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On the Behavior of Solutions of the System of Rational Difference Equations $x_{n+1}=\frac{x_{n-1}}{y_nx_{n-1}-1},\ y_{n+1}=\frac{y_{n-1}}{x_ny_{n-1}-1},\ z_{n+1}=\frac{x_n}{y_nz_{n-1}}$

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Abstract

In this paper, we investigate the solutions of the system of difference equations $x_{n+1}=\frac{x_{n-1}}{y_nx_{n-1}-1}$, $y_{n+1}=\frac{y_{n-1}}{x_ny_{n-1}-1}$, $z_{n+1}=\frac{x_n}{y_nz_{n-1}}$ where $x_0,x_{-1},y_0,y_{-1},z_0,z_{-1}\in\Re$.

Keywords: Difference Equation, Difference Equation Systems, Solutions

1. Introduction

Recently, there has been great interest in studying difference equation systems. One of the reasons for this is a necessity for some techniques which can be used in investigating equations arising in mathematical models describing real life situations in population biology, economic, probability theory, genetics, psychology etc. Although difference equations are very simple in form, it is extremely difficult to understand throughly the global behavior of their solutions; for example, see Refs. [1-28].

In this paper, we investigated the periodicity of the solutions of the difference equation system

$$x_{n+1}=\frac{x_{n-1}}{y_nx_{n-1}-1},\quad y_{n+1}=\frac{y_{n-1}}{x_ny_{n-1}-1},\quad z_{n+1}=\frac{x_n}{y_nz_{n-1}} \tag{1.1}$$

where the initial conditions are arbitrary real numbers.

2. Main Results

Theorem 1. Let $y_0=a, y_{-1}=b, x_0=c, x_{-1}=d, z_0=e, z_{-1}=f$ be arbitrary real numbers and let $\{x_n,y_n,z_n\}$ be a solutions of the system (1.1). Also, assume that $ad\neq1$ and $cb\neq1$, $a\neq0, b\neq0, e\neq0$ and $f\neq0$. Then all solutions of (1.1) are

$$x_n=\begin{cases}\dfrac{d}{(ad-1)^n}, & n-odd\\ c(cb-1)^n, & n-even\end{cases} \tag{1.2}$$

$$y_n=\begin{cases}\dfrac{b}{(cb-1)^n}, & n-odd\\ a(ad-1)^n, & n-even\end{cases} \tag{1.3}$$

$$z_n=\begin{cases}\dfrac{c(cb-1)^{n-1}}{af(ad-1)^{n-1}}, & n=4k-3, k=1,2,3,\cdots\\ \dfrac{d(cb-1)^{n-1}}{be(ad-1)^{n-1}}, & n=4k-2, k=1,2,3,\cdots\\ \dfrac{f(cb-1)^{n-1}}{(ad-1)^{n-1}}, & n=4k-1, k=1,2,3,\cdots\\ \dfrac{e(cb-1)^{n-1}}{(ad-1)^{n-1}}, & n=4k \quad ,k=1,2,3,\cdots\end{cases} \tag{1.4}$$

Proof: For $n=1,2,3,$ we have

$$x_1=\frac{x_{-1}}{y_0x_{-1}-1}=\frac{d}{ad-1}$$

$$y_1=\frac{y_{-1}}{x_0y_{-1}-1}=\frac{b}{cb-1}$$

$$z_1=\frac{x_0}{y_0z_{-1}}=\frac{c}{af},$$

$$x_2=\frac{x_{01}}{y_1x_0-1}=\frac{c}{\frac{b}{cb-1}c-1}=c(cb-1)$$

$$y_2=\frac{y_0}{x_1y_0-1}=\frac{a}{\frac{d}{ad-1}a-1}=a(ad-1)$$

$$z_2=\frac{x_1}{y_1z_0}=\frac{\frac{d}{ad-1}}{\frac{b}{cb-1}e}=\frac{d(cb-1)}{be(ad-1)},$$

$$x_3=\frac{x_1}{y_2x_1-1}=\frac{\frac{d}{ad-1}}{a(ad-1)\frac{d}{ad-1}-1}=\frac{d}{(ad-1)^2},$$

$$y_3=\frac{y_1}{x_2y_1-1}=\frac{\frac{b}{cb-1}}{c(cb-1)\frac{b}{cb-1}-1}=\frac{b}{(cb-1)^2}$$

$$z_3=\frac{x_2}{y_2z_1}=\frac{c(cb-1)}{a(ad-1)\frac{c}{af}}=\frac{f(cb-1)}{(ad-1)},$$

$$x_4=\frac{x_2}{y_3x_2-1}=\frac{c(cb-1)}{\frac{b}{(cb-1)^2}c(cb-1)-1}=\frac{c(cb-1)}{\frac{cb}{cb-1}-1}$$
$$=c(cb-1)^2$$

$$y_4=\frac{y_2}{x_3y_2-1}=\frac{a(ad-1)}{\frac{d}{(ad-1)^2}a(ad-1)-1}=\frac{a(ad-1)}{\frac{ad}{ad-1}-1}$$
$$=a(ad-1)^2$$

$$z_4=\frac{x_3}{y_3z_2}=\frac{\frac{d}{(ad-1)^2}}{\frac{b}{(cb-1)^2}\frac{d(cb-1)}{be(ad-1)}}=\frac{e(cb-1)}{(ad-1)}$$

for $n=k$ assume that

$$x_{2k-1}=\frac{x_{2k-3}}{y_{2k-2}x_{2k-3}-1}=\frac{d}{(ad-1)^k}$$

$$x_{2k}=\frac{x_{2k-2}}{y_{2k-1}x_{2k-2}-1}=c(cb-1)^k$$

$$y_{2k-1}=\frac{y_{2k-3}}{x_{2k-2}y_{2k-3}-1}=\frac{b}{(cb-1)^k}$$

$$y_{2k}=\frac{y_{2k-2}}{x_{2k-1}y_{2k-2}-1}=a(ad-1)^k$$

and

$$z_{4k-3}=\frac{x_{4k-4}}{y_{4k-4}z_{4k-5}}=\frac{c(cb-1)^{k-1}}{af(ad-1)^{k-1}}$$

$$z_{4k-2}=\frac{x_{4k-3}}{y_{4k-3}z_{4k-4}}=\frac{d(cb-1)^k}{be(ad-1)^k}$$

$$z_{4k-1}=\frac{x_{4k-2}}{y_{4k-2}z_{4k-3}}=\frac{f(cb-1)^k}{(ad-1)^k}$$

$$z_{4k}=\frac{x_{4k-1}}{y_{4k-1}z_{4k-2}}=\frac{e(cb-1)^k}{(ad-1)^k}$$

are true. Then $n=k+1$ we will show that (1.2), (1.3) and (1.4) are true. From (1.1), we have

$$x_{2k+1}=\frac{x_{2k-1}}{y_{2k}x_{2k-1}-1}=\frac{\frac{d}{(ad-1)^k}}{a(ad-1)^k\frac{d}{(ad-1)^k}-1}$$
$$=\frac{d}{(ad-1)^{k+1}}$$

$$y_{2k+1}=\frac{y_{2k-1}}{x_{2k}y_{2k-1}-1}=\frac{\frac{b}{(cb-1)^k}}{c(cb-1)^k\frac{b}{(cb-1)^k}-1}.$$
$$=\frac{b}{(cb-1)^{k+1}}$$

Also, similarly from (1.1), we have

$$z_{4k+1}=\frac{x_{4k}}{y_{4k}z_{4k-1}}=\frac{c(cb-1)^{2k}}{a(ad-1)^{2k}\frac{f(cb-1)^k}{(ad-1)^k}}=\frac{c(cb-1)^k}{af(ad-1)^k}$$

$$z_{4k+2}=\frac{x_{4k+1}}{y_{4k+1}z_{4k}}=\frac{\frac{d}{(ad-1)^{2k+1}}}{\frac{b}{(cb-1)^{2k+1}}\frac{e(cb-1)^k}{(ad-1)^k}}=\frac{d(cb-1)^{k+1}}{be(ad-1)^{k+1}}$$

Also, we have

$$x_{2k+2}=\frac{x_{2k}}{y_{2k+1}x_{2k-1}}=\frac{c(cb-1)^k}{\frac{b}{(cb-1)^{k+1}}c(cb-1)^k-1}=\frac{c(cb-1)^k}{\frac{b}{cb-1}c-1}$$

$$=c(cb-1)^{k+1}$$

$$y_{2k+2}=\frac{y_{2k}}{x_{2k+1}y_{2k}-1}=\frac{a(ad-1)^k}{\frac{d}{(ad-1)^{k+1}}a(ad-1)^k-1}=\frac{a(ad-1)^k}{\frac{d}{ad-1}a-1}$$

$$=a(ad-1)^{k+1}$$

and

$$z_{4k+3}=\frac{x_{4k+2}}{y_{4k+2}z_{4k+1}}=\frac{c(cb-1)^{2k+1}}{a(ad-1)^{2k+1}\frac{c(cb-1)^k}{af(ad-1)^k}}$$

$$=\frac{f(cb-1)^{k+1}}{(ad-1)^{k+1}}$$

$$z_{4k+4}=\frac{x_{4k+3}}{y_{4k+3}z_{4k+2}}=\frac{\frac{d}{(ad-1)^{2k+2}}}{\frac{b}{(cb-1)^{2k+2}}\frac{d(cb-1)^{k+1}}{be(ad-1)^{k+1}}}$$

$$=\frac{e(cb-1)^{k+1}}{(ad-1)^{k+1}}$$

Corollary 1. *Let* a,b,c,d,e,f *be arbitrary real numbers and let* $\{x_n,y_n,z_n\}$ *be a solution of the system* (1.1). *If* $0<a,b,c,d,e,f<1$ *then we have*

$$\lim_{n\to\infty}x_{2n-1}=\lim_{n\to\infty}y_{2n-1}=\infty$$

$$\lim_{n\to\infty}z_{4n-3}=\begin{cases}0, cb<ad\\ \infty, cb>ad\\ \frac{c}{af}, cb=ad\end{cases},\quad \lim_{n\to\infty}z_{4n-1}=\begin{cases}0, & cb<ad\\ \infty, & cb>ad\\ f, & cb=ad\end{cases}$$

and

$$\lim_{n\to\infty}x_{2n}=\lim_{n\to\infty}y_{2n}=0$$

$$\lim_{n\to\infty}z_{4n-2}=\begin{cases}0, & cb<ad\\ \infty, & cb>ad\\ \frac{d}{be}, & cb=ad\end{cases},\quad \lim_{n\to\infty}z_{4n}=\begin{cases}0, & cb<ad\\ \infty, & cb>ad\\ e, & cb=ad\end{cases}$$

Proof. From $0<a,b,c,d<1$ we have

$$0<ad<1\Rightarrow -1<ad-1<0$$

and

$$0<cb<1\Rightarrow -1<cb-1<0$$

Hence, we obtain

$$\lim_{n\to\infty}x_{2n-1}=\lim_{n\to\infty}\frac{d}{(ad-1)^n}=d\lim_{n\to\infty}\frac{1}{(ad-1)^n}$$

$$=d.\infty=\begin{cases}-\infty, & n-odd\\ +\infty, & n-even\end{cases}$$

$$\lim_{n\to\infty}y_{2n-1}=\lim_{n\to\infty}\frac{d}{(cb-1)^n}=b\lim_{n\to\infty}\frac{1}{(cb-1)^n}$$

$$=b.\infty=\begin{cases}-\infty, & n-odd\\ +\infty, & n-even\end{cases}$$

and

$$\lim_{n\to\infty}z_{4n-3}=\lim_{n\to\infty}\frac{c(cb-1)^{n-1}}{af(ad-1)^{n-1}}=\lim_{n\to\infty}\frac{c}{af}\left(\frac{cb-1}{ad-1}\right)^{n-1}$$

$$=\begin{Bmatrix}ad>cb\Rightarrow ad-1>cb-1\Rightarrow\frac{cb-1}{ad-1}<1\\ ad<cb\Rightarrow ad-1<cb-1\Rightarrow\frac{cb-1}{ad-1}>1\\ ad=cb\Rightarrow ad-1=cb-1\Rightarrow\frac{cb-1}{ad-1}=1\end{Bmatrix}$$

$$=\begin{cases}0, & cb<ad\\ \infty, & cb>ad\\ \frac{c}{af}, & cb=ad\end{cases}$$

$$\lim_{n\to\infty}z_{4n-1}=\lim_{n\to\infty}\frac{f(cb-1)^n}{(ad-1)^n}=\lim_{n\to\infty}f\left(\frac{ab-1}{ad-1}\right)^n$$

$$=\begin{Bmatrix}ad>cb\Rightarrow ad-1>cb-1\Rightarrow\frac{cb-1}{ad-1}<1\\ ad<cb\Rightarrow ad-1<cb-1\Rightarrow\frac{cb-1}{ad-1}>1\\ ad=cb\Rightarrow ad-1=cb-1\Rightarrow\frac{cb-1}{ad-1}=1\end{Bmatrix}$$

$$=\begin{cases}0, & cb<ad\\ \infty, & cb>ad\\ f, & cb=ad\end{cases}$$

Similarly, we have

$$\lim_{n\to\infty}x_{2n}=\lim_{n\to\infty}c(cd-1)^n=c\lim_{n\to\infty}(cd-1)^n=c\cdot 0=0$$

$$\lim_{n\to\infty} y_{2n} = \lim_{n\to\infty} c\left(af-1\right)^n = a\lim_{n\to\infty}\left(af-1\right)^n = a\cdot 0 = 0$$

and

$$\lim_{n\to\infty} z_{4n-2} = \lim_{n\to\infty}\frac{d\left(cb-1\right)^n}{be\left(ad-1\right)^n} = \lim_{n\to\infty}\frac{d}{be}\left(\frac{cb-1}{ad-1}\right)^n$$

$$= \begin{cases} ad > cb \Rightarrow ad-1 > cb-1 \Rightarrow \dfrac{cb-1}{ad-1} < 1 \\ ad < cb \Rightarrow ad-1 < cb-1 \Rightarrow \dfrac{cb-1}{ad-1} > 1 \\ ad = cb \Rightarrow ad-1 = cb-1 \Rightarrow \dfrac{cb-1}{ad-1} = 1 \end{cases}$$

$$= \begin{cases} 0, & cb < ad \\ \infty, & cb > ad \\ \dfrac{d}{be}, & cb = ad \end{cases}$$

$$\lim_{n\to\infty} z_{4n} = \lim_{n\to\infty}\frac{e\left(cb-1\right)^n}{\left(ad-1\right)^n} = \lim_{n\to\infty} e\left(\frac{cb-1}{ad-1}\right)^n$$

$$= \begin{cases} ad > cb \Rightarrow ad-1 > cb-1 \Rightarrow \dfrac{cb-1}{ad-1} < 1 \\ ad < cb \Rightarrow ad-1 < cb-1 \Rightarrow \dfrac{cb-1}{ad-1} > 1 \\ ad = cb \Rightarrow ad-1 = cb-1 \Rightarrow \dfrac{cb-1}{ad-1} = 1 \end{cases}$$

$$= \begin{cases} 0, & cb < ad \\ \infty, & cb > ad \\ e, & cb = ad \end{cases}$$

Corollary 2. *Let* a,b,c,d,e,f *be arbitrary real numbers and let* $\{x_n, y_n, z_n\}$ *be a solution of the system* (1.1). *If* $1 < ad$, $cb < 2$, $e \neq 0$ *and* $f \neq 0$ *then we have*

$$\lim_{n\to\infty} x_{2n-1} = \lim_{n\to\infty} y_{2n-1} = \infty$$

$$\lim_{n\to\infty} z_{4n-3} = \begin{cases} 0, & cb < ad \\ \infty, & cb > ad \\ \dfrac{c}{af}, & cb = ad \end{cases}$$

$$\lim_{n\to\infty} z_{4n-1} = \begin{cases} 0, & cb < ad \\ \infty, & cb > ad \\ f, & cb = ad \end{cases}$$

and

$$\lim_{n\to\infty} x_{2n} = \lim_{n\to\infty} y_{2n} = 0$$

$$\lim_{n\to\infty} z_{4n-2} = \begin{cases} 0, & cb < ad \\ \infty, & cb > ad \\ \dfrac{d}{be}, & cb = ad \end{cases}$$

$$\lim_{n\to\infty} z_{4n} = \begin{cases} 0, & cb < ad \\ \infty, & cb > ad \\ e, & cb = ad \end{cases}$$

Proof. From $1 < ad < 2 \Rightarrow 0 < ad-1 < 1$ we have

$$\lim_{n\to\infty}\left(ad-1\right)^n = 0$$

from $1 < cb < 2 \Rightarrow 0 < cb-1 < 1$ we have

$$\lim_{n\to\infty}\left(cb-1\right)^n = 0$$

Hence, we have

$$\lim_{n\to\infty} x_{2n-1} = \lim_{n\to\infty}\frac{d}{\left(ad-1\right)^n} = d\lim_{n\to\infty}\frac{1}{\left(ad-1\right)^n} = d\cdot\infty$$

$$= \begin{cases} -\infty, & d < 0 \\ +\infty, & d > 0 \end{cases}$$

$$\lim_{n\to\infty} y_{2n-1} = \lim_{n\to\infty}\frac{b}{\left(cb-1\right)^n} = b\lim_{n\to\infty}\frac{1}{\left(cb-1\right)^n} = b\cdot\infty$$

$$= \begin{cases} -\infty, & b < 0 \\ +\infty, & b > 0 \end{cases}$$

and

$$\lim_{n\to\infty} z_{4n-3} = \lim_{n\to\infty}\frac{c\left(cb-1\right)^{n-1}}{af\left(ad-1\right)^{n-1}} = \lim_{n\to\infty}\frac{c}{af}\left(\frac{cb-1}{ad-1}\right)^{n-1}$$

$$= \begin{cases} ad > cb \Rightarrow ad-1 > cb-1 \Rightarrow \dfrac{cb-1}{ad-1} < 1 \\ ad < cb \Rightarrow ad-1 < cb-1 \Rightarrow \dfrac{cb-1}{ad-1} > 1 \\ ad = cb \Rightarrow ad-1 = cb-1 \Rightarrow \dfrac{cb-1}{ad-1} = 1 \end{cases}$$

$$= \begin{cases} 0, & cb < ad \\ \infty, & cb > ad \\ \dfrac{c}{af}, & cb = ad \end{cases}$$

$$\lim_{n\to\infty} z_{4n-1} = \lim_{n\to\infty} \frac{f(cb-1)^n}{(ad-1)^n} = \lim_{n\to\infty} f\left(\frac{cb-1}{ad-1}\right)^n$$

$$= \begin{cases} ad > cb \Rightarrow ad-1 > cb-1 \Rightarrow \frac{cb-1}{ad-1} < 1 \\ ad < cb \Rightarrow ad-1 < cb-1 \Rightarrow \frac{cb-1}{ad-1} > 1 \\ ad = cb \Rightarrow ad-1 = cb-1 \Rightarrow \frac{cb-1}{ad-1} = 1 \end{cases}$$

$$= \begin{cases} 0, & cb < ad \\ \infty, & cb > ad \\ f, & cb = ad \end{cases}$$

Similarly, we have

$$\lim_{n\to\infty} x_{2n} = \lim_{n\to\infty} c(cb-1)^n = c\lim_{n\to\infty}(cb-1)^n = c\cdot 0 = 0$$

$$\lim_{n\to\infty} y_{2n} = \lim_{n\to\infty} a(ad-1)^n = a\lim_{n\to\infty}(ad-1)^n = a\cdot 0 = 0$$

and

$$\lim_{n\to\infty} z_{4n-2} = \lim_{n\to\infty} \frac{d(cb-1)^n}{be(ad-1)^n} = \lim_{n\to\infty} \frac{d}{be}\left(\frac{cb-1}{ad-1}\right)^n$$

$$= \begin{cases} ad > cb \Rightarrow ad-1 > cb-1 \Rightarrow \frac{cb-1}{ad-1} < 1 \\ ad < cb \Rightarrow ad-1 < cb-1 \Rightarrow \frac{cb-1}{ad-1} > 1 \\ ad = cb \Rightarrow ad-1 = cb-1 \Rightarrow \frac{cb-1}{ad-1} = 1 \end{cases}$$

$$= \begin{cases} 0, & cb < ad \\ \infty, & cb > ad \\ \frac{d}{be}, & cb = ad \end{cases}$$

$$\lim_{n\to\infty} z_{4n} = \lim_{n\to\infty} \frac{e(cb-1)^n}{(ad-1)^n} = \lim_{n\to\infty} e\left(\frac{cb-1}{ad-1}\right)^n$$

$$= \begin{cases} ad > cb \Rightarrow ad-1 > cb-1 \Rightarrow \frac{cb-1}{ad-1} < 1 \\ ad < cb \Rightarrow ad-1 < cb-1 \Rightarrow \frac{cb-1}{ad-1} > 1 \\ ad = cb \Rightarrow ad-1 = cb-1 \Rightarrow \frac{cb-1}{ad-1} = 1 \end{cases}$$

$$= \begin{cases} 0, & cb < ad \\ \infty, & cb > ad \\ e, & cb = ad \end{cases}$$

Corollary 3. *Let* $\{x_n, y_n, z_n\}$ *be the solutions of* (1.1). *If* $cb, ad \in (-\infty, 0)$, $e \neq 0$ *and* $f \neq 0$ *then*

$$\lim_{n\to\infty} x_{2n-1} = \lim_{n\to\infty} y_{2n-1} = 0$$

$$\lim_{n\to\infty} z_{4n-3} = \begin{cases} 0, & cb < ad \\ \infty, & cb > ad \\ \frac{c}{af}, & cb = ad \end{cases}$$

$$\lim_{n\to\infty} z_{4n-1} = \begin{cases} 0, & cb < ad \\ \infty, & cb > ad \\ f, & cb = ad \end{cases}$$

and

$$\lim_{n\to\infty} x_{2n} = \lim y_{2n} = \infty$$

$$\lim_{n\to\infty} z_{4n-2} = \begin{cases} 0, & cb < ad \\ \infty, & cb > ad \\ \frac{d}{be}, & cb = ad \end{cases}$$

$$\lim_{n\to\infty} z_{4n} = \begin{cases} 0, & cb < ad \\ \infty, & cb > ad \\ e, & cb = ad \end{cases}$$

Proof. From $-\infty < cb < 0 \Rightarrow -\infty < cb-1 < -1$ and $-\infty < ad < 0 \Rightarrow -\infty < ad-1 < -1$ we have

$$\lim_{n\to\infty}(cb-1)^n = \begin{cases} -\infty, & n-odd \\ +\infty, & n-even \end{cases}$$

and

$$\lim_{n\to\infty}(ad-1)^n = \begin{cases} -\infty, & n-odd \\ +\infty, & n-even \end{cases}$$

Hence, we have

$$\lim_{n\to\infty} x_{2n-1} = \lim_{n\to\infty} \frac{d}{(ad-1)^n} = d\lim_{n\to\infty} \frac{1}{(ad-1)^n} = d\cdot 0 = 0$$

$$\lim_{n\to\infty} y_{2n-1} = \lim_{n\to\infty} \frac{b}{(cb-1)^n} = b\lim_{n\to\infty} \frac{1}{(cb-1)^n} = b\cdot 0 = 0$$

and

$$\lim_{n\to\infty} z_{4n-3} = \lim_{n\to\infty}\frac{c(cb-1)^{n-1}}{af(ad-1)^{n-1}} = \lim_{n\to\infty}\frac{c}{af}\left(\frac{cb-1}{ad-1}\right)^{n-1}$$

$$= \begin{Bmatrix} ad > cb \Rightarrow ad-1 > cb-1 \Rightarrow \dfrac{cb-1}{ad-1} < 1 \\ ad < cb \Rightarrow ad-1 < cb-1 \Rightarrow \dfrac{cb-1}{ad-1} > 1 \\ ad = cb \Rightarrow ad-1 = cb-1 \Rightarrow \dfrac{cb-1}{ad-1} = 1 \end{Bmatrix}$$

$$= \begin{cases} 0, & cb < ad \\ \infty, & cb > ad \\ \dfrac{c}{af}, & cb = ad \end{cases}$$

$$\lim_{n\to\infty} z_{4n-1} = \lim_{n\to\infty}\frac{f(cb-1)^{n}}{(ad-1)^{n}} = \lim_{n\to\infty} f\left(\frac{cb-1}{ad-1}\right)^{n}$$

$$= \begin{Bmatrix} ad > cb \Rightarrow ad-1 > cb-1 \Rightarrow \dfrac{cb-1}{ad-1} < 1 \\ ad < cb \Rightarrow ad-1 < cb-1 \Rightarrow \dfrac{cb-1}{ad-1} > 1 \\ ad = cb \Rightarrow ad-1 = cb-1 \Rightarrow \dfrac{cb-1}{ad-1} = 1 \end{Bmatrix}$$

$$= \begin{cases} 0, & cb < ad \\ \infty, & cb > ad \\ f, & cb = ad \end{cases}$$

Similarly, we have

$$\lim_{n\to\infty} x_{2n} = \lim_{n\to\infty} c(cb-1)^n = c\lim_{n\to\infty}(cb-1)^n = c\cdot\infty$$

$$= \begin{cases} -\infty, & c > 0 \quad and \quad n-odd \\ +\infty, & c < 0 \quad and \quad n-odd \\ +\infty, & c > 0 \quad and \quad n-even \\ -\infty, & c < 0 \quad and \quad n-even \end{cases}$$

$$\lim_{n\to\infty} y_{2n} = \lim_{n\to\infty} a(ad-1)^n = a\lim_{n\to\infty}(ad-1)^n = a\cdot\infty$$

$$= \begin{cases} -\infty, & a > 0 \quad and \quad n-odd \\ +\infty, & a < 0 \quad and \quad n-odd \\ +\infty, & a > 0 \quad and \quad n-even \\ -\infty, & a < 0 \quad and \quad n-even \end{cases}$$

and

$$\lim_{n\to\infty} z_{4n-2} = \lim_{n\to\infty}\frac{d(cb-1)^{n}}{be(ad-1)^{n}} = \lim_{n\to\infty}\frac{d}{be}\left(\frac{cb-1}{ad-1}\right)^{n}$$

$$= \begin{Bmatrix} ad > cb \Rightarrow ad-1 > cb-1 \Rightarrow \dfrac{cb-1}{ad-1} < 1 \\ ad < cb \Rightarrow ad-1 < cb-1 \Rightarrow \dfrac{cb-1}{ad-1} > 1 \\ ad = cb \Rightarrow ad-1 = cb-1 \Rightarrow \dfrac{cb-1}{ad-1} = 1 \end{Bmatrix}$$

$$= \begin{cases} 0, & cb < ad \\ \infty, & cb > ad \\ \dfrac{d}{be}, & cb = ad \end{cases}$$

$$\lim_{n\to\infty} z_{4n} = \lim_{n\to\infty}\frac{e(cb-1)^{n}}{(ad-1)^{n}} = \lim_{n\to\infty} e\left(\frac{cb-1}{ad-1}\right)^{n}$$

$$= \begin{Bmatrix} ad > cb \Rightarrow ad-1 > cb-1 \Rightarrow \dfrac{cb-1}{ad-1} < 1 \\ ad < cb \Rightarrow ad-1 < cb-1 \Rightarrow \dfrac{cb-1}{ad-1} > 1 \\ ad = cb \Rightarrow ad-1 = cb-1 \Rightarrow \dfrac{cb-1}{ad-1} = 1 \end{Bmatrix}$$

$$= \begin{cases} 0, & cb < ad \\ \infty, & cb > ad \\ e, & cb = ad \end{cases}$$

Corollary 4. *Let* $\{x_n, y_n\}$ *be the solutions of* (1.1). *If* $a,b,c,d \in \Re$ *and* $ad, cb \in (2,+\infty)$, $e \neq 0$ *and* $f \neq 0$ *then we have*

$$\lim_{n\to\infty} x_{2n-1} = \lim_{n\to\infty} y_{2n-1} = 0$$

$$\lim_{n\to\infty} z_{4n-3} = \begin{cases} 0, & cb < ad \\ \infty, & cb > ad \\ \dfrac{c}{af}, & cb = ad \end{cases}$$

$$\lim_{n\to\infty} z_{4n-1} = \begin{cases} 0, & cb < ad \\ \infty, & cb > ad \\ f, & cb = ad \end{cases}$$

and

$$\lim_{n\to\infty} x_{2n} = \lim_{n\to\infty} y_{2n} = \infty$$

$$\lim_{n\to\infty} z_{4n-2} = \begin{cases} 0, & cb < ad \\ \infty, & cb > ad \\ \dfrac{d}{be}, & cb = ad \end{cases}$$

$$\lim_{n\to\infty} z_{4n} = \begin{cases} 0, & cb < ad \\ \infty, & cb > ad \\ e, & cb = ad \end{cases}$$

Proof. The proof is clear from Corollary 3.

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