.16)$$

that is, $\left\{K^{\frac{p-1}{p}}(x)\frac{f(u_n)}{u_n^{p-1}} w_n^{p-1}\right\}$ is bounded in $L^{\frac{p}{p-1}}(\mathbb{R}^N)$.

Let

$$\Omega_+ = \{x \in \mathbb{R}^N : w(x) > 0\} \text{ and}$$
$$\Omega_0 = \{x \in \mathbb{R}^N : w(x) = 0\}.$$

Noting that

$$w_n = \frac{u_n}{\|u_n\|} \to w \text{ a.e. and } \|u_n\| \to +\infty,$$

then $\|u_n\| \to +\infty$ a.e. in $x \in \Omega_+$. Hence by (F_4), we have

$$K^{\frac{p-1}{p}}(x)\frac{f(u_n)}{u_n^{p-1}}w_n^{p-1} \to K^{\frac{p-1}{p}}(x)lw^{p-1} \quad \text{a.e. in } x \in \Omega_+. \tag{3.17}$$

Since $w_n \to 0$ a.e. in $x \in \Omega_0$, it follows from (3.7) that

$$K^{\frac{p-1}{p}}(x)\frac{f(u_n)}{u_n^{p-1}}w_n^{p-1} \to 0 \equiv K^{\frac{p-1}{p}}(x)lw^{p-1} \quad \text{a.e. in } x \in \Omega_0. \tag{3.18}$$

Thus, (3.16)–(3.18) imply that

$$K^{\frac{p-1}{p}}(x)\frac{f(u_n)}{u_n^{p-1}}w_n^{p-1} \to K^{\frac{p-1}{p}}(x)lw^{p-1} . \quad \text{weakly in } L^{\frac{p}{p-1}}(\mathbb{R}^N). \tag{3.19}$$

From $\varphi \in C_0^\infty(\mathbb{R}^N)$ and $K \in L^\infty(\mathbb{R}^N)$, we know that $K^{1/p}\varphi \in L^\infty(\mathbb{R}^N)$, then (3.19) leads to (3.15).

Lemma 3.3. If $0 < \alpha \le p$, $l > \mu^*$ and $(V_1),(K_1)$, hold, then problem (3.3) has no any nontrivial non-negative solutions.

Proof. Since $l > \mu^*$, there is a constant $\delta > 0$ such that $\mu^* < \mu^* + \delta < l$.

By the definition of μ^* in (1.10), there exists a $v_\delta \in W$ such that $\int_{\mathbb{R}^N} K(x)v_\delta^p \mathrm{d}x = 1$ and

$$\mu^* < \|v_\delta\|^p < \mu^* + \delta.$$

Since $C_0^\infty(\mathbb{R}^N)$ is dense in W, we may assume $v_\delta \in C_0^\infty(\mathbb{R}^N)$ Now, let $R > 0$ be such that $\operatorname{supp} v_\delta \subset B_R$ and define

$$\mu_R = \inf\left\{\int_{\mathbb{R}^N}\left[|\nabla u|^p + V(x)u^p\right]\mathrm{d}x : \int_{\mathbb{R}^N} K(x)u^p\mathrm{d}x = 1, u \in W_0^{1,p}(B_R)\right\}$$

Then, $v_\delta \in W_0^{1,p}(B_R)$. and

$$\mu_R < \|v_\delta\|^p < \mu^* + \delta < l. \tag{3.20}$$

By the compactness of the embedding $W_0^{1,p}(B_R) \hookrightarrow L^p(B_R)$, it is not difficult to see that there exists $w_R \in W_0^{1,p}(B_R)$ with $w_R \ge 0$ and $\int_{\mathbb{R}^N} K(x)w_R^p\mathrm{d}x = 1$ such that

$$-\Delta_p w_R + V(x)|w_R|^{p-2} w_R = \mu_R K(x) w_R^{p-1}, \ x \in B_R.$$

Since $w_R \ge 0$ and $K(x)$ is a positive continuous function, by the definition of μ_R, we have that

$$-\Delta_p w_R + V(x)|w_R|^{p-2} w_R \ge 0.$$

From (V_1), we get $V(x)|w_R|^{p-2} w_R > 0$ for all $w_R > 0$ and

$$\int_0^1\left(V(x)|w_R|^{p-2} w_R w_R\right)^{-\frac{1}{p}} \mathrm{d}w_R = \infty.$$

Thus by the strong maximum principle in [15], we have

$$w_R > 0, \ \forall x \in B_R; \ \frac{\partial w_R}{\partial n} < 0, \ \forall |x| \in R.$$

If $0 \not\equiv u \in W$ is a nonnegative solution of (3.3), then integrating by parts.

$$\begin{aligned}
&\mu_R \int_{B_R} K(x) w_R^{p-1} u\mathrm{d}x \\
&= \int_{B_R}\left[-\Delta_p w_R + V(x)| w_R |^{p-2} w_R\right]u\mathrm{d}x \\
&= \int_{B_R}|\nabla w_R|^{p-1}\nabla u\mathrm{d}x \\
&-\int_{\partial B_R}\left|\frac{\partial w_R}{\partial n}\right|^{p-2}\frac{\partial w_R}{\partial n}u\mathrm{d}S + \int_{B_R} V(x)|w_R|^{p-2} w_R u\mathrm{d}x \\
&= \int_{B_R} lK(x) w_R{}^{p-1} u\mathrm{d}x - \int_{\partial B_R}\left|\frac{\partial w_R}{\partial n}\right|^{p-2}\frac{\partial w_R}{\partial n}u\mathrm{d}S \\
&\ge \int_{B_R} lK(x) w_R{}^{p-1} u\mathrm{d}x.
\end{aligned}$$

Using $u \in W, u \ge 0$ and $u \not\equiv 0$, we may choose $R > 0$ large enough such that $\int_{B_R} lK(x)w_R^{p-1}u\mathrm{d}x > 0$, thus the above calculation shows that $\mu_R \ge l$ in contradiction with (3.20). This complete the proof.

Proof of Theorem 1.1. Clearly, if $\|u_n\| \to \infty$, as $n \to +\infty$, from Lemmas 3.2 and 3.3 we get a contradiction. Hence, $\{u_n\}$ is bounded in W. Then by Lemma 2.1 we see that problem (2.1) has a positive solution $u_1 \in W$ with $I(u_1) > 0$. So, the proof is complete by Theorem 2.1.

4. References

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