Applied Mathematics, 2011, 2, 959-964
doi:10.4236/am.2011.28132 Published Online August 2011 (http://www.SciRP.org/journal/am)

Numerical Treatment of Nonlinear Third Order Boundary Value Problem

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Received January 25, 2011; revised June 20, 2011; accepted June 27, 2011

Abstract

In this paper, the boundary value problems for nonlinear third order differential equations are treated. A generic approach based on nonpolynomial quintic spline is developed to solve such boundary value problem. We show that the approximate solutions of such problems obtained by the numerical algorithm developed using nonpolynomial quintic spline functions are better than those produced by other numerical methods. The algorithm is tested on a problem to demonstrate the practical usefulness of the approach.

Keywords: Nonlinear Third Order Boundary Value Problem, Nonpolynomial Quintic Spline, Draining and Coating Flows

1. Introduction

Engineering problems that are time-dependent are often described in terms of differential equations with conditions imposed at single point (initial/final value problems); while engineering problems that are position dependent are often described in terms of differential equations with conditions imposed at more than one point (boundary value problems). Boundary value problems are encountered in many engineering fields including optimal control, beam deflections, heat flow, draining and coating flows, and various dynamic systems. In this paper we are concerned with general third-order nonlinear boundary value problems, such problems arise in the study of draining and coating flows. Many authors have studied and solved such type of third order boundary value problems with various types of boundary conditions. A. Khan and Tariq Aziz [1] solved a third-order linear and non-linear boundary value problem of the type

$$y'''(x) = f(x, y), a \le x \le b \tag{1}$$

Subject to

$$y(a) = k_1, \quad y'(a) = k_2, \quad y(b) = k_3 \tag{2}$$

by deriving a fourth order method using polynomial quintic splines. S. Valarmathi and N. Ramanujam [2], T. Y. Na [3], N. S. Asaithambi [4], Xueqin Li and Minggen Cui [5] and N. H. Shuaib *et al*. [6] used some other computational methods for solving boundary value problems for third-order ordinary differential equations. A. Cabada *et al.* [7] studied external solutions for third-order nonlinear problems with upper and lower solutions in reversed order.

In this paper Nonpolynomial quintic spline functions are applied to obtain a numerical solution of the following nonlinear third order two-point boundary value problems

$$y''' - f(x, y, y', y'') = g(x) y^2, \quad x \in [0,1] \tag{3}$$

subject to the boundary conditions:

$$y(1) = A_1, \quad y^{(1)}(0) = A_2, \quad y^{(1)}(1) = A_3 \tag{4}$$

where $A_i, \ i = 1,2,3$ are finite real constants.

The existence theorems for the solution of (3) subjected to boundary condition (4) are derived by Xueqin Li and Minggen Cui [5]. M. Pei *et al*. and F. M. Minhos [8,9] also discussed the existence of nonlinear third order boundary value problems. Xueqin Li and Minggen Cui consider the problem in the reproducing kernel Hilbert space $W_1[0,1]$. In order to derive the existence theorems for solution of (3) they make the following assumptions H:

$(H1)$ $f(x, y, z, w) \in [0,1] XR^3$ is completely continuous function;

$(H2)$ $f(x, y, z, w)$, $f_x(x, y, z, w)$, $f_y(x, y, z, w)$ and $f_w(x, y, z, w)$ are bounded;

$(H3)$ $f(x, y, z, w) > 0$ on $[0,1] XR^3$, where $f(x,y,z,w) \in W_1[0,1]$ as

$$y = y(x) \in W_1[0,1], \quad z = z(x) \in W_1[0,1],$$

$$w = w(x) \in W_1[0,1], \quad (0 \le x \le 1, -\infty < y, z, w < \infty)$$

Here the reproducing kernel Hilbert space $W_1[0,1]$ is the inner product space $W_1[0,1]$ which is defined by,

$$W_1[0,1] = \{u(x) : u(x) \textit{is absolutely continuous real value function in} [0,1], u'(x) \in L^2[0,1]\}$$

The inner product and norm in $W_1[0,1]$ are given, respectively, by

$$u(x), v(x)_{W_1} = u(0)v(0) + \int_0^1 u'(x)v'(x)\mathrm{d}x$$

$$u_{W_1} = u(x),\ u(x)^{1/2}$$

M. Cui *et al*. and C. I. Li *et al*. [10,11] had proved that $W_1[0,1]$ is a complete reproducing kernel space. That is, there exists a reproducing kernel function

$$Q_x(y) \in W_1[0,1], y \in [0,1]$$

for each fixed $x \in [0,1]$ and any $u(y) \in W_1[0,1]$, such that $u(y), Q_x(y)_{W_1} = u(x)$ The reproducing kernel $Q_x(y)$ can be denoted by

$$Q_x(y) = \begin{cases} 1+y, & y \le x, \\ 1+x, & y > x. \end{cases}$$

In the present paper, our main objective is to apply non-polynomial quintic spline function [12-14] that has a polynomial and trigonometric parts to develop a new numerical method for obtaining smooth approximations to the solution of nonlinear third-order differential equations of the system of form (3) subjected to (4). Here algorithms are developed and the approximate solutions obtained by these algorithms are compared with the solutions obtained by iterative method [5]. The paper is organized as follows—In Section 2, we have given a brief introduction of nonpolynomial quintic spline. In Section 3, we give a brief derivation of this non-polynomial quintic spline. We present the spline relations to be used for discretization of the given system (3). In Section 4, we present our numerical method for a system of nonlinear third-order boundary-value problems and development of boundary conditions, truncation error and class of the method are discussed, in Section 5, numerical evidence is included to compare and demonstrate the efficiency of the methods, in which we have shown that our algorithm performs better than an iterative method. Finally, in Section 6 we have concluded the paper with some remarks.

2. Nonpolynomial Quintic Spline

A quintic spline function $S_\Delta(x)$, interpolating to a function $u(x)$ defined on $[a,b]$ is such that

1) In each subinterval $[x_{j-1}, x_j]$, $S_\Delta(x)$ is a polynomial of degree at most five.

2) The first, second, third and fourth derivatives of $S_\Delta(x)$ are continuous on $[a,b]$.

To be able to deal effectively with such problems we introduce "spline functions" containing a parameter τ. These are "non-polynomial splines" defined through the solution of a differential equation in each subinterval. The arbitrary constants are being chosen to satisfy certain smoothness conditions at the joints. These "splines" belong to the class C^2 and reduce into polynomial splines as parameter $\tau \to 0$. The exact form of the spline depends upon the manner in which the parameter is introduced. We have studied parametric spline functions: spline under compression, spline under tension and adaptive spline. A number of spline relations have been obtained for subsequent use.

A function $S_\Delta(x,\tau)$ of class $C^4[a,b]$ which interpolate $u(x)$ at the mesh points $\{x_j\}$ depends on a parameter τ, reduces to ordinary quintic spline $S_\Delta(x)$ in $[a,b]$ as $\tau \to 0$ is termed as parametric quintic spline function. The three parametric quintic splines derived from quintic spline by introducing the parameter in three different ways are termed as "parametric quintic spline-I", "parametric quintic spline-II" and "adaptive quintic spline".

The spline function we propose in this paper has the following form

$$\text{span}\{1, x, x^2, x^3, \sin|\tau|x, \cos|\tau|x\},$$

$$\text{or} \quad \text{span}\{1, x, x^2, x^3, \sinh|\tau|x, \cosh|\tau|x\},$$

$$\text{or} \quad \text{span}\{1, x, x^2, x^3, x^4, x^5\}, \text{ where } \tau = 0$$

The above fact is evident when correlation between polynomial and non-polynomial splines basis is investigated in the following manner:

$$T_5 = \text{span}\{1, x, x^2, x^3, \sin(\tau x), \cos(\tau x)\},$$

$$= \text{span}\left\{1, x, x^2, x^3, \frac{24}{\tau^4}\left(\cos(\tau x) - 1 + \frac{(\tau x)^2}{2}\right), \frac{120}{\tau^5}\left(\sin(\tau x) - (\tau x) + \frac{(\tau x)^3}{6}\right)\right\}$$

From the above equation it follows that

$$\lim_{\tau\to 0} T_5 = \{1, x, x^2, x^3, x^4, x^5\}$$

where τ is the frequency of the trigonometric part of the splines function which can be real or pure imaginary and which will be used to raise the accuracy of the method. This approach has the advantage over finite difference methods that it provides continuous approximation to not only for $y(x)$, but also for y', y'' and higher derivatives at every point of the range of integration. Also, the $C^\infty -$differentiability of the trigonometric part of non-polynomial splines compensates for the loss of smoothness inherited by polynomial splines in this paper.

3. Development of the Method

Without loss of generality in order to develop the numerical method for approximating solution of a differential Equation (3), we consider a uniform mesh Δ with nodal points x_i on $[a,b]$ such that

$$\Delta : a = x_0 < x_1 < x_2 < x_3 < \cdots < x_N = b$$

$$x_i = a + ih, \quad i = 0,1,2,\cdots,N$$

where, $h = \dfrac{b-a}{N}$.

Let us consider a non-polynomial function $S_\Delta(x)$ of class $C^4[a,b]$ which interpolates $y(x)$ at the mesh points x_i, $i = 0,1,2,\cdots,N,$ depends on a parameter τ, and reduces to ordinary quintic spline $S_\Delta(x)$ in $[a,b]$ as $\tau \to 0$.

For each segment $[x_i, x_{i+1}]$, $i = 0,1,2,\cdots,N-1,$ the non-polynomial, $S_\Delta(x)$, define by

$$\begin{aligned} S_\Delta(x) &= a_i + b_i(x - x_i) + c_i(x-x_i)^2 + d_i(x-x_i)^3 \\ &\quad + e_i \sin(x - x_i) + f_i \cos(x - x_i) \qquad (5) \\ i &= 0,1,2,\cdots,N-1 \end{aligned}$$

where a_i, b_i, c_i, d_i, e_i and f_i are constants and τ is arbitrary parameter.

Let y_i be an approximation to $y(x_i)$, obtained by the segment $S_\Delta(x)$ of the mixed splines function passing through the points (x_i, y_i) and (x_{i+1}, y_{i+1}), to obtain the necessary conditions for the coefficients introduced in (5), we do not only require that $S_\Delta(x)$ satisfies interpolatory conditions at x_i and x_{i+1}, but also the continuity of first, second and third derivatives at the common nodes (x_i, y_i) are fulfilled.

To derive expression for the coefficients of (5) in terms of y_i, y_{i+1}, D_i, D_{i+1}, T_i, T_{i+1}, F_i and F_{i+1} we first denote:

$$\begin{aligned} S_\Delta(x_i) &= y_i, & S_\Delta(x_{i+1}) &= y_{i+1} \\ S'_\Delta(x_i) &= D_i, & S'_\Delta(x_{i+1}) &= D_{i+1} \\ S^{(3)}_\Delta(x_i) &= T_i, & S^{(3)}_\Delta(x_{i+1}) &= T_{i+1} \\ S^{(4)}_\Delta(x_i) &= F_i, & S^{(4)}_\Delta(x_{i+1}) &= F_{i+1} \end{aligned} \qquad (6)$$

From algebraic manipulation we get the following expression:

$$\begin{aligned} a_i &= y_i - \frac{F_i}{\tau^4}, \\ b_i &= D_i - \frac{F_{i+1} - F_i \cos\theta}{\tau^3 \sin\theta}, \\ c_i &= \frac{y_{i-1} - 2y_i + y_{i+1}}{2h^2}, \\ d_i &= \frac{\tau[T_{i+1} - T_i\cos\theta] - F_i \sin\theta}{6(1-\cos\theta)}, \\ e_i &= \frac{F_{i+1} - F_i\cos\theta}{\tau^4 \sin\theta}, \\ f_i &= \frac{F_i}{\tau^4}. \end{aligned} \qquad (7)$$

where $\theta = \tau h$ and $i = 0,1,2,\cdots,N-1$.

Using the continuity of the first and third derivatives at (x_i, y_i), that is $S'_{\Delta-i}(x_i) = S'_{\Delta i}(x_i)$ and $S'''_{\Delta-i}(x_i) = S'''_{\Delta i}(x_i)$, we obtain the following relations:

$$\begin{aligned} &(\alpha+\beta)\frac{y_{i-2} - 3y_{i-1} + 3y_i - y_{i+1}}{h^3} \\ &= \frac{1}{h^2}[hF_{i-1} - T_{i-1} - T_i] - [\alpha T_i - \beta T_{i-1}] \\ &T_{i-1} - 2T_i + T_{i+1} = h(\alpha+\beta)[F_{i+1} - F_{i-1}], i = 1(1)N-1 \end{aligned} \qquad (8)$$

The operator Λ is defined by for any function

$$\wedge w_i = p(w_{i+2} + w_{i-2}) + q(w_{i+1} + w_{i-1}) + s w_i$$

for any function w evaluated at the mesh points. Then we have the following relations connecting y and its derivatives:

1) $\wedge T_i =$

$$\frac{1}{h^3}\left[(\alpha+\beta)(y_{i+2} + y_{i-2}) + (2\alpha - 4\beta)(y_{i+1} - y_{i-1})\right] \qquad (9)$$

2) $\wedge F_i = \dfrac{1}{h^4}\delta^4 y_i$

where $p = \alpha_1 + \dfrac{\alpha}{6}$,

$$q = 2\left[\frac{1}{6}(2\alpha+\beta) - (\alpha_1 - \beta_1)\right],$$

$$s = 2\left[\frac{1}{6}(\alpha + 4\beta) + (\alpha_1 - 2\beta_1)\right],$$

$$\alpha = \left(\frac{1}{\theta^2}\right)(\theta csc\theta - 1),$$

$$\beta = \left(\frac{1}{\theta^2}\right)(1 - \theta cot\theta),$$

$$\beta_1 = \frac{1}{\theta^2}\left(\frac{1}{3} - \beta\right),$$

$$\theta = \tau h,$$

$$T_i = S'''_\Delta(x_i)$$

and $F_i = S^{(4)}_\Delta(x_i)$.

4. Description of the Method and Development of Boundary Conditions

At the mesh point x_i the proposed differential equation

$$y''' + f(x, y, y', y'') = g(x)y^2, \quad x \in [a, b] \tag{10}$$

subjected to boundary conditions (4), may be discretized by

$$T_i + f_i = g_i y_i^2 \tag{11}$$

where $T_i = S'''_\Delta(x_i)$ and $g_i = g(x_i)$.

Using the spline relation (9) (*i*), in (11) we have

$$\begin{aligned}&(-\alpha - \beta)y_{i-2} + (2\alpha - 4\beta)y_{i-1} + (-2\alpha + 4\beta - h^2 qf_{i+1})y_{i+1}\\&+(-\alpha - \beta - ph^2 f_{i+2})y_{i+2} - h^2(pf_{i-2} + qf_{i-1} + sf_i)\\&= -h^3(pg_{i-2}y_{i-2}^2 + qg_{i-1}y_{i-1}^2 + sg_i y_i^2 + qg_{i+1}y_{i+1}^2\\&+ pg_{i+2}y_{i+2}^2), \quad i = 2(1)N-2\end{aligned} \tag{12}$$

To obtain unique solution we need two more equations to be associated with (12) so that we use the following boundary conditions:

1) To obtain the second-order boundary formula we define:

$$\begin{aligned}&(\alpha+\beta)[y_1 - 3y_2 + 3y_3 - y_4]\\&= h(-y_2''' - y_3''') - h^3[\alpha y_3''' - \beta y_2'''], \quad i = 1\\&(\alpha+\beta)[y_{N-3} - 3y_{N-2} + 3y_{N-1} - y_N]\\&= h(-y_{N-2}''' - y_{N-1}') - h^3[\alpha y_{N-1}''' - \beta y_{N-2}'''],\\&i = N-1\end{aligned} \tag{13}$$

for any choice of α and β, $\alpha + \beta = 1/2$. Using Equation (3) we have:

$$\begin{aligned}&(\alpha+\beta)[y_1 - 3y_2 - 3y_3 - y_4]\\&-(hf_2 + \beta h^3 f_2 - hf_3 + \alpha h^3 f_3)\\&= [-h + \beta h^3]g_2 y_i^2 - [h + \alpha h^3]g_3 y_i^2\end{aligned}$$

$$\begin{aligned}&(\alpha+\beta)[y_{N-3} - 3y_{N-2} - 3y_{N-1} - y_N]\\&-(hf_{N-2} + \beta h^3 f_{N-2} - hf_{N-3} + \alpha h^3 f_{N-3})\\&= [-h + \beta h^3]g_{N-2}y_{N-2}^2 - [h + \alpha h^3]g_{N-1}y_{N-1}^2\end{aligned}$$

2) To obtain the fourth-order boundary formula we define:

$$\begin{aligned}&(\alpha+\beta)[y_1 - 3y_2 + 3y_3 - y_4]\\&= \frac{h}{2}(-y_2''' - y_3''') - \frac{h^3}{8}[\alpha y_3''' - \beta y_2'''], \quad i = 1,\\&(\alpha+\beta)[y_{N-3} - 3y_{N-2} + 3y_{N-1} - y_N]\\&= \frac{h}{2}(-y_{N-2}''' - y_{N-1}''') - \frac{h^3}{8}[\alpha y_{N-1}''' - \beta y_{N-2}'''], \quad i = N-1\end{aligned} \tag{14}$$

for any choice of α and β, $\alpha + \beta = 1/2$. Using Equation (3) we have:

$$\begin{aligned}&(\alpha+\beta)[y_1 - 3y_2 - 3y_3 - y_4]\\&-\left(\frac{h}{2}f_2 + \beta\frac{h^3}{8}f_2 - \frac{h}{2}f_3 + \alpha\frac{h^3}{8}f_3\right)\\&= \left[-\frac{h}{2} + \beta\frac{h^3}{8}\right]g_2 y_i^2 - \left[\frac{h}{2} + \alpha\frac{h^3}{8}\right]g_3 y_i^2\end{aligned}$$

$$\begin{aligned}&(\alpha+\beta)[y_{N-3} - 3y_{N-2} - 3y_{N-1} - y_N]\\&-\left(\frac{h}{2}f_{N-2} + \beta\frac{h^3}{8}f_{N-2} - \frac{h}{2}f_{N-1} + \alpha\frac{h^3}{8}f_{N-1}\right)\\&= \left[-\frac{h}{2} + \beta\frac{h^3}{8}\right]g_{N-2}y_{N-2}^2 - \left[\frac{h}{2} + \alpha\frac{h^3}{8}\right]g_{N-1}y_{N-2}^2\end{aligned}$$

By expanding (8) in Taylor series about x_i, we obtain the following local truncation error:

$$\begin{aligned}T_i &= \left[\frac{1}{6}(9\alpha - 3\beta) - (4p + q)\right]h^4 y_i^{(4)}\\&+\left[\frac{1}{1806}(33\alpha - 3\beta) - \frac{1}{12}(16p + q)\right]h^6 y_i^{(6)}\\&+\left[\frac{1}{131040}(1613\alpha + 27\beta) - \frac{1}{360}(4p + q)\right]h^8 y_i^{(8)}\\&+O(h^9)\end{aligned} \tag{15}$$

For any choice of α and β, provided that $\alpha + \beta = 1/2$.

Remark 1): Second-order method

For $\alpha = 1/4$, $\beta = 1/4$,

and $p = 0.04063489941134321703$,

$q = 0.25412730690212937985$,

$s = 0.41047570631347259688$.

gives $T_i = O(h^4)$.

Remark 2): Fourth-order method

For $\alpha = \frac{1}{6}, \beta = \frac{1}{3}, p = \frac{1}{120}, q = \frac{26}{120}$,

and $s = \frac{66}{120}$, gives $T_i = O(h^6)$.

Clearly, the family of numerical methods is described by the Equation (12), boundary equations and the solution vector $Y = [y_1, y_2, \cdots, y_N]^T$, T denoting transpose, is obtained by solving a non-linear algebraic system of order *N*.

To ensure cost effectiveness, better accuracy and simple applicability of the new method, the best way is to find the unknown parameters *α* and *β*, which are the expressions containing the actual parameter τ. The hall mark of the new approach is that it gives family of fourth- and second-order methods by running the code once and also skips the multiplications involved in the expressions *α* and *β*.

5. Numerical Example

We now consider a numerical example illustrating the comparative performance of nonpolynomial quintic spline algorithms over an iterative method [5].

Example: Consider the boundary value problem

$$y'''^{(x)} = xy''(x) + y'(x) + xy(x) + y^2(x) \quad (16)$$

under the boundary condition

$$y(1) = y'(0) = y'(1) = 0 \quad (17)$$

The analytic solution of (16) is

$$y(x) = x(x-1) \quad (18)$$

Nonpolynomial Quintic Spline Solution of Example

The maximum observed errors (in absolute value) by our algorithm (of second order) and iterative method (Xueqin Li *et al*. [5]) for the example considered are presented in **Table 1**.

6. Discussion and Conclusions

In this paper we used a nonpolynomial Quintic spline function to develop numerical algorithms of system of nonlinear third order boundary value problems. Here the result obtained by our algorithm is better than that obtained by some other method as compared in **Tables 1** and **2**. The approximate solutions obtained by the present algorithms are very encouraging and it is a powerful tool for solution of nonlinear third order boundary value problems.

Table 1. Comparison of our algorithm of second order with iterative method.

Node	(Maximum Absolute Error) (By Xueqin Li *et al.* [5]) N = 10	(Maximum Absolute Error) (Our Method)	
		N = 10	N = 20
0.1	4.24272E-04	2.52718E-04	6.01631E-05
0.2	1.97294E-04	1.25174E-04	3.2713E-05
0.3	3.66041E-04	2.52833E-04	5.3529E-05
0.4	1.020626E-03	2.62819E-04	6.52289E-05
0.5	1.52747E-03	3.82917E-04	8.82427E-05

Table 2. Comparison of our algorithm of fourth order with iterative method.

Node	(Maximum Absolute Error) (By Xueqin Li *et al.* [5]) N = 10	(Maximum Absolute Error) (Our Method)	
		N = 10	N = 20
0.1	4.24272E-04	6.28192E-05	3.58192E-06
0.2	1.97294E-04	4.62816E-05	2.23147E-06
0.3	3.66041E-04	3.52842E-05	2.18372E-06
0.4	1.020626E-03	8.16252E-05	4.98326E-06
0.5	1.52747E-03	1.93165E-05	1.28429E-06

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