Applied Mathematics, 2011, 2, 1011-1018
doi:10.4236/am.2011.28140 Published Online August 2011 (http://www.SciRP.org/journal/am)

On Generalized Multivalued Random Variational-Like Inclusions

Mohammad Kalimuddin Ahmad, Salahuddin
Department of Mathematics, Aligarh Muslim University, Aligarh, India
E-mail: *ahmad_kalimuddin@yahoo.co.in*, *salahuddin12@mailcity.com*
Received April 9, 2011; *revised May* 28, 2011; *accepted June* 6, 2011

Abstract

In this paper, we posed a random iterative algorithm for generalized multivalued random variational like inclusions. We define the random relaxed Lipschitz and relaxed monotone mappings and prove the existence and convergence of solutions of the random iterative sequences generated by a random iterative algorithm.

Keywords: Generalized Multivalued Random Variational Like Inclusions, Random Iterative Sequences, Measurable Space, Separable Real Hilbert Space, η-Subdifferential, Hausdorff Metric

1. Introduction

It is well know that the study of random equations involving random operators in view of their need in dealing with probabilistic models in applied sciences is very important. It has also been well documented that the introduction of the randomness leads to several questions including the measurability and probabilistic aspect of solutions [1-4].

The systematic study of random equations employing the techniques of functional analysis was first introduced by Prague school of probabilistic by Spacek [5] and Hans [6]. This has received considerable attention from numerous authors, e.g. Adomian [7], Tsokos and Padgett [8], Cho et al. [9], and Chang and Huang [10]. The main question concerning random operator equations are essen- tially the same as those of deterministic operator equa- tions, that is question of existence, uniqueness, characteri- zation, contraction and approximation of solutions. The theory of randomness, however leads to several new questions like measurability of solutions, probabilistic and statistical aspects of random solutions, estimate for the difference between the mean value of the solutions of the random equations and deterministic solutions of the averaged equations.

The theory of variational inequality provides a natural and elegant framework for study of many seemingly unrelated free boundary value problems arising in various branches of engineering, mathematics and financial sciences. Variational inequalities have many deep results dealing with nonlinear partial differential equations which play important and fundamental role in general equilibrium theory, economics, managerial sciences and operation research, see [11,12].

Motivated and inspired by the recent research work going in these fields [13-17], we consider the generalized multivalued random variational like inclusions and construct its random iterative algorithm. Then we prove the existence and convergence of random solutions of the problem and establish its equivalence with original problem.

2. Preliminaries

Let (Ω,Σ) be a measurable space and H a separable real Hilbert space whose inner product and norm are designed by $\langle x,x\rangle=\|x\|^2$. We denote by $\mathrm{B}(H), 2^H$ and $C(H)$, the class of Borel σ-field in H, the family of all nonempty power subsets of H and the family of all nonempty compact subsets of H, respectively.

A mapping $x:\Omega\to H$ is said to be measurable if for any $B\in\mathrm{B}(H)$, $\{t\in\Omega, x(t)\in B\}\in\Sigma$. A mapping $T:\Omega\times H\to H$ is called random operator if for any $x\in H$, $T(t,x)=x(t)$ is measurable. A random operator T is said to be continuous if for any $t\in\Omega$, the mapping $T(t,\cdot):H\to H$ is continuous. A multivalued mapping $v:\Omega\to 2^H$ is said to be measurable if for any

$$B\in\mathrm{B}(H), v^{-1}(B)=\{t\in\Omega: v(t)\cap B\neq\varnothing\}\in\Sigma.$$

Let $T, A, E : \Omega \times H \to 2^H$ be the random multivalued mappings and $G, g : \Omega \times H \to H$ are single valued mappings.

Let $\eta, N : \Omega \times H \times H \to H$ be two random bifunctions. We consider the problem of finding measurable mappings $x, u, v, w : \Omega \to H$ such that for all

$$t \in \Omega,\ u(t) \in T(t, x(t)),\ v(t) \in A(t, x(t)),\ w(t) \in E(t, x(t))$$

and

$$\left\langle G(t, w(t)) - N(t, u(t), v(t)), \eta(t, y(t), g(t, x(t)))\right\rangle \geq \phi(g(t, x(t))) - \phi(y(t)), \forall\, y(t) \in H, \tag{1}$$

where $\phi : H \to R \cup \{+\infty\}$ and

$$\operatorname{dom}\phi = \{z \in H : \phi(z) < \infty\}.$$

The inequality (1) is called the generalized multivalued random variational-like inclusions (GMRVLI).

Let us recall some basic concepts and results.

Definition 1. A random operator $g : \Omega \times H \to H$ is said to be

1) randomly strong monotone if there exists a measurable function $\alpha : \Omega \to (0, \infty)$ such that

$$\left\langle g(t, x(t)) - g(t, x(t)), x(t) - y(t)\right\rangle \geq \alpha(t)\left\|x(t) - y(t)\right\|^2$$
$$\forall\, x(t), y(t) \in H \text{ and fixed } t \in \Omega,$$

2) randomly Lipschitz continuous if there exists a measurable function $\beta : \Omega \to (0, \infty)$ such that

$$\left\|g(t, x(t)) - g(t, y(t))\right\| \leq \beta(t)\left\|x(t) - y(t)\right\|,$$
$$\forall\, x(t), y(t) \in H \text{ and fixed } t \in \Omega.$$

Definition 2. A random mapping $\eta : \Omega \times H \times H \to H$ is called

1) randomly monotone if

$$\left\langle x(t) - y(t), \eta(t, x(t), y(t))\right\rangle \geq 0,\quad \forall\, x(t), y(t) \in H \text{ and for every fixed } t \in \Omega, \tag{2}$$

2) randomly strict monotone if equality holds in (2) only when $x(t) = y(t)$ and for each fixed $t \in \Omega$,

3) randomly strong monotone if there exists a measurable function $\sigma : \Omega \to (0, \infty)$ such that

$$\left\langle x(t) - y(t), \eta(t, x(t), y(t))\right\rangle \geq \sigma(t)\left\|x(t) - y(t)\right\|^2,$$
$$\forall\, x(t), y(t) \in H \text{ and each fixed } t \in \Omega,$$

4) randomly Lipschitz continuous if there exists a measurable function $\delta : \Omega \to (0, \infty)$ such that

$$\left\|\eta(t, x(t), y(t))\right\| \leq \delta(t)\left\|x(t) - y(t)\right\|,$$
$$\forall\, x(t), y(t) \in H \text{ and for each fixed, } t \in \Omega$$

Definition 3. Let $\eta : \Omega \times H \times H \to H$ be a random bifunction. A proper functional $\phi : H \to R \cup \{+\infty\}$ is said to be η-subdifferentiable at a point $x(t) \in H$, for each fixed $t \in \Omega$, if there exists a point $f^*(t) \in H$, for every fixed $t \in \Omega$, such that

$$\left\langle f^*(t), \eta(t, y(t), x(t))\right\rangle \leq \phi(y(t)) - \phi(x(t)),\quad \forall\, y(t) \in H \tag{3}$$

where $f^*(t)$ is called η-subgradient of ϕ at $x(t)$, for each fixed $t \in \Omega$.

The set of all η-subgradient of ϕ at $x(t)$ for each fixed $t \in \Omega$, denoted by $\partial_\eta \phi : H \to 2^H$, is defined by

$$\partial_\eta \phi(x(t)) = \begin{cases} \left\{ f^*(t) \in H : \left\langle f^*(t), \eta(t, y(t), x(t))\right\rangle \leq \phi(y(t)) - \phi(x(t))\right\}, \forall\, y(t) \in H \text{ if } x(t) \in H \\ \varnothing, \text{ if } x(t) \notin H \end{cases}$$

Theorem 1 [18]: Let $\phi : H \to R \cup \{+\infty\}$ be a function with $\operatorname{dom}\phi = \varnothing$. Then for each fixed

$$t \in \Omega,\ x(t) \in H,\ u(t) \in T(t, x(t)),\ v(t) \in A(t, x(t)),\ w(t) \in E(t, x(t))$$

is a solution set of problem (1) if and only if

$$g(t, x(t)) \in \operatorname{dom}\phi$$

and

$$N(t, u(t), v(t)) - G(t, w(t)) \in \partial_\eta \phi(g(t, x(t))).$$

Assumption 1. A random mapping $\eta : \Omega \times H \times H \to H$ satisfies the condition

$$\eta(t, y(t), x(t)) + \eta(t, x(t), y(t)) = 0$$
$$\forall\, x(t), y(t) \in H \text{ and for each fixed } t \in \Omega.$$

Let $Q : \Omega \times H \to 2^H$ be a random multivalued mapping. Then the graph of Q denoted by $\operatorname{Graph}(Q)$ is defined as follows:

$$\operatorname{Graph}(Q) = \left\{(x(t), y(t)) \in H \times H; y(t) \in Q(t, x(t)), \text{ for fixed } t \in \Omega\right\}.$$

Definition 4. Let $\eta : \Omega \times H \times H \to H$ be a given random mapping. Then a random multivalued mapping $Q : \Omega \times H \to 2^H$ is called randomly η-monotone if

$\forall x(t), y(t) \in H$ and fixed $t \in \Omega$,

$$\langle a(t)-b(t), \eta(t, x(t), y(t))\rangle \geq 0,$$
$$\forall\, a(t) \in Q(t, x(t)), b(t) \in Q(t, y(t)).$$

Q is called randomly maximal η-monotone if and only if it is randomly η-monotone and there is no other randomly η-monotone multivalued mapping whose graph strictly contains the graph of Q.

Proposition 1. Let $\eta: \Omega \times H \times H \to H$ be randomly strict monotone and $Q: \Omega \times H \to 2^H$ be randomly η-monotone multivalued mappings. If the range of $(I+\lambda(t)Q)$, $R(I+\lambda(t)Q) = H$, for measurable mapping $\lambda: \Omega \to (0,\infty)$, where I is the identity operator, then Q is randomly maximal η-monotone. Furthermore, the inverse random operator $(I+\lambda(t)Q)^{-1}: H \to H$ is single valued.

Proof. Suppose that Q is not a randomly maximal η-monotone, then there exists $(x_0(t), a_0(t)) \notin \text{Graph}(Q)$ such that

$$\langle a_0(t)-b(t), \eta(t, x_0(t), y(t))\rangle \geq 0$$
$$\forall (y(t), b(t)) \in \text{Graph}(Q). \tag{4}$$

By assumption that $R(I+\lambda(t)Q) = H$, there exists $(x_1(t), a_1(t)) \in \text{Graph}(Q)$ such that

$$x_1(t)+\lambda(t)a_1(t) = x_0(t)+\lambda(t)a_0(t). \tag{5}$$

Since (4) is true for all fixed $t \in \Omega$, $(y(t), b(t)) \in \text{Graph}(Q)$. We have

$$\langle a_0(t)-a_1(t), \eta(t, x_0(t), x_1(t))\rangle \geq 0.$$

But from (5), we have

$$\lambda(t)(a_0(t)-a_1(t)) = x_1(t)-x_0(t)$$

and hence,

$$\frac{1}{\lambda(t)}\langle x_1(t)-x_0(t), \eta(t, x_0(t), x_1(t))\rangle \geq 0.$$

Multiplying by $\lambda(t) > 0$, we have

$$\langle x_0(t)-x_1(t), \eta(t, x_0(t), x_1(t))\rangle \leq 0.$$

Since η is randomly strict monotone, we have $x_0(t) = x_1(t)$ for fixed $t \in \Omega$ and hence from (5), we get $a_1(t) = a_0(t)$. So we reach the contradiction that $(x_1(t), a_1(t)) \in \text{Graph}(Q)$ or $(x_0(t), a_0(t)) \in \text{Graph}(Q)$. Therefore Q is randomly maximal η-monotone. For the second part, for each fixed

$$t \in \Omega, x(t), y(t) \in (I+\lambda(t)Q)^{-1}(z(t)),$$

$$\frac{z(t)-x(t)}{\lambda(t)} \in Q(x(t)) \text{ and } \frac{z(t)-y(t)}{\lambda(t)} \in Q(y(t)).$$

We set for each fixed $t \in \Omega$, $a(t) = \frac{z(t)-x(t)}{\lambda(t)}$ and $b(t) = \frac{z(t)-y(t)}{\lambda(t)}$.

Therefore, for each fixed $t \in \Omega$, $z(t) = \lambda(t)a(t)+x(t)$ and $z(t) = \lambda(t)b(t)+y(t)$.

By randomly η-monotonicity of Q, we have

$$\begin{aligned}
0 &= \langle z(t)-z(t), \eta(t, x(t), y(t))\rangle \\
&= \langle \lambda(t)a(t)+x(t)-(\lambda(t)b(t)+y(t)), \eta(t, x(t), y(t))\rangle \\
&= \lambda(t)\langle a(t)-b(t), \eta(t, x(t), y(t))\rangle \\
&\quad + \langle x(t)-y(t), \eta(t, x(t), y(t))\rangle \\
&\geq \langle x(t)-y(t), \eta(t, x(t), y(t))\rangle.
\end{aligned}$$

Since η is randomly strict monotone, we have for every fixed $t \in \Omega$, $x(t) = y(t)$ and hence $(I+\lambda(t)Q)^{-1}$ is a randomly single valued mapping.

Remark 1. If $\eta: \Omega \times H \times H \to H$ satisfies **Assumption 1** and $\phi: H \to R \cup \{+\infty\}$, then it is easy to see that the randomly multivalued map $\partial_\eta \phi: H \to 2^H$ is randomly η-monotone.

3. Random Iterative Algorithm

In this section, we use the proximal point technique to suggest a random iterative algorithm for solving the problem (1). For this purpose, we assume that $\eta: \Omega \times H \times H \to H$ is randomly strict monotone, satisfies **Assumption 1** and $\phi: H \to R \cup \{+\infty\}$ such that

$$R(I+\lambda(t)\partial_\eta \phi) = H$$

for a measurable function $\lambda: \Omega \to (0,\infty)$.

From **Proposition 1**, we have

$$J^{\phi}_{\lambda(t)}(x(t)) = (I+\lambda(t)\partial_\eta \phi)^{-1}(x(t)), \forall\, x(t) \in H$$

and for each fixed $t \in \Omega$ is a single-valued.

Lemma 1. Measurable mappings $x, u, v, w: \Omega \to H$ are solutions of the problem (1) if and only if for each fixed $t\in\Omega,\ u(t)\in T(t,x(t)),\ v(t)\in A(t,x(t)),$ $w(t)\in E(t,x(t)),$

$$g(t,x(t)) = J^{\phi}_{\lambda(t)}\left[g(t,x(t))-\lambda(t)\left\{G(t,w(t))-N(t,u(t),v(t))\right\}\right] \tag{6}$$

where $\lambda:\Omega\to(0,\infty)$ is a measurable function, $J^{\phi}_{\lambda(t)}=\left(I+\lambda(t)\partial_\eta\phi\right)^{-1}$ is so-called random proximal mapping and I stands for identity mapping on H, $\forall\, x(t)\in H$ and $t\in\Omega$.

Proof. From the definition of $J^{\phi}_{\lambda(t)}$, it follows that

$$g(t,x(t))-\lambda(t)\left[G(t,w(t))-N(t,u(t),v(t))\right] \in g(t,x(t))+\lambda(t)\partial_\eta\phi(g(t,x(t)))$$

for all measurable mapping $\lambda:\Omega\to(0,\infty)$ and hence

$$N(t,u(t),v(t))-G(t,w(t))\in\partial_\eta\phi(g(t,x(t))).$$

From the definition of $\partial_\eta\phi$, we have

$$\phi(y(t))\geq\phi(g(t,x(t)))+\left\langle N(t,u(t),v(t))-G(t,w(t)),\ \eta(t,y(t),g(t,x(t)))\right\rangle,\ \forall\, y(t)\in H$$

and for each fixed $t\in\Omega$. Thus the measurable mappings $x,u,v,w:\Omega\to(0,\infty)$ are solutions of (1). For finding the approximate solutions of (1), we can apply a successive approximation method to the problem of finding

$$x(t)\in F(t,x(t)),\ \forall\text{ fixed } t\in\Omega$$

where,

$$F(t,x(t))=x(t)-g(t,x(t)) +J^{\varphi}_{\lambda(t)}\left[g(t,x(t))-\lambda(t)\left\{G(t,w(t))-N(t,u(t),v(t))\right\}\right]$$

for measurable mapping $\lambda:\Omega\to(0,\infty)$.

On the basis of the above observation, we propose the following random iterative algorithm to compute the approximate solutions of (1).

Algorithm 1. Assume that $N,\eta:\Omega\times H\times H\to H$ are two random bifunctions and $G,g:\Omega\times H\to H$ the random single valued mappings. Let $T,A,E:\Omega\times H\to C(H)$ be the random mappings. For given $t\in\Omega, x_0(t)\in H,$ we take $u_0(t)\in T(t,x_0(t))$, $v_0(t)\in A(t,x_0(t))$ and $w_0(t)\in E(t,x_0(t))$ and let

$$x_1(t)=x_0(t)-g(t,x_0(t)) +J^{\phi_0}_{\lambda(t)}\left[g(t,x_0(t))-\lambda(t)\cdot\left\{G(t,w_0(t))-N(t,u_0(t),v_0(t))\right\}\right]$$

$\forall\, t\in\Omega$ and $\lambda:\Omega\to(0,\infty)$.

Since

$$u_0(t)\in T(t,x_0(t))\in C(H)$$

$$v_0(t)\in A(t,x_0(t))\in C(H)$$

and

$$w_0(t)\in E(t,x_0(t))\in C(H)$$

there exists

$$u_1(t)\in T(t,x_1(t)),\ v_1(t)\in A(t,x_1(t))$$

and

$$w_1(t)\in E(t,x_1(t))$$

such that

$$\|u_0(t)-u_1(t)\|\leq \mathrm{H}(T(t,x_0(t)),T(t,x_1(t)))$$

$$\|v_0(t)-v_1(t)\|\leq \mathrm{H}(A(t,x_0(t)),A(t,x_1(t)))$$

$$\|w_0(t)-w_1(t)\|\leq \mathrm{H}(E(t,x_0(t)),E(t,x_1(t)))$$

where $H(\cdot,\cdot)$ is a Hausdorff metric on $C(H)$.

Let,

$$x_2(t)=x_1(t)-g(t,x_1(t)) +J^{\phi_1}_{\lambda(t)}\left[g(t,x_1(t))-\lambda(t)\cdot\left\{G(t,w_1(t))-N(t,u_1(t),v_1(t))\right\}\right].$$

Again, since

$$u_1(t)\in T(t,x_1(t))\in C(H)$$

$$v_1(t)\in A(t,x_1(t))\in C(H)$$

and

$$w_1(t)\in E(t,x_1(t))\in C(H),$$

there exists

$$u_2(t)\in T(t,x_2(t)),\ v_2(t)\in A(t,x_2(t))$$

and

$$w_2(t)\in E(t,x_2(t))$$

such that

$$\|u_1(t)-u_2(t)\|\leq \mathrm{H}(T(t,x_1(t)),T(t,x_2(t)))$$

$$\|v_1(t)-v_2(t)\|\leq \mathrm{H}(A(t,x_1(t)),A(t,x_2(t)))$$

$$\|w_1(t)-w_2(t)\|\leq \mathrm{H}(E(t,x_1(t)),E(t,x_2(t)))$$

Continuing in this way, we can obtain random sequences $\{x_n(t)\}$, $\{u_n(t)\}$, $\{v_n(t)\}$ and $\{w_n(t)\}$ as

$$\begin{aligned} x_{n+1}(t) &= x_n(t) - g(t, x_n(t)) \\ &\quad + J_{\lambda(t)}^{\phi_n}\left[g(t, x_n(t)) - \lambda(t) \right. \\ &\quad \left. \cdot \{ G(t, w_n(t)) - N(t, u_n(t), v_n(t)) \} \right], \end{aligned} \tag{7}$$

$$u_n(t) \in T(t, x_n(t)),$$
$$\|u_n(t) - u_{n+1}(t)\| \le \mathrm{H}(T(t, x_n(t)), T(t, x_{n+1}(t)))$$

$$v_n(t) \in A(t, x_n(t)),$$
$$\|v_n(t) - v_{n+1}(t)\| \le \mathrm{H}(A(t, x_n(t)), A(t, x_{n+1}(t)))$$

$$w_n(t) \in E(t, x_n(t)),$$
$$\|w_n(t) - w_{n+1}(t)\| \le \mathrm{H}(E(t, x_n(t)), E(t, x_{n+1}(t)))$$

$n = 0, 1, 2, \cdots$.

Definition 5. For each $t \in \Omega, x(t), y(t) \in H$, a random bifunction $N : \Omega \times H \times H \to H$ is said to be

1) randomly relaxed Lipschitz continuous with respect to $T : \Omega \times H \to 2^H$ if there exists a measurable function $\kappa : \Omega \to (-\infty, 0]$ such that

$$\begin{aligned} &\langle N(t, u_1(t), \cdot) - N(t, u_2(t), \cdot), x(t) - y(t) \rangle \\ &\le \kappa(t) \|x(t) - y(t)\|^2 \end{aligned}$$

$\forall\, u_1(t) \in T(t, x(t)), u_2(t) \in T(t, y(t))$, and fixed $t \in \Omega$;

2) randomly relaxed monotone with respect to $A : \Omega \times H \to 2^H$ if there exists a measurable function $c : \Omega \to (0, \infty)$ such that

$$\begin{aligned} &\langle N(t, \cdot, v_1(t)) - N(t, \cdot, v_2(t)), x(t) - y(t) \rangle \\ &\ge -c(t) \|x(t) - y(t)\|^2 \end{aligned}$$

$\forall\, v_1(t) \in A(t, x(t)), v_2(t) \in A(t, y(t))$, and fixed $t \in \Omega$;

3) randomly Lipschitzian if for any $r : \Omega \to (0, \infty)$ such that

$$\|N(t, u_1(t), \cdot) - N(t, u_2(t), \cdot)\| \le r(t) \|x(t) - y(t)\|$$

$\forall\, u_1(t) \in T(t, x(t)), u_2(t) \in T(t, y(t))$, and fixed $t \in \Omega$.

Lemma 2. Let $\eta : \Omega \times H \times H \to H$ be a randomly strong monotone, and randomly Lipschitz continuous with measurable coefficients $\sigma : \Omega \to (0, \infty)$ and $\delta : \Omega \to (0, \infty)$ respectively which satisfies Assumption 1. Then

$$\begin{aligned} &\left\| J_{\lambda(t)}^{\phi}(x(t)) - J_{\lambda(t)}^{\varphi}(y(t)) \right\| \le \tau(t) \|x(t) - y(t)\|, \\ &\forall\, x(t), y(t) \in H, t \in \Omega \end{aligned}$$

where $\tau(t) = \dfrac{\delta(t)}{\sigma(t)}$ and $\lambda : \Omega \to (0, \infty)$.

Proof. From the definition of $J_{\lambda(t)}^{\phi}$, we have

$$J_{\lambda(t)}^{\phi}(x(t)) = (I + \lambda(t) \partial_\eta \phi)^{-1}(x(t))$$

and hence

$$\frac{1}{\lambda(t)}\left(x(t) - J_{\lambda(t)}^{\phi}(x(t)) \right) \in \partial_\eta \phi\left(J_{\lambda(t)}^{\phi}(x(t)) \right)$$

and

$$\frac{1}{\lambda(t)}\left(y(t) - J_{\lambda(t)}^{\phi}(y(t)) \right) \in \partial_\eta \phi\left(J_{\lambda(t)}^{\phi}(y(t)) \right)$$
$$\forall\, x(t), y(t) \in H$$

and each fixed $t \in \Omega$.

Since $\partial_\eta \phi$ is random η-monotone, we have

$$\begin{aligned} &\frac{1}{\lambda(t)} \left\langle x(t) - J_{\lambda(t)}^{\phi}(x(t)) - \left(y(t) - J_{\lambda(t)}^{\phi}(y(t)) \right), \right. \\ &\qquad \left. \eta\left(t, J_{\lambda(t)}^{\phi}(x(t)), J_{\lambda(t)}^{\phi}(y(t))\right) \right\rangle \ge 0. \end{aligned}$$

Multiplying by measurable function $\lambda : \Omega \to (0, \infty)$, we get

$$\begin{aligned} &\left\langle x(t) - y(t) - \left(J_{\lambda(t)}^{\phi}(x(t)) - J_{\lambda(t)}^{\phi}(y(t)) \right), \right. \\ &\qquad \left. \eta\left(t, J_{\lambda(t)}^{\phi}(x(t)), J_{\lambda(t)}^{\phi}(y(t))\right) \right\rangle \ge 0 \end{aligned}$$

or,

$$\begin{aligned} &\left\langle x(t) - y(t), \eta\left(t, J_{\lambda(t)}^{\phi}(x(t)), J_{\lambda(t)}^{\phi}(y(t))\right) \right\rangle \\ &\ge \left\langle J_{\lambda(t)}^{\phi}(x(t)) - J_{\lambda(t)}^{\phi}(y(t)), \eta\left(t, J_{\lambda(t)}^{\phi}(x(t)), J_{\lambda(t)}^{\phi}(y(t))\right) \right\rangle \end{aligned} \tag{8}$$

Since η is randomly strong monotone, we have

$$\begin{aligned} &\left\langle J_{\lambda(t)}^{\phi}(x(t)) - J_{\lambda(t)}^{\phi}(y(t)), \eta\left(t, J_{\lambda(t)}^{\phi}(x(t)), J_{\lambda(t)}^{\varphi}(y(t))\right) \right\rangle \\ &\ge \sigma(t) \left\| J_{\lambda(t)}^{\phi}(x(t)) - J_{\lambda(t)}^{\phi}(y(t)) \right\|^2. \end{aligned} \tag{9}$$

From randomly Lipschitz continuity of η, we get

$$\begin{aligned} &\left\langle x(t) - y(t), \eta\left(t, J_{\lambda(t)}^{\phi}(x(t)), J_{\lambda(t)}^{\varphi}(y(t))\right) \right\rangle \\ &\le \|x(t) - y(t)\| \left\| \eta\left(t, J_{\lambda(t)}^{\phi}(x(t)), J_{\lambda(t)}^{\varphi}(y(t))\right) \right\| \\ &\le \delta(t) \|x(t) - y(t)\| \left\| J_{\lambda(t)}^{\phi}(x(t)) - J_{\lambda(t)}^{\varphi}(y(t)) \right\|. \end{aligned} \tag{10}$$

From (8)-(10), we have

$$\left\| J^{\phi}_{\lambda(t)} x(t) - J^{\phi}_{\lambda(t)} y(t) \right\| \le \tau(t) \left\| x(t) - y(t) \right\|,$$

$$\forall\, x(t), y(t) \in H \text{ where } \tau(t) = \frac{\delta(t)}{\sigma(t)}.$$

Definition 6. A random multivalued mapping $A : \Omega \times H \to C(H)$ is called randomly H-Lipschitz continuous if there exists a measurable function $\xi : \Omega \to (0,\infty)$ such that

$$\mathrm{H}\left(A(t,x(t)), A(t,y(t))\right) \le \xi(t) \left\| x(t) - y(t) \right\|,$$

$$\forall\, x(t), y(t) \in H, \text{ and fixed } t \in \Omega.$$

Theorem 2. Let $\eta : \Omega \times H \times H \to H$ be a randomly strong monotone and randomly Lipschitz continuous with corresponding random coefficients $\sigma(t)$ and $\delta(t)$ and satisfy the **Assumption 1**. Let $T, A, E : \Omega \times H \to 2^H$ be the randomly H-Lipschitz continuous with corresponding random coefficients $\mu(t)$, $\xi(t)$ and $\gamma(t)$ respectively. Let $g, G : \Omega \times H \to H$ be a randomly Lipschitz continuous with random coefficients $\beta(t)$, $\zeta(t)$ respectively and randomly strong monotone with random coefficient $\alpha(t)$. Let $N : \Omega \times H \times H \to H$ be a randomly relaxed Lipschitz continuous with respect to random operator T with random coefficients $\kappa(t)$, randomly relaxed monotone with respect to random operator A with random coefficient $c(t)$ and randomly Lipschitz continuous with respect to first and second arguments with random coefficients $r(t)$ and $s(t)$ respectively. For each n let $\phi_n : H \to R \cup \{+\infty\}$ and $\phi : H \to R \cup \{+\infty\}$ be the mappings such that

$$R\left(I + \lambda(t)\partial_\eta \phi_n\right) = R\left(I + \lambda(t)\partial_\eta \phi\right) = H$$

for $\lambda(t) > 0$.

Assume that for fixed $t \in \Omega$,

$$\lim_{n\to\infty} \left\| J^{\phi_n}_{\lambda(t)}\left(z(t)\right) - J^{\phi_{n-1}}_{\lambda(t)}\left(z(t)\right) \right\| = 0, \text{ for } z(t) \in H \qquad (11)$$

If

$$\left| \lambda(t) - \frac{\tau^2(t)\left(c(t)-\kappa(t)\right) - \tau(t)\zeta(t)\gamma(t)(1-p(t))}{\left(\tau(t)\left(r(t)\mu(t)+s(t)\xi(t)\right)\right)^2 - \left(\tau(t)\zeta(t)\gamma(t)\right)^2} \right|$$

$$< \frac{\sqrt{\left(\tau^2(t)\left(c(t)-\kappa(t)\right) - \tau(t)\zeta(t)\gamma(t)\left(1-p(t)\right)\right)^2 - B}}{\left(\tau(t)\left(r(t)\mu(t)+s(t)\xi(t)\right)\right)^2 - \left(\tau(t)\zeta(t)\gamma(t)\right)^2}$$

where,

$$B = \left(\left(\tau(t)\left(r(t)\mu(t)+s(t)\xi(t)\right)\right)^2 - \left(\tau(t)\zeta(t)\gamma(t)\right)^2\right) \cdot \left(\tau(t)+p(t)-1\right)\left(\tau(t)-p(t)+1\right),$$

$$c(t) - \kappa(t) > \frac{\zeta(t)\gamma(t)\left(1-p(t)\right)}{\tau(t)} + \frac{\sqrt{B}}{\tau^2(t)}$$

$$\begin{aligned} & r(t)\mu(t) + s(t)\xi(t) > \zeta(t)\gamma(t), \\ & p(t) < 1,\ 1 - p(t) < \tau(t),\ c(t) > \kappa(t), \\ & p(t) = \left(1+\tau(t)\right)\sqrt{1 - 2\alpha(t) + \beta^2(t)}. \end{aligned} \qquad (12)$$

Then there exist, for any fixed $t \in \Omega$, $x(t) \in H$, $u(t) \in T(t, x(t))$, $v(t) \in A(t, x(t))$, $w(t) \in E(t, x(t))$ satisfying the generalized multivalued random variational like inclusions (1) and $u_n(t) \to u(t)$, $v_n(t) \to v(t)$, $w_n(t) \to w(t)$, $x_n(t) \to x(t)$ in H for fixed $t \in \Omega$, $(n \to \infty)$, where the random iterative sequences $\{u_n(t)\}$, $\{v_n(t)\}$, $\{w_n(t)\}$ and $\{x_n(t)\}$ are generated by random iterative **Algorithm 1**.

Proof. From **Algorithm 1**, we have

$$\begin{aligned}
& \left\| x_{n+1}(t) - x_n(t) \right\| \\
&= \Big\| x_n(t) - g(t, x_n(t)) + J^{\phi}_{\lambda(t)}\Big[g(t, x_n(t)) \\
&\quad - \lambda(t)\left\{ G(t, w_n(t)) - N(t, u_n(t), v_n(t)) \right\}\Big] - x_{n-1}(t) \\
&\quad + g(t, x_{n-1}(t)) - J^{\phi_{n-1}}_{\lambda(t)}\Big[g(t, x_{n-1}(t)) - \lambda(t) \\
&\quad \cdot \left\{ G(t, w_{n-1}(t)) - N(t, u_{n-1}(t), v_{n-1}(t)) \right\}\Big]\Big\| \\
&\le \left\| x_n(t) - x_{n-1}(t) - \left(g(t, x_n(t)) - g(t, x_{n-1}(t)) \right) \right\| \\
&\quad + \Big\| J^{\phi_n}_{\lambda(t)}\Big[g(t, x_n(t)) - \lambda(t)\left\{ G(t, w_n(t)) \right. \\
&\quad \left. - N(t, u_n(t), v_n(t)) \right\}\Big] - J^{\phi_{n-1}}_{\lambda(t)}\Big[g(t, x_{n-1}(t)) \\
&\quad - \lambda(t)\left\{ G(t, w_{n-1}(t)) - N(t, u_{n-1}(t), v_{n-1}(t)) \right\}\Big]\Big\| \\
&\le \left\| x_n(t) - x_{n-1}(t) - \left(g(t, x_n(t)) - g(t, x_{n-1}(t)) \right) \right\| \\
&\quad + \tau(t)\Big\| g(t, x_n(t)) - g(t, x_{n-1}(t)) - \lambda(t)\Big[G(t, w_n(t)) \\
&\quad - G(t, w_{n-1}(t)) - N(t, u_n(t), v_n(t)) \\
&\quad + N(t, u_{n-1}(t), v_{n-1}(t)) \Big]\Big\| + \left\| J^{\phi_n}_{\lambda(t)}\left(h(x_{n-1}(t)) \right) \right. \\
&\quad \left. - J^{\phi_{n-1}}_{\lambda(t)}\left(h(x_{n-1}(t)) \right) \right\| \le \left(1 + \tau(t)\right)\Big\| x_n(t) - x_{n-1}(t) \\
&\quad - \left(g(t, x_n(t)) - g(t, x_{n-1}(t)) \right)\Big\| + \tau(t)\Big\| x_n(t) - x_{n-1}(t) \\
&\quad + \lambda(t)\left(N(t, u_n(t), v_n(t)) - N(t, u_{n-1}(t), v_{n-1}(t)) \right)\Big\| \\
&\quad + \tau(t)\lambda(t)\left\| G(t, w_n(t)) - G(t, w_{n-1}(t)) \right\| \\
&\quad + \left\| J^{\phi_n}_{\lambda(t)}\left(h(x_{n-1}(t)) \right) - J^{\phi_{n-1}}_{\lambda(t)}\left(h(x_{n-1}(t)) \right) \right\|
\end{aligned} \qquad (13)$$

where,

$$h(x_n(t)) = g(t, x_n(t)) - \lambda(t)\big(G(t, w_n(t)) - N(t, u_n(t), v_n(t))\big).$$

Since g is randomly strong monotone and randomly Lipschitz continuous, we have

$$\begin{aligned}
&\left\|x_n(t) - x_{n-1}(t) - \big(g(t, x_n(t)) - g(t, x_{n-1}(t))\big)\right\|^2 \\
&\le \|x_n(t) - x_{n-1}(t)\|^2 \\
&-2\langle g(t, x_n(t)) - g(t, x_{n-1}(t)), x_n(t) - x_{n-1}(t)\rangle \\
&+\|g(t, x_n(t)) - g(t, x_{n-1}(t))\|^2 \\
&\le \|x_n(t) - x_{n-1}(t)\|^2 - 2\alpha(t)\|x_n(t) - x_{n-1}(t)\|^2 \\
&+\beta^2(t)\|x_n(t) - x_{n-1}(t)\|^2 \\
&\le \left[1 - 2\alpha(t) + \beta^2(t)\right]\|x_n(t) - x_{n-1}(t)\|^2.
\end{aligned} \tag{14}$$

Since T, A, E are randomly H-Lipschitz continuous, N a randomly Lipschitz continuous with respect to first and second arguments and g also a randomly Lipschitz continuous, we have

$$\begin{aligned}
&\|G(t, w_n(t)) - G(t, w_{n-1}(t))\| \\
&\le \zeta(t)\|w_n(t) - w_{n-1}(t)\| \\
&\le \zeta(t)\mathrm{H}\big(E(t, x_n(t)), E(t, x_{n-1}(t))\big) \\
&\le \zeta(t)\gamma(t)\|x_n(t) - x_{n-1}(t)\|,
\end{aligned} \tag{15}$$

$$\begin{aligned}
&\|N(t, u_n(t), v_n(t)) - N(t, u_{n-1}(t), v_n(t))\| \\
&\le r(t)\|u_n(t) - u_{n-1}(t)\| \\
&\le r(t)\mathrm{H}\big(T(t, x_n(t)), T(t, x_{n-1}(t))\big) \\
&\le r(t)\mu(t)\|x_n(t) - x_{n-1}(t)\|,
\end{aligned} \tag{16}$$

$$\begin{aligned}
&\|N(t, u_{n-1}(t), v_n(t)) - N(t, u_{n-1}(t), v_{n-1}(t))\| \\
&\le s(t)\|v_n(t) - v_{n-1}(t)\| \\
&\le s(t)\mathrm{H}\big(A(t, x_n(t)), A(t, x_{n-1}(t))\big) \\
&\le s(t)\xi(t)\|x_n(t) - x_{n-1}(t)\|.
\end{aligned} \tag{17}$$

Further, since N is randomly relaxed Lipschitz continuous with respect to random operator T and randomly relaxed monotone with respect to random operator A, we have

$$\begin{aligned}
&\big\|x_n(t) - x_{n-1}(t) + \lambda(t) \\
&\cdot\big(N(t, u_n(t), v_n(t)) - N(t, u_{n-1}(t), v_{n-1}(t))\big)\big\|^2 \\
&\le \|x_n(t) - x_{n-1}(t)\|^2 + 2\lambda(t)\langle N(t, u_n(t), v_n(t)) \\
&-N(t, u_{n-1}(t), v_n(t)), x_n(t) - x_{n-1}(t)\rangle \\
&+2\lambda(t)\langle N(t, u_{n-1}(t), v_n(t)) \\
&-N(t, u_{n-1}(t), v_{n-1}(t)), x_n(t) - x_{n-1}(t)\rangle \\
&+\lambda^2(t)\|N(t, u_n(t), v_n(t)) - N(t, u_{n-1}(t), v_{n-1}(t))\|^2 \\
&\le \|x_n(t) - x_{n-1}(t)\|^2 + 2\lambda(t)\kappa(t)\|x_n(t) - x_{n-1}(t)\|^2 \\
&-2\lambda(t)c(t)\|x_n(t) - x_{n-1}(t)\|^2 \\
&+\lambda^2(t)\big(r(t)\mu(t) + s(t)\xi(t)\big)^2\|x_n(t) - x_{n-1}(t)\|^2 \\
&\le \Big[1 - 2\lambda(t)\big(c(t) - \kappa(t)\big) + \lambda^2(t) \\
&\cdot\big(r(t)\mu(t) + s(t)\xi(t)\big)^2\Big]\|x_n(t) - x_{n-1}(t)\|^2.
\end{aligned} \tag{18}$$

Combining (13) and (18), we get

$$\begin{aligned}
&\|x_{n+1}(t) - x_n(t)\| \\
&\le \Big[\big(1 + \tau(t)\big)\sqrt{1 - 2\alpha(t) + \beta^2(t)} + \tau(t) \\
&\cdot\sqrt{1 - 2\lambda(t)\big(c(t) - \kappa(t)\big) + \lambda^2(t)\big(r(t)\mu(t) + s(t)\xi(t)\big)^2} \\
&+\tau(t)\lambda(t)\zeta(t)\gamma(t)\Big]\|x_n(t) - x_{n-1}(t)\| \\
&+\left\|J_{\lambda(t)}^{\phi_n}\big(h(x_{n-1}(t))\big) - J_{\lambda(t)}^{\phi_{n-1}}\big(h(x_{n-1}(t))\big)\right\| \\
&\le \Big[p(t) + \tau(t) \\
&\cdot\sqrt{1 - 2\lambda(t)\big(c(t) - \kappa(t)\big) + \lambda^2(t)\big(r(t)\mu(t) + s(t)\xi(t)\big)^2} \\
&+\tau(t)\lambda(t)\zeta(t)\gamma(t)\Big]\|x_n(t) - x_{n-1}(t)\| \\
&+\left\|J_{\lambda(t)}^{\phi_n}\big(h(x_{n-1}(t))\big) - J_{\lambda(t)}^{\phi_{n-1}}\big(h(x_{n-1}(t))\big)\right\| \\
&\le \theta(t)\|x_n(t) - x_{n-1}(t)\| \\
&+\left\|J_{\lambda(t)}^{\phi_n}\big(h(x_{n-1}(t))\big) - J_{\lambda(t)}^{\varphi_{n-1}}\big(h(x_{n-1}(t))\big)\right\|
\end{aligned} \tag{19}$$

where $p(t) = \big(1 + \tau(t)\big)\sqrt{1 - 2\alpha(t) + \beta^2(t)}$ and

$$\begin{aligned}
\theta(t) &= p(t) + \tau(t) \\
&\cdot\sqrt{1 - 2\lambda(t)\big(c(t) - \kappa(t)\big) + \lambda^2(t)\big(r(t)\mu(t) + s(t)\xi(t)\big)^2} \\
&+\tau(t)\lambda(t)\zeta(t)\gamma(t).
\end{aligned}$$

Now, from (12), we have $\theta(t)<1$, and for fixed $t\in\Omega$,

$$\lim_{n\to\infty}\left\|J_{\lambda(t)}^{\phi_n}\left(h\left(x_{n-1}(t)\right)\right)-J_{\lambda(t)}^{\phi_{n-1}}\left(h\left(x_{n-1}(t)\right)\right)\right\|=0,$$

it follows from (19), that $\{x_n(t)\}$ is a Cauchy sequence in H. Since H is complete, we may suppose that $x_n(t)\to x(t)\in H$. Now we prove that $u_n(t)\to u(t)\in T(t,x(t))$, $v_n(t)\to v(t)\in A(t,x(t))$ and $w_n(t)\to w(t)\in E(t,x(t))$.

From **Algorithm 1**, we have

$$\begin{aligned}\left\|u_{n+1}(t)-u_n(t)\right\| &\le \mathrm{H}\left(T\left(t,x_{n+1}(t)\right),T\left(t,x_n(t)\right)\right)\\ &\le \mu(t)\left\|x_{n+1}(t)-x_n(t)\right\|\end{aligned}$$

$$\begin{aligned}\left\|v_{n+1}(t)-v_n(t)\right\| &\le \mathrm{H}\left(A\left(t,x_{n+1}(t)\right),A\left(t,x_n(t)\right)\right)\\ &\le \xi(t)\left\|x_{n+1}(t)-x_n(t)\right\|\end{aligned}$$

$$\begin{aligned}\left\|w_{n+1}(t)-w_n(t)\right\| &\le \mathrm{H}\left(E\left(t,x_{n+1}(t)\right),E\left(t,x_n(t)\right)\right)\\ &\le \gamma(t)\left\|x_{n+1}(t)-x_n(t)\right\|,\end{aligned}$$

which imply that the random sequences $\{u_n(t)\}$, $\{v_n(t)\}$ and $\{w_n(t)\}$ are Cauchy random sequences in H. Let $u_n(t)\to u(t)$, $v_n(t)\to v(t)$ and $w_n(t)\to w(t)$. Now we will show that $u(t)\in T(t,x(t))$, $v(t)\in A(t,x(t))$ and $w(t)\in E(t,x(t))$. In fact,

$$\begin{aligned}&d\left(u(t),T\left(t,x(t)\right)\right)\\ &=\inf\left\{\left\|u(t)-z(t)\right\|:\text{for fixed } t\in\Omega,\ z(t)\in T\left(t,x(t)\right)\right\}\\ &\le\left\|u(t)-u_n(t)\right\|+d\left(u_n(t),T\left(t,x(t)\right)\right)\\ &\le\left\|u(t)-u_n(t)\right\|+\mathrm{H}\left(T\left(t,x_n(t)\right)T\left(t,x(t)\right)\right)\\ &\le\left\|u(t)-u_n(t)\right\|+\mu(t)\left\|x_n(t)-x(t)\right\|\to 0 \text{ as } n\to\infty.\end{aligned}$$

Hence $d\left(u(t),T\left(t,x(t)\right)\right)=0$, since sequence of measurable map is also measurable and therefore $u(t)\in T\left(t,x(t)\right)$ for fixed $t\in\Omega$, similarly we can prove $v(t)\in A\left(t,x(t)\right)$ for fixed $t\in\Omega$ and $w(t)\in E\left(t,x(t)\right)$. This completes the proof.

4. References

[1] A. T. Bharucha-Reid, "Random Integral Equations," Academic Press, New York, 1972.

[2] S. S. Chang, "Variational Inequality and Complementarity Problem Theory with Applications," Shanghai Scientific and Technology Literature Publishing House, Shanghai, 1991.

[3] D. O'Regan and N. Shahzad, "Random Approximation and Random Fixed Point Theory for Random Nonself Multimaps," *New Zeal Land Journal of Mathematics*, Vol. 34, No. 2, 2005, pp. 103-123.

[4] Salahuddin and M. K. Ahmad, "Collectively Random Fixed Point Theorems and Application," *Pan American Mathematical Journal*, Vol. 20, No. 3, 2010, pp. 69-84.

[5] A. Spacek, "Zufallige Gleichungen," *Czechoslovak Mathematical Journal*, Vol. 5, No. 4, 1955, pp. 462-466.

[6] O. Hans, "Reduzierende Zufallige Transformalionen," *Czechoslovak Mathematics Journal*, Vol. 7, 1957, pp. 154-158.

[7] G. Adomian, "Random Operator Equations in Mathematical Physics," *International Journal of Mathematical Physics*, Vol. 11, 1970, pp. 1069-1084. doi:10.1063/1.1665198

[8] C. P. Tsokos, V. J. Padgett, "Random Integral Equations with Applications to Life Science and Engineering," Academic Press, New York, 1974.

[9] Y. J. Cho, M. F. Khan and Salahuddin, "Notes on Random Fixed Point Theorems," *Journal of Korean Society of Mathematics Education, Series B*: *Pure and Applied Mathematics*, Vol. 13, No. 3, 2006, pp. 227-236.

[10] S. S. Chang and N. J. Huang, "Generalized Random Multivalued Quasi Complementarity Problems," *Indian Journal of Mathematics*, Vol. 35, 1993, pp. 396-405.

[11] C. Baiocchi and A. Capelo, "Variational and Quasi- Variational Inequalities, Applications to Free Boundary Problems," John Wiley, New York, 1984.

[12] Salahuddin, "Some Aspects of Variational Inequalities," Ph.D. Thesis, Aligarh Muslim University, Aligarh, 2000.

[13] N. J. Huang, X. Long and Y. J. Cho, "Random Generalized Nonlinear Variational Inclusions," *Bulletin of Korean Mathematics Society*, Vol. 34, No. 4, 1997, pp. 603-615.

[14] T. Hussain, E. Tarafdar and X. Z. Yuan, "Some Results in Random Generalized Games and Random Quasi Variational Inequalities," *Far East Journal Mathematical Sciences*, Vol. 2, 1994, pp. 35-55.

[15] N. X. Tan, "Random Quasi-Variational Inequalities," *Mathematische Nachrichten*, Vol. 125, 1986, pp. 319-328.

[16] R. U. Verma, "On Generalized Variational Inequalities involving Relaxed Lipschitz and Relaxed Monotone Operators," *Journal Mathematical Analysis Application*, Vol. 213, 1997, pp. 387-392. doi:10.1006/jmaa.1997.5556

[17] X. Z. Yuan, "Noncompact Random Generalized Games and Random Quasi Variational Inequalities," *Journal Applied Stochastics Analysis*, Vol. 7, No. 4, 1994, pp. 467-486. doi:10.1155/S1048953394000377

[18] C. H. Lee, Q. H. Ansari and J. C. Yao, "A Perturbed Algorithm for Strongly Nonlinear Variational Inclusions," *Bulletin Australian Mathematical Society*, Vol. 62, No. 3, 2000, pp. 417-426. doi:10.1017/S0004972700018931

