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Enveloping Lie Algebras of Low Dimensional Leibniz Algebras

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Abstract

We calculate the enveloping Lie algebras of Leibniz algebras of dimensions two and three. We show how these Lie algebras could be used to distinguish non-isomorphic (nilpotent) Leibniz algebras of low dimension in some cases. These results could be used to associate geometric objects (loop spaces) to low dimensional Leibniz algebras.

Keywords: Leibniz Algebra, Enveloping Lie Algebra, Nilpotent Algebra

1. Introduction

In this paper, we work with vector spaces (and algebras) over a field F of characteristic 0, although our results can be extended in obvious way to the case of vector spaces over a field of positive characteristic (not equal 2), or even over a commutative ring with unit. By an *algebra* $(L,\cdot)$, we mean a vector space L over F with a (not necessarily associative) bilinear operation $\cdot : L\times L \to L$. For $x\in L,\ \lambda(x): L\to L$; $y \to x\cdot y$ denotes the left multiplication map. Let $Der(L)$ denotes the Lie sub-algebra of $gl(L)$ consisting of the derivations on L. Recall that a linear map $\xi \in gl(L)$ is a *derivation* of $(L,\cdot)$ if and only if

$$\left[\xi,\lambda(x)\right]=\lambda(\xi x)$$

for all $x\in L$. Here, we work with a class of algebras in which the left multiplication map has a stronger compatibility relation with derivations. These are Leibniz algebras, introduced by J. L. Loday [1], as non-anti-symmetric generalizations of Lie algebras.

Definition 1.1. A Leibniz algebra L is a vector space over a field F equipped with a bilinear map

$$\cdot : L\times L\to L$$

satisfying the Leibniz identity

$$x\cdot(y\cdot z)=(x\cdot y).z+y\cdot(x\cdot z),\ \text{for all}\ x,y,z\in L.$$

Obviously, a Lie algebra is a Leibniz algebra. A Leibniz algebra is a Lie algebra if and only if

$$x\cdot x=0\ \ (x\in L)$$

Also, an algebra $(L,\cdot)$ is a Leibniz algebra if and only if $\lambda(x)\subseteq Der(L)$ or equivalently,

$$\lambda:(L,\cdot)\to\left(gl(L),[\cdot,\cdot]\right)$$

is a homomorphism. Thus we have a homomorphism $\lambda:(L,\cdot)\to\left(gl(L),[\cdot,\cdot]\right)$ when $(L,\cdot)$ is a Leibniz algebra.

Definition 1.2. If $(L,\cdot)$ is a Leibniz algebra. We may define $L^1=L, L^k=L\cdot L^{k-1} (k>1)$. The series

$$L^1\supseteq L^2\supseteq L^3\supseteq\cdots$$

is called the *descending central series* of L. If the series terminates for some positive integer *s*, then the Leibniz algebra L is said to be *nilpotent.*

2. Methods

The main tool to classify Low dimensional Leibniz algebras is to find the corresponding enveloping Lie algebra and fit them in the Beck-Kolman list of low dimensional Lie algebras. Since these Lie algebras are realized as certain quotients of the given Leibniz algebras, we first need the following fact.

Theorem 1.1. *Let* $(L_1,\cdot)$ *and* $(L_2,\cdot)$ *be Leibniz algebras. If* $L_1 \cong L_2$ *then* $L_1/J_1 \cong L_2/J_2$ *, where* $J_i = \langle x\cdot x; x\in L_i\rangle$ *is the ideal generated by squares in* L_i*, for i* = 1, 2.

Proof. Let $\phi: L_1 \to L_2$ be an isomorphism. We define $\psi: L_1/J_1 \to L_2/J_2$ such that $\psi(x+J_1)=\phi(x)+J_2$. It is easy to show that ψ is well defined and onto, and

$$\psi\left((x_1+J_1)\cdot(x_2+J_2)\right)$$
$$=\psi(x_1\cdot x_2+J_1)=\phi(x_1\cdot x_2)+J_2$$
$$=\phi(x_1)\cdot\phi(x_2)+J_2=\phi(x_1+J_2)\cdot\phi(x_2+J_2)$$
$$=\psi(x_1+J_1)\cdot\psi(x_2+J_2)$$

Also,

$$Ker(\psi)=\{x+J_1;\phi(x)+J_2=0\}$$
$$=\{x+J_1;\phi(x)\in J_2=\phi(J_1)\}$$
$$=\{x+J_1;x\in J_1\}=0 \qquad \square$$

This theorem could be used to prove that some (nilpotent) Leibniz algebras are non-isomorphic. This is important, as the nilpotent low dimensional Lie algebras are already classified [2].

Example 1.1. *Let* $L_1: e_1\cdot e_1=e_2, e_2\cdot e_1=e_3$, *and* $L_2: e_1\cdot e_1=e_3, e_1\cdot e_2=e_3, e_2\cdot e_1=e_3$.

Then L_1/J_1 *is a one-dimensional abelian Lie algebra*, *but* L_2/J_2 *is a two-dimensional abelian Lie algebra*, *therefore* L_1/J_1 is not isomorphic to L_2/J_2. *By the previous theorem*, L_1 *is not isomorphic to* L_2.

We noted that Leibniz algebras are non-antisymmetric in general. Hence, it is natural to consider the skew-symmetrization of a Leibniz algebra $(L,\cdot)$. This is done through the skew-symmetrized binary operation

$$[[\cdot,\cdot]]=\frac{1}{2}(x\cdot y-y\cdot x)$$

for $x,y\in L$. Note that, in general, $(L,[[\cdot,\cdot]])$ is not a Lie algebra. On the other hand, by definition of Leibniz algebra, $\lambda(x)\in Der(L,[[\cdot,\cdot]])$ for all $x\in L$, and $\lambda:(L,[[\cdot,\cdot]])\to Der(L)$ is a homomorphism of anti-commutative algebras. Let

$$J=\langle x\cdot x; x\in L\rangle$$

be the two-sided ideal of $(L,\cdot)$ generated by all squares. Then J contains all symmetric products $x\cdot y-y\cdot x$, for $x,y\in L$, and since $\lambda(x\cdot x)=[\lambda(x),\lambda(x)]$ for all $x\in L$, we have

$$J\subseteq \ker(\lambda)$$

Let $M\subseteq L$ be any ideal containing J, then since $x\cdot y+M=-y\cdot x+(x\cdot y+y\cdot x)=-y\cdot x+M$ for $x,y\in L$, the Leibniz product in L lifts to a Lie bracket $[\cdot,\cdot]$ in L/M. Conversely, if $J\subseteq M\subseteq L$ is an ideal, then the quotient $h=L/M$ is a Lie algebra. In particular, J is the smallest two-sided ideal of L such that L/J is a Lie algebra.

Let $(h,[\cdot,\cdot])$ be a Lie, and $(L,\cdot)$ be a Leibniz algebra, we define a binary operation on the semidirect product $h\times L$ by

$$\left[(\bar{e}_i,x),(\bar{e}_j,y)\right]=\left([\bar{e}_i,\bar{e}_j],e_i\cdot y-e_j\cdot x\right)$$

for $\bar{e}_i,\bar{e}_i\in h$, *and* $x,y\in L$, where $\bar{e}_i=e_i+M$ [3]. Since M contains all squares, it is clear that this operation is well defined.

Proposition 1.1. *If* L *is a nilpotent Leibniz algebra, and* M *is an ideal of* L *such that* $J\subseteq M\subseteq \ker(\lambda)$ *and* $h=L/M$ *, then*

1) h *is nilpotent Lie algebra.*

2) $N=h\times L$ *is a nilpotent Lie algebra.*

Proof. 1) L is a nilpotent Leibniz algebra, then there exist $n\in N$ such that $L\supseteq L^2\supseteq L^3\supseteq\cdots\supseteq L^n=\{0\}$ Therefore $L/M\supseteq L^2/M\supseteq L^3/M\supseteq\cdots\supseteq L^n/M=\{0\}$ Then $h\supseteq h^2\supseteq h^3\supseteq\cdots\supseteq h^n=\{0\}$ Thus h is a nilpotent Lie algebra.

2) Clearly if h and L are nilpotent, then N is a nilpotent Lie algebra. $\square$

The above proposition associates two Lie algebras h and N to a Leibniz algebra L. Here h is a quotient of L, whereas N is its extension. The corresponding Lie groups could be employed to associate a geometric object to L [3]. We would consider the problem of classification of these geometric objects (loop spaces) in a forthcoming paper.

3. Results

Next let us remind the classification results for Leibniz algebras of dimension two and three [4]. We use the convention to denote the j^{th} algebra of dimension *i* by $L_{i,j}$.

Theorem 1.2. *In dimension two*, *there are two non-isomorphic nilpotent Leibniz algebras*, *where* $L_{2,1}$ *is abelian*, *and* $L_{2,2}$ *is given by the table* $e_1\cdot e_1=e_2$.

Theorem 1.3. *In dimension three*, *there are five concrete and one parametric family of pairwise non isomorphic algebras.*

$L_{3,1}$: *Abelian*,

$L_{3,2}: e_1\cdot e_1=e_2$,

$L_{3,3}: e_1\cdot e_2=e_3, e_2\cdot e_1=-e_3$,

$L_{3,4} : e_1 \cdot e_1 = e_3, e_2 \cdot e_2 = \alpha e_3, e_1 \cdot e_2 = e_3 (\alpha \in C),$

$L_{3,5} : e_1 \cdot e_1 = e_3, e_1 \cdot e_2 = e_3, e_2 \cdot e_1 = e_3,$

$L_{3,6} : e_1 \cdot e_1 = e_2, e_2 \cdot e_1 = e_3.$

4. Discussion

In this section, we classify the enveloping Lie algebras of Leibniz algebras of dimension two and three. This is not a trivial task, as in each case we have to identify the resulting Lie algebra as one of the known low dimensional Lie algebras [2], by carefully defining the appropriate change of basis.

In dimension two, we have $L_{2,2} : e_1 \cdot e_1 = e_2$. Then

$$J_{2,2} := \{\lambda e_2 : \lambda \in F\}$$

and

$$h_{2,2} = L_{2,2} / J_{2,2} = span\{e_1 + J_{2,2}, e_1 \in L_{2,2}\}$$
$$= span\{\overline{e}_1, e_1 \in L_{2,2}\}$$

Therefore $h_{2,2}$ is 1-dimensional abelian Lie algebra. Now, we consider $N_{2,2} = h_{2,2} \times L_{2,2}$ with basis elements $E_1 = (\overline{e}_1, 0), E_2 = (0, e_1), E_3 = (0, e_3)$. The multiplication table of $N_{2,2}$ is given by

$[\cdot,\cdot]$	E_1	E_2	E_3
E_1	0	E_3	0
E_2	$-E_3$	0	0
E_3	0	0	0

For example

$$[E_1, E_1] = [(e_1 + J_{2,2}, 0), (e_1 + J_{2,2}, 0)]$$
$$= (e_1 \cdot e_1 + J_{2,2}, 0) = (e_2 + J_{2,2}, 0)$$
$$= 0$$

and

$$[E_1, E_2] = [(e_1 + J_{2,2}, 0), (0, e_1)]$$
$$= (0, e_1 \cdot e_1) = (0, e_2) = E_2$$

Thus, $N_{2,2}$ is a 3-dimensional Lie algebra with $[E_1, E_2] = E_3$. Briefly, we have **Table 1** for dimension 2 (where the last column identifies the Lie algebra in the Beck-Kolman list [2]).

In dimension three, for each $L_{3,k}, k = 2,3,4,5,6$ we want to obtain corresponding Lie algebra $N_{3,k}$. For $L_{3,2}$, one can show that $h_{3,2}$ is 2-dimensional abelian Lie algebra and $N_{3,2}$ is given by $[E_1, E_3] = E_4$. For $L_{3,3}$, one can prove that $J_{3,3} = \{0\}$, therefore $h_{3,3}$ is given by

Table 1. The enveloping Lie algebras of Leibniz algebras of dimension two.

$L_{i,j}$	$h_{i,j}$	$N_{i,j} = h_{i,j}$¡Á$L_{i,j}$	$N_{i,j}$
$L_{2,2}$	1-dimensional abelian Lie algebra	$[E_1, E_2] = E_3$	g_3

$$[e_1, e_2] = e_3, [e_2, e_1] = -e_3,$$

and $N_{3,3}$ by

$$[E_1, E_2] = E_3, \quad [E_1, E_5] = E_6, \quad [E_2, E_4] = -E_6$$

Finally, for $L_{3,4}$, one can show that $L_{3,4}$ is a 2-dimensional abelian Lie algebra and $N_{3,4}$ is given by $[E_1, E_3] = E_5, [E_1, E_4] = E_5, [E_2, E_4] = \alpha E_5 (\alpha \in C)$. Theref ore for 3-dimensional Leibniz algebras, we get the **Table 2**.

Note that in rows three and four, the enveloping Lie algebras are isomorphic, while the original Leibniz algebras are not isomorphic. The isomorphism $N_{3,2} \cong g_3 \oplus C^2$ is given by

$e_1 \mapsto E_1, e_2 \mapsto E_3, e_3 \mapsto E_4, e_4 \mapsto E_2, e_5 \mapsto E_5,$

$N_{3,3} \cong g_{6,21}$ by

$e_1 \mapsto -E_2, e_2 \mapsto E_1, e_3 \mapsto E_4, e_4 \mapsto E_5, e_5 \mapsto E_3, e_6 \mapsto E_6.$

$N_{3,4} \cong g_{5,2}$ by

$e_1 \mapsto -\alpha E_1, e_2 \mapsto -E_3 + E_4, e_3 \mapsto E_3, e_4 \mapsto E_2, e_5 \mapsto -\alpha E_5.$

$N_{3,5} \cong g_{5,2}$ by

$e_1 \mapsto -E_1 + E_4, e_2 \mapsto E_1, e_3 \mapsto E_2, e_4 \mapsto E_4, e_5 \mapsto E_5.$

Also, for the 2-dimensional abelian Lie algebra $L_{2,1}, J_{2,1} = \{0\}$, and $h_{2,1}$ is an abelian Lie algebra. Therefore $N_{2,1} = h_{2,1} \times L_{2,1}$ is 4-dimensional Lie algebra. Finally, for the 3-dimensional abelian Lie algebra

Table 2. The enveloping Lie algebras of Leibniz algebras of dimension three.

$L_{i,j}$	$h_{i,j}$	$N_{i,j} = h_{i,j}$¡Á$L_{i,j}$	$N_{i,j}$
$L_{3,2}$	2-dimensional abelian Lie algebra	$[E_1, E_3] = E_4$	$g_3 \oplus C^2$
$L_{3,3}$	$[e_1, e_2] = e_3,$ $[e_2, e_1] = -e_3,$	$[E_1, E_2] = E_3, [E_1, E_5] = E_6,$ $[E_2, E_4] = -E_6$	$g_{6,21}$
$L_{3,4}$	2-dimensional abelian Lie algebra	$[E_1, E_3] = E_5, [E_1, E_4] = E_5,$ $[E_2, E_4] = \alpha E_5 (\alpha \in C).$	$g_{5,2}$
$L_{3,5}$	2-dimensional abelian Lie algebra	$[E_1, E_3] = E_5, [E_1, E_4] = E_5,$ $[E_2, E_3] = -E_5$	$g_{5,2}$
$L_{3,6}$	1-dimensional abelian Lie algebra	$[E_1, E_2] = E_3$	$g_3 \oplus C^2$

$L_{3,1}, J_{3,1} = \{0\}$ and $h_{3,1}$ is an abelian Lie algebra. Therefore

$$N_{3,1} = h_{3,1} \times L_{3,1}$$

is a 6-dimensional Lie algebra.

5. Conclusions

We have classified the enveloping Lie algebras of Leibniz algebras of dimension two and three. In each case, we have identified the corresponding Lie algebra as one of the known low dimensional Lie algebras, by defining the appropriate change of basis which implements the canonical isomorphism.

There is one two dimensional Leibniz algebra (up to isomorphism) whose corresponding Lie quotient is a 1-dimensional abelian Lie algebra. On the other hand, there are exactly five non-isomorphic three dimensional Leibnitz algebra, which correspond to three 2-dimensional abelian Lie algebras (two of which are isomorphic), one 1-dimensional abelian Lie algebra, and a 3-dimensional non-abelian Lie algebra.

6. References

[1] J. L. Loday, “Une version non-commutative des algebras de Lie, Les algebres de Leibniz,” *Mathematics at Ecole Normale Supérieure*, Vol. 39, 1993, pp. 269-293.

[2] R. E. Beck and B. Kolman, “Constructions of Nilpotent Lie Algebras over Arbitrary Fields,” In: P. S. Wang, Ed., *Proceedings of* 1981 *ACM Symposium on Symbolic and Algebraic Computation*, New York, 1981, pp. 169-174.

[3] M. K. Kinyon and A. Weinestein, “Leibniz Algebras, Courant Algebroids, and Multiplications on Homogeneous Spaces,” *American Journal of Mathematics*, Vol. 123, No. 3, 2001, pp. 525-550. doi:10.1353/ajm.2001.0017

[4] S. Albeverio, B. A. Omirov and I. S. Rakhimov, “Varieties of Nilpotent Complex Leibniz Algebras of Dimension Less Than Five,” *Communications in Algebra*, Vol. 33, No. 5, 2005, pp. 1575-1585. doi:10.1081/AGB-200061038