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A Posteriori Error Estimate for Streamline Diffusion Method in Solving a Hyperbolic Equation

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Abstract

In this article, we use streamline diffusion method for the linear second order hyperbolic initial-boundary value problem. More specifically, we prove a posteriori error estimates for this method for the linear wave equation. We observe that this error estimates make finite element method increasingly powerful rather than other methods.

1. Introduction

The wave equation based on rigorous a posteriori error estimates is a largely subject, despite the importance of these problems in the modeling of a number of physical phenomena. A posteriori have made every method increasingly powerful; such that there are various approaches to a posteriori error estimates and it has new successfully applied to varied problems by several authors (see Ainsworth and Tinsley Oden [1]; Asadzadeh [2]; Gergouli [3]; Johnson [4] and [5]).

Gergouli *et al*. [3] and his teammates applied finite element method for linear wave equation and obtained a posteriori error estimates in L (L^2) norm in Johnson proved existence solution for second order hyperbolic problems and used discontinuous Galerkin method for them and obtained a priori and a posteriori error estimates. In this paper, we do new work and use streamline diffusion method (SD-method) for solving the linear second order hyperbolic initial-boundary value problem.

Streamline diffusion methods (Asadzadeh [6]; Asadzadeh and Kowalczyk [7]; Eriksson and Johnson [8]; Brenner [9]; Dubois [10]; Fuhrer [11]) perform slightly better than the standard finite element methods for smooth solutions and non-smooth solutions hyperbolic problems as a two-dimensional one which both is higher order accurate and has good stability properties. Due to the fact that artificial diffusion is added only in the characteristic direction so that internal layers are not smeared out, while the added diffusion removes oscillations near boundary layers.

We consider the linear second order hyperbolic initial boundary value problem (see Codina [12]; Haws, [13]; Gergoulus *et al*., [3]; Iraniparast [14]; Kalmenov [15]; in Sobolov space Adams [16]; Shermenew [17]) as follows:

$$\begin{cases} u_{tt} - \nabla \cdot (a\nabla u) = f & \text{in } \Omega \times (0,T) \\ u(x,0 = u_0(x)) & \text{on } \Omega \times \{0\} \\ u_t(x,0) = u_1(x,0) & \text{on } \Omega \times \{0\} \\ u(x,0) = 0 & \text{on } \partial\Omega \times (0,T] \end{cases} \quad (1)$$

Here, $\Omega \subset \mathfrak{R}^d$ is a bounded open polygonal domain with boundary $\partial\Omega$ and we have $u_0 \in H_0^1(\Omega), u_1 \in L^2(\Omega)$, a is a scalar-value function in $C(\bar{\Omega})$ and $f \in L^2(0,T; L^2(\Omega))$.

For (1), we use one variable changing and obtain a new problem. We apply SD-method for new problem and obtain a posteriori error estimates. A posteriori error bound provides a computable upper bound on the error in some norm using the computed finite element solution (see Ainsworth and Tinsley Oden [1]; Asadzadeh [2]; Burman [18]; Johnson and Szepessy [19]; Sandboge [20]).

In order to make use of the theory of Semigroups we write the system (1) in the following abstract form:

$$\begin{cases} w_t + Aw = F & \text{in } \left(\Omega \times (0,T)\right)^2 \\ w(x,0) = w_0(x) & \text{on } \left(\Omega \times \{0\}\right)^2 \\ w(0,t) = 0 & \text{on } \left(\partial\Omega \times (0,T]\right)^2 \end{cases} \quad (2)$$

Here, we assume $v = u_t$ for $x \in \mathbb{R}^d$ and $t \in [0,T]$, also:

$$w(x,t) = \left(u(x,t), v(x,t)\right)^{\mathrm{T}}, \quad w_t(x,t) = \left(u_t(x,t), v_t(x,t)^{\mathrm{T}}\right)$$

$$A = \begin{pmatrix} 0 & -I \\ \nabla \cdot (a\nabla) & 0 \end{pmatrix} \quad w_0(x) = \left(u_0(x), u_1(x)\right)^{\mathrm{T}}$$

and

$$F(x,t) = \left(0, f(x,t)\right)^{\mathrm{T}}$$

where, I is identity matrix.

The rest of this study is organized as follows. In Section 2, we define slabs for space-time domain and obtain SD-method for (2) this slabs. In Section 3, we consider a posteriori error estimates for SD-method form of Section 2 and obtain dual problem. In Section 4, we define interpolation estimates for dual problem. In Section 5, we complete proof for a posteriori error estimates by using definitions in Section 4.

2. The Streamline Diffusion Method

In this section, we consider the SD-method for solving (2) that is based on using finite element over the space-time domain $\Omega \times [0,T]$. To define this method, let $0 = t_0 < t_1 < \cdots < t_N = T$ be a subdivision of the time interval $[0,T]$ into intervals $I_n = (t_n, t_{n+1})$, with time steps $k_n = t_{n+1} - t_n$, $n = 0,1,\cdots,N-1$ and introduce the corresponding space-time slabs, i.e.:

$$S_{n=} \left\{(x,t) : x \in, t_n < t < t_{n+1}\right\} \quad (3)$$

for $n = 0,1,\cdots,N-1$. Further, for each n let W^n be a finite element subspace of $H^1(S_n) \times H^1(S_n)$, (see Adams, [16]) and let:

$$\dot{W}^n = \left\{w \in W^n \mid w(0, t) = 0, \text{ for } t \in I_n\right\} \quad (4)$$

We can formulate SD-method on the slab S_n for (2), as follows:

For $n = 0,\cdots,N-1$, find $w^n \in \dot{W}$ such that:

$$\begin{aligned} &\left(w_t^n + Aw^n, g + \delta(g_t + Ag)\right)_n + <w_+^n, g+>_n \\ &= \left(F, g + \delta(g_t + Ag)\right)_n + <w_-^n, g_+>_n \end{aligned} \quad (5)$$

where, $\delta = \bar{C}h$ with $\bar{C}$ is a suitable chosen (sufficiently small, see Johnson, [18]) positive constant and parameter h is defined in the following. Further, we de-

fine the following notations for (6):

$$(w,g)_n = \int_{Sn} w^T \cdot g \, dx dt$$

$$<w,g>_n = \int_\Omega w^T(x,t_n) . g(x,t_n) dx$$

$$w+(x,t) = \lim_{s\to 0+} w(x,t+s) \lim_{s\to 0+}$$

$$w-(x,t) = \lim_{s\to 0-} w(x,t+s)$$

The terms including $<,>$ in the above formula is a jump conditions which imposes a weakly enforced continuity condition across the slab interfaces, at t_n and is the mechanism by which information is propagated from one slab to another. For more concisely, after summing over n, we may rewrite (5) as follow:

We assume $\dot{w} = \coprod_{n=0}^{N-1} \dot{W}^n$ and find $| w \in \dot{w}$, such that:

$$B(w,g) = L(g) \quad (6)$$

For $g \in \dot{w}$ and where the bilinear form $B(.,.)$ and the linear form $L(.)$ define by:

$$\begin{aligned} B(w,g) = &\sum_{n=0}^{N-1} \left(w_t^n + Aw^n, g + \delta(g_t + Ag)\right)_n \\ &+ \sum_{n=1}^{N-1} <[w^n], g_+>_n + <w_+^n g_+>_0 \end{aligned}$$

where, we define $w = (w_1, w_2)^{\mathrm{T}}$ such that for $i = 1,2$:

$$[w_i] = w_{i,+} - w_{i,-}, \quad [w] = \left([w_1],[w_2]\right)^{\mathrm{T}}$$

and

$$L(g) = \sum_{n=0}^{N-1} \left(F, g + \delta(gt + Ag)\right)_n + <w_0, g+>_0$$

For $h > 0$, we define T_h^n such that be a triangulation of the slab S_n into triangles K satisfying as usual the minimum angle condition (Ciarlet [21]) and assume that the parameter h is represented with the maximum diameter of the triangles $K \in T_h^n$. We introduce:

$$\begin{aligned} W_h^n = &\Big\{w \in H^1(S_n) \times H^1(S_n) : w\big|_k \in P_k(K) \times P_k(K) \\ &\text{for } \in T_h^n, w(0,t) = 0 \text{ for } t \in I_n \Big\} \end{aligned}$$

where, $P_k(K)$ denotes the set of polynomials in K of degree less than or equal k and:

$$W_h = \prod_{n=0}^{N-1} W_h^n$$

Thus (6) can be formulated as follows:
Find $w_h \in W_h$ such that:

$$B(w_h, g) = L(g) \quad (7)$$

for $g \in W_h$. Moreover, we know that the exact solution of (6) satisfies:

$$B(w, g) = L(g)$$

for $g \in \dot{w}$ and by use (6) and (7), we have the Galerkin orthogonality relation:

$$B(e, g) = 0 \tag{8}$$

where, $e = w - w_h$.

3. An a Posteriori Error Estimate for the SD-Method

In this section, we shall consider the following simplified version of SD-method for (7) with $\delta = 0$: Find $w_h \in W_h$, such that for $n = 0, 1, \cdots, N-1$:

$$\sum_{n=0}^{N-1} \left(w_{h,t}^n + Aw_h^n, g\right)_n + \sum_{n=1}^{N-1} <[w_h^n], g+>_n + <w_{h,+}^n, g+>_0$$
$$= \sum_{n=0}^{N-1} (F, g)_n + <w_0, g+>_0 \tag{9}$$

where, $g \in W_h$ and $w_{h,-}^0 = 0$.

For simplicity, we take $w_0 = 0$ and $F = 0$. In order to obtain a representation of the error, we consider the following auxiliary problem, referred to as the linearized dual problem: Find Φ such that:

$$\begin{cases} L^*\Phi \equiv -\Phi_t + A^T\Phi = \Psi^{-1}e & \text{in}\Omega \\ \Phi(0, t) = 0, t \in (0, T) \\ \Omega(x, T) = 0, x \in \Omega \end{cases} \tag{10}$$

and L^* denotes the adjoint of the operator L defined in (2) and ψ is a positive weight function. Note that this problem is computed "backward", but there is a corresponding change in sign. Further, we shall first introduce the following notation:

$$\|w\|_{L_2^\psi(\Omega)} = (w, \Psi w)_\Omega^{1/2} \tag{11}$$

Multiplying (10) by e and integrating by parts and summing over n, we obtain the following error representation formula:

$$\|e\|_{L_2^{\psi-1}(\Omega)} = (e, \Psi^{-1}e)_\Omega = (e, L^*\Phi)_\Omega$$
$$= \sum_{n=0}^{N-1} \left(e, -\Phi_t + A^T\Phi\right)_n \tag{12}$$
$$= \sum_{n=0}^{N-1} \left(e, -\Phi_t\right)_n + \sum_{n=0}^{N-1} \left(e, A^T\Phi\right)_n$$

We have for $n = 0, 1, \cdots, N-1$ by part integrating:

$$\left(e, \Phi_t\right)_n = -\int_{S_n} e^T \cdot \Phi_t \,\mathrm{d}x\mathrm{d}t$$
$$= -\sum_{n=0}^{N-1} \int_\Omega e^T\left(x, t_n\right) \cdot \Phi\left(x, t_n\right)\mathrm{d}x + \int_{S_n} e_t^T \cdot \Phi \mathrm{d}x\mathrm{d}t$$
$$= -\sum_{n=0}^{N-1} \int_\Omega e^T\left(x, t_n\right) \cdot \Phi\left(x, t_n\right)\mathrm{d}x + \left(e_t, \Phi\right)_n \tag{13}$$

We define $e = (e_1, e_2)^T$ and $\Phi = (\phi_1, \phi_2)^T$ and obtain for $n = 0, 1, \cdots, N-1$:

$$\left(e, A^T\Phi\right)_n = \int_{S_n} e^T \cdot A^T\Phi \mathrm{d}x\mathrm{d}t$$
$$= \int_{S_n} e^T \cdot \begin{pmatrix} 0 & -\nabla \cdot (a\nabla) \\ -1 & 0 \end{pmatrix} \begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix} \mathrm{d}x\mathrm{d}t$$
$$= \int_{S_n} (e_1 e_2) \cdot \begin{pmatrix} -\nabla.(a\nabla\phi_2) \\ -\phi_1 \end{pmatrix} \mathrm{d}x\mathrm{d}t$$
$$= \int_{S_n} (-e_1\nabla \cdot (a\nabla\phi_2) - e_2\phi_1)\mathrm{d}x\mathrm{d}t \tag{14}$$
$$= \int_{S_n} (a\nabla\phi_2\nabla e_1 - e_2\phi_1)\mathrm{d}x\mathrm{d}t$$
$$= \int_{S_n} (-\nabla \cdot (a\nabla\phi_2)e_1 - e_2\phi_1)\mathrm{d}x\mathrm{d}t$$
$$= \int_{S_n} (Ae)^T \cdot \Phi\mathrm{d}x\mathrm{d}t = (Ae, \Phi)_n$$

We define:

$$J = \sum_{n=0}^{N-1} \int_\Omega e^T\left(x, t_n\right) \cdot \Phi\left(x, t_n\right)dx$$
$$= \left(<e_-, \Phi_->_1 - <e_+, \Phi_+>_0\right)$$
$$+ \left(<e_-, \Phi_->_2 - <e_+, \Phi_+>_1\right)$$
$$+ \cdots + \left(<e_-, \Phi_->_{N-1} - <e_+, \Phi_+>_{N-2}\right)$$
$$+ \left(<e_-, \Phi_->_N - <e_+, \Phi_+>_{N-1}\right)$$
$$= -\left[<e_-, \Phi_->_N + <e_+, \Phi_+>_0 \right.$$
$$\left. + \sum_{n=0}^{N-1} <[e], \Phi_+>_n + \sum_{n=0}^{N-1} <e_-, [\Phi]>_n \right]$$

According to (9), $\phi(., t_N = T) = 0$ and since $e_-^0 = [w_0] = 0$, we get:

$$J = \sum_{n=0}^{N-1} <[w_h], \Phi_+>_n \tag{15}$$

Then in (12), by use (15) and (16), we have:

$$\|e\|_{L_2^{\psi+1}(\Omega)}^2 = \sum_{n=0}^{N-1} (e_t, \Phi) + \sum_{n=0}^{N-1} (Ae, \Phi)_n + \sum_{n=0}^{N-1} <[w_h], \Phi_+>_n$$

So that recalling (9) and using the Galerkin orthogonality (8), we obtain:

$$\|e\|^2_{L_2^{\Psi+1}(\Omega)} = \sum_{n=0}^{N=1}(e_t,\Phi) + \sum_{n=0}^{N=1}(Ae,\Phi)_n + \sum_{n=0}^{N=1}<[w_h],\Phi_+>_n$$

$$= \sum_{n=0}^{N-1}\left((w-w_h)_t + A(w-w_h),\Phi\right)_n$$

$$+ \sum_{n=0}^{N-1}<[w_h],\Phi_+>_n = \sum_{n=0}^{N-1}(F-w_{h,t}-Aw_h,\Phi)_n$$

$$+ \sum_{n=0}^{N-1}<[w_h],\Phi_+>_n$$

(16)

where, $\hat{\Phi} \in W_h$ is an interpolant of Φ. The idea is now to estimate $\hat{\Phi}-\Phi$ in terms of $\psi^{-1}e$ using a strong stability estimates for solution Φ of the dual problem.

4. Interpolation Estimates for the Dual Solution

We shall now consider our interpolant $\hat{\Phi} \in W_h$ in (16) to be the space-time L_2-projection of Φ, namely if the first, we define the L_2-projections:

$$P_n : L_2(\Omega) \mapsto W_h^n$$

and

$$\pi_n : L_2(S_n) \mapsto \prod{}_{0,n} = \left\{w \in L_2(S_n) : w(x,.)\right.$$

$$\left. \text{is constant on } I_n, \ x \in \Omega\right\}$$

in space and in space time, respectively, by:

$$\int_\Omega (P_n,\Phi)^T \cdot w\mathrm{d}x = \int_\Phi \Phi^T \cdot w\mathrm{d}x, \quad \forall w \in W_h^n$$

$$\pi_n w \mid S_n = \frac{1}{k_n}\int_{I_n} w(.,t)\mathrm{d}t, \quad \forall w \in \prod{}_{0,n}$$

Then, we can define $\hat{\Phi}\big|S_n \in W_h^n$ by letting:

$$\hat{\Phi}\big|S_n = P_n\pi_n\Phi = \pi_nP_n\Phi \in W_h^n$$

where, $\Phi = \Phi\big|S_n$. Further, if we introduce P and π defined by:

$$(P\Phi)\big|S_n = P_n(\Phi\big|S_n)$$

and

$$(\pi\Phi)\big|S_n = \pi_n(\Phi\big|S_n)$$

respectively, then we can let $\hat{\Phi} \in W_h$ to be:

$$\hat{\Phi} = P\pi\Phi = \pi P\Phi \in W_h$$

Now, we define residual of computed solution w_h by:

$$R_0 = F - w_{h,t} - Aw_h$$

$$R_1 = w_{h,+}^n - w_{h,-}^n, \text{on } S_n$$

$$R_2 = \frac{(P_n-I)w_{h,-}^n}{k_n}, \text{on } S_n$$

where, I is the identity operator.

In the end of this section, we shall give a lemma for interpolation estimates by the projection operators P, leaving the overall of I and II to next section.

Lemma 1: There is a constant C such that for residual $R \in L_2(\Omega)$:

$$\left|(R,\Phi-P\Phi)_\Omega\right| \le C\left\|h^2(I-P)R\right\|_{L_2^{\Psi-1}(\Omega)}\left\|\Phi_{xx}\right\|_{L_2^{\Psi}(\Omega)}$$ (17)

Proof: (see Johnson and Szepessy [19] and Sandboge [20]).

5. The Completion of the Proof of a Posteriori Error Estimates

In this section we state and prove a posteriori error estimate by estimating of the terms I and II in the error representation formula (16). To this approach we introduce the stability factors (see Burman [18]) associated with discretization in time and space, defined by:

$$\gamma_e^t = \frac{\left\|\Phi_t\right\|_{L_2^{\Psi}(\Omega)}}{\left\|e\right\|_{L_2^{\Psi-1}(\Omega)}}$$ (18)

and

$$\gamma_e^x = \frac{\left\|\Phi_{xx}\right\|_{L_2^{\Psi}(\Omega)}}{\left\|e\right\|_{L_2^{\Psi-1}(\Omega)}}$$ (19)

respectively. We now apply the result of the previous sections; using Catchy-Schwartz inequality in (16) coupled with the interpolation estimate (17) and the strong stability factors (18) and (19), to derive the $L^2(L^2)$ a posteriori error estimates for the scheme (9).

Theorem 1: The error $e = w - w_h$, where w is the solution of the continuous problems (2) and w_h that of (9), satisfies the following stability estimate:

$$\|e\|_{L_2^{\Psi-1}(\Omega)} \le C\gamma_e^x\left\|h^2(I-P)R_0\right\|_{L_2^{\Psi-1}(\Omega)} + C\gamma_e^x\left\|k_nR_1\right\|_{L_2^{\Psi-1}(\Omega)}$$

$$+ \gamma_e^x\left\|h^2R_2\right\|_{L_2^{\Psi-1}(\Omega)} + \gamma_e^x\left\|k_nR_2\right\|_{L_2^{\Psi-1}(\Omega)}$$

(20)

Proof: Using the notation introduce above, we may write (17) as:

$$\|e\|_{L_2^{\Psi-1}(\Omega)}^2 = \sum_{n=0}^{N-1}(R_0,\hat{\Phi}-\Phi)_n + \sum_{n=0}^{N-1}<k_n\frac{[w_h]}{k_n},(\hat{\Phi}-\Phi)_+>_n$$

$$= I + II$$

Below we shall estimate the terms I and II separately.

Splitting the interpolation error by writing
$\hat{\Phi} - \Phi = \hat{\Phi} - P\Phi + P\Phi - \Phi$ and $\hat{\Phi}_n = \pi_n P\Phi$, we have:

$$I = \sum_{n=0}^{N-1} \left(R_0, \hat{\Phi} - P\Phi + P\Phi - \Phi\right)_n$$

$$= \sum_{n=0}^{N-1} \left(R_0, \hat{\Phi}_n - P\Phi\right)_n + \sum_{n=0}^{N-1} \left(R_0, P\Phi - \Phi\right)_n$$

$$\leq C \left\| h^2 \left(I - P\right) R_0 \right\|_{L_2^{\Psi-1}(\Omega)} \left\| \Phi_{xx} \right\|_{L_2^{\Psi}(\Omega)}$$

where we have used the fact that R_0 is constant in the time, (making the first integral zero) and then using interpolation estimate (17) in the second integral. It remains to estimate the terms *II*, to this end, we need the following notation:

$$\Phi_+^n(x) = \Phi(x,t) - \int_{t_n}^{t} \frac{\partial}{\partial \tau} \Phi(x,\tau) d\tau dt$$

so that:

$$k_n \Phi_+^n(x) = \int_{I_n} \Phi(x,t) dt - \int_{I_n} \int_{I_n}^{t} \Phi_\tau(x,\tau) d\tau dt \quad (21)$$

where, $\Phi_\tau = \dfrac{\partial \Phi}{\partial \tau}$ and $\hat{\Phi}_n = \hat{\Phi}(.,t_n)$:

$$II = \sum_{n=0}^{N-1} \left\langle k_n \left| \frac{[w_h]}{k_n}, \left(\hat{\Phi} - \Phi\right)_+ \right\rangle_n$$

$$= \sum_{n=0}^{N-1} \left\langle k_n \frac{[w_h]}{k_n}, \left(\hat{\Phi}_n - P\Phi + P\Phi - \Phi\right)_+ \right\rangle_n$$

$$= \sum_{n=0}^{N-1} \left\langle k_n \frac{w_h}{k_n}, \left(\hat{\Phi}_n - P\Phi\right) + \right\rangle_n$$

$$+ \sum_{n=0}^{N-1} \left\langle k_n \frac{[w_h]}{k_n}, \left(P\Phi - \Phi\right)_+ \right\rangle_n : II_1 + II_2$$

To estimate II_1, we use (21) to get:

$$II_1 = \sum_{n=0}^{N-1} \left\langle k_n R_1, \left(\hat{\Phi}_n\right)_+ - P\Phi_+ \right\rangle_n$$

$$= \sum_{n=0}^{N-1} \left\langle R_1 k_n \hat{\Phi}_n - P k_n \Phi_+ \right\rangle_n$$

$$= \sum_{n=0}^{N-1} \left\langle R_1, k_n \hat{\Phi}_n - \int_{I_n} P\Phi(.,t) dt + \int_{I_n} \int_{t_n}^{t} P\Phi_\tau(.,\tau) d\tau dt \right\rangle_n$$

$$= \sum_{n=0}^{N-1} \int_{I_n} \int_{t_n}^{t} \left\langle R_1 P\Phi_\tau(.,\tau) \right\rangle_n d\tau dt$$

$$\leq \left\| k_n R_2 \right\|_{L_2^{\Psi-1}(\Omega)} \left\| P\Phi_t \right\|_{L_2^{\Psi}(\Omega)}$$

$$\leq \left\| k_n R_1 \right\|_{L_2^{\Psi-1}(\Omega)} \left\| \Phi_t \right\|_{L_2^{\Psi}(\Omega)}$$

As for the II_2-terms we can write:

$$II_2 = \sum_{n=0}^{N-1} \left\langle k_n \frac{[w_h]}{k_n} \left(P\Phi - \Phi\right)_+ \right\rangle_n$$

$$\cdot \sum_{n=0}^{N-1} \left\langle \frac{w_{h,+}^n - w_{h,-}^n}{k_n}, \left(P_n - I\right) k_n \Phi_+ \right\rangle_n$$

$$= \sum_{n=0}^{N-1} \left\langle \frac{P_n w_{h,-}^n - w_{h,-}^n}{k_n}, \left(P_n - I\right) \left(\int_{I_n} \Phi(.,t) dt \right. \right.$$

$$\left. \left. - \int_{I_n} \int_{t_n}^{t} \Phi_\tau(.,\tau) d\tau dt \right) \right\rangle_n$$

$$\leq \sum_{n=0}^{N-1} \int_{I_n} \left\langle \frac{\left(P_n - I\right) w_{h,-}^n}{k_n}, \left(P_n - I\right) \Phi(.,t) \right\rangle_n dt$$

$$+ \sum_{n=0}^{N-1} \int_{I_n} \int_{I_n} \left\langle \frac{\left(P_n - I\right) w_{h,-}^n}{k_n}, \left(P_n - I\right) \Phi_\tau(.,t) d\tau dt \right\rangle_n$$

$$\leq \left\| k_n R_2 \right\|_{L_2^{\Psi-1}(\Omega)} \left\| \Phi_{xx} \right\|_{L_2^{\Psi}(\Omega)} + \left\| k_n R_2 \right\|_{L_2^{\Psi-1}(\Omega)} \left\| \Phi_t \right\|_{L_2^{\Psi}(\Omega)}$$

The a posteriori error estimate now follows immediately after collecting the terms and using the definition of the stability factors (18) and (19).

For $\delta \neq 0$ in (7), we can obtain a posteriori error estimates with similar way.

6. References

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