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Analysis of an Il'in Scheme for a System of Singularly Perturbed Convection-Diffusion Equations

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Abstract

In this paper, a numerical solution for a system of singularly perturbed convection-diffusion equations is studied. The system is discretized by the Il'in scheme on a uniform mesh. It is proved that the numerical scheme has first order accuracy, which is uniform with respect to the perturbation parameters. We show that the condition number of the discrete linear system obtained from applying the Il'in scheme for a system of singularly perturbed convection-diffusion equations is O(N) and the relevant coefficient matrix is well conditioned in comparison with the matrices obtained from applying upwind finite difference schemes on this problem. Numerical results confirm the theory of the method.

Keywords: Convection-Diffusion, Il'in Scheme, Uniform Convergence, Singular Perturbation, Condition Number

1. Introduction

Consider the following system of l coupled singularly perturbed convection-diffusion equations: Find $\mathbf{u} = (u_1,\cdots,u_l) \in \left(C^2(0,1)\cap C[0,1]\right)^l$ such that

$$
\begin{aligned}
&L\mathbf{u} := -E\mathbf{u}'' - B\mathbf{u}' + A\mathbf{u} = \mathbf{f},\\
&x \in \Omega = (0,1), \mathbf{u}(0) = \mathbf{u}(1) = 0,
\end{aligned}
\tag{1.1}
$$

With $E = \operatorname{diag}\{\varepsilon_1, \varepsilon_2, \cdots, \varepsilon_l\}$, where ε_i's for $i = 1,\cdots,l$ are known small positive diffusion co-efficients, $A = (a_{ij}(x))$ is an $l \times l$ matrix, and $\mathbf{f} = f_i(x)$ for $i = 1,\cdots,l$ is a vector-valued right hand side. Furthermore, we shall assume that B is diagonal with diagonal elements $b_i(x)$ and define

$$\beta_k = \min_{x\in[0,1]} |b_k(x)| > 0 \text{ for } k = 1,\cdots,l. \tag{1.2}$$

Some results for systems of singularly perturbed of differential equations can be found in: Linss and Madden [1], Madden and Stynes [2] and Gracia and Lisbona [3].

Bellew and O'Riordan [4], Cen [5], Amiraliyev [6] and Andreev [7] used the finite difference method for a coupled system of two singularly perturbed convection – diffusion equations.

T. Linss [8] considered an upwind finite difference scheme on special layer adapted Shishkin and Bakhvalov meshes. He showed that the error in the discrete maximum norm is bounded by $CN^{-1}\ln N$ and CN^{-1} for Shishkin and Bakhvalov meshes respectively, where C is independent of the perturbation parameters ε_k for $k = 1,\cdots,l$, and N is the number of mesh points used.

The discrete linear systems that arise from Shishkin or Bakhvalov meshes do not have a good condition number. H. G. Roos in [9] showed that the condition number of the discrete linear system associated with the upwind schemes on Shishkin meshes for a single equation is $O\left(\varepsilon^{-2}N^2\ln^{-2}N\right)$, which is not good when ε is small. In fact, if A is the coefficient matrix of the linear system associated with the upwind schemes on Shishkin meshes, then $\|A\| \le C\dfrac{N^2}{\varepsilon^2\ln^2 N}$ and $\|A^{-1}\| \le C$, where C is a constant independent of ε and the norm is the discrete maximum norm for matrices. Nevertheless, he proposed a precondition which has reduced this condition number to $O\left(N^2\ln^{-1}N\right)$.

In this paper, we study the Il'in scheme (see [10]), for problem (1.1). We show that for this method the error in the discrete maximum norm is bounded by CN^{-1}, where C is independent of the perturbation parameters ε_k $(k = 1,\cdots,l)$, and we prove that if A is the coefficients matrix of the linear system associated with the Il'in scheme on uniform meshes, then $\|A\| \le CN$ and $\|A^{-1}\| < C$. So the condition number of the discrete linear system associated with the Il'in scheme on uniform meshes for single equation is $O(N)$, which is better in comparison with the precondition of upwind schemes on Shishkin meshes.

The paper is organized as follows: in section 2, we give some properties of the solution of (1.1) and in section 3, we state the difference approximation Il'in scheme. In section 4, we analyze the error of Il'in scheme applied to (1.1), and some numerical examples are presented in section 5.

2. Properties of the Exact Solutions

In this section, we analyze the exact solution of (1.1). Assume that $v \in L^\infty([0,1])$ and $\omega = \{x_i \mid 0 = x_0 < x_1 < \cdots < x_N = 1\}$, where $x_i = ih$ and $h = 1/N$. Consider the following norms:

$$\|v\|_\infty = \max_{x\in(0,1)} |v(x)|, \|v\|_{\infty,\omega} = \max_{x_i\in\omega} |v_i|,$$

$$\|v\|_1 = \int_0^1 |v(x)| \mathrm{d}x, \|v\|_{1,\omega} = h\sum_{i=0}^{N} | |v_i|,$$

where $v_i = v(x_i)$ for $i = 0,\cdots,N$. Consider the following discrete norm:

$$\|v\|_{W_{1,1}^h} = \|v\|_{1,\omega} + h\sum_{i=1}^{N} |D^- v_i|,$$

where the space $W_{1,1}^h$ contains $v \in R^{N+1}$, such that $\|v\|_{W_{1,1}^h} < \infty$. Difference operators are defined as follows:

$$D^+ u_i = \frac{u_{i+1} - u_i}{h}, D^- u_i = \frac{u_i - u_{i-1}}{h}, D^0 u_i = \frac{u_{i+1} - u_{i-1}}{2h}.$$

Let $R_0^{N+1} = \{v \in R^{N+1} \mid v_0 = v_N = 0\}$, and consider the space $\overset{0}{W}{}_{1,1}^h$ on R_0^{N+1}. We define the following norm on $\overset{0}{W}{}_{1,1}^h$ as $\|v\|_{\overset{0}{W}{}_{1,1}^h} = h\sum_{i=1}^{N} |D^- v_i|$.

Also for the dual of $\overset{0}{W}{}_{1,1}^h$, which is denoted by $W_{-1,\infty}^h$, we define

$$\|f\|_{W_{-1,\infty}^h} = \max \frac{|\langle v, f \rangle|}{\|v\|_{\overset{0}{W}{}_{1,1}^h}},$$

where $\langle v, f\rangle = h\sum_{i=1}^{N-1} v_i f_i$. In [11], it has been shown that

$$\|D^+ f\|_{W_{-1,\infty}^h} = \min_{c\in R} \|f - c\|_{\infty,\omega}, \tag{2.1}$$

or equivalently

$$\|f\|_{W_{-1,\infty}^h} = \min_{F: D^+ F = f} \|F\|_{\infty,\omega}. \tag{2.2}$$

For the vector-valued function $\mathbf{v}(x) = [v_1(x),\cdots,v_l(x)]^T$, consider the infinity norm as

$$\|\mathbf{v}(x)\|_\infty = \max_{i=1,\cdots,l} \|v_i\|_\infty.$$

To estimate the error in our difference approximation, we shall require some bounds for the derivatives of the solution of (1.1), under the assumptions

$$\begin{gathered}\sum_{j=1}^{l} a_{ij}(x) \ge 0,\ a_{ii}(x) > 0 \text{ and } a_{ij}(x) \le 0 \\ \text{for } i \ne j,\ x \in [0,1] \text{ and } i, j = 1,\cdots,l\end{gathered} \tag{2.3}$$

and strict inequality hold at least for one k *i.e.*,

$$\sum_{j=1}^{l} a_{kj}(x) > 0 \tag{2.4}$$

Lemma 2.1 If $\mathbf{y} = (y_1(x),\cdots,y_l(x))^T$, $Ly \ge 0$ in Ω and $\mathbf{y}(0) \ge 0, \mathbf{y}(1) \ge 0$ then $\mathbf{y}(x) \ge 0$ in Ω.

Proof. Let $y_i(x)$ be minimum at t_i for $i = 1,2,\cdots,l$, i.e., $y_i(t_i) = \min_{x\in\Omega} y_i(x)$ and also assuming

$$y_j(t_j) = \min_{i=1,\cdots,l} y_i(t_i) \tag{2.5}$$

If $y_j(t_j) \ge 0$ the lemma is proved. So let $y_j(t_j) < 0$. If $y_j(t_j) = y_k(t_j)$ for $k = 1,2,\cdots,l$, then it follows that $\mathbf{y}'(t_j) = 0$ and $\mathbf{y}''(t_j) \ge 0$. By (1.1)

$$\begin{aligned}L\mathbf{y}\big|_{t_j} &:= -E\mathbf{y}''(t_j) - B\mathbf{y}'(t_j) + A\mathbf{y}(t_j) \\ &\le A\mathbf{y}(t_j) = y_k(t_j) A.\mathbf{1}\end{aligned}$$

In this case according to (2.4) since $y_k(t_j) < 0$, the k th component of $A\mathbf{y}(t_j)$ is negative, which is a contradiction to the assumption of the lemma. If there is a k with $1 \le k \le l$ such that $y_j(t_j) < y_k(t_j)$ then

$$\begin{aligned}&-\varepsilon_j y_j''(t_j) - b_j(t_j) y_j'(t_j) + \sum_{m=1}^{l} a_{jm}(t_j) y_m(t_j) \\ &= -\varepsilon_j y_j''(t_j) + \sum_{m=1}^{l} a_{jm}(t_j) y_j(t_j) \\ &+ \sum_{m=1, m\ne j}^{l} a_{jm}(t_j)\big(y_m(t_j) - y_j(t_j)\big) \\ &\le \max_{m=1,\cdots,l} \big(y_m(t_j) - y_j(t_j)\big) \sum_{m=1, m\ne j}^{l} a_{jm}(t_j) + y_j(t_j) \sum_{m=1}^{l} a_{jm}(t_j).\end{aligned}$$

If $\sum_{m=1,m\ne j}^{l} a_{jm}(t_j) < 0$, it is obvious that the right hand side of the above inequality is negative. If $\sum_{m=1,m\ne j}^{l} a_{jm}(t_j) = 0$, then since $a_{jj} > 0$ we have $\sum_{m=1}^{l} a_{jm}(t_j) > 0$ so the right hand side is negative and again we reach a contradiction. So the lemma is proved.

Theorem 2.2 Suppose u solves (1.1) and assume that a_{ij} for $i, j = 1,\cdots,l$ satisfies (2.3) and (2.4), then $\|u\|_\infty \le C\|\mathbf{f}\|_\infty$.

By theorem 2.2, T.Linss [10] has proved the following lemma.

Lemma 2.3 Let **u** be the solution of (1.1) and suppose (1.2), (2.3) and (2.4) hold. Then for $x \in [0,1]$ and $n = 0,1$

$$\left|u_k^{(n)}(x)\right| \le \begin{cases} C\left[1+\varepsilon_k^{-n}\exp\left(-\dfrac{\beta_k x}{\varepsilon_k}\right)\right] & b_k \ge \beta_k \\ C\left[1+\varepsilon_k^{-n}\exp\left(-\dfrac{\beta_k(1-x)}{\varepsilon_k}\right)\right] & b_k \le -\beta_k. \end{cases}$$

By lemma (2.3) and the application of the technique mentioned in [12], the following lemma can be proved in a similar way.

Lemma 2.4 Let **u** satisfy (1.1). If

$$u_k(x) = v_k(x) + z_k(x), \text{ for } k = 1,\cdots,l,$$

where

$$v_k(x) = \begin{cases} \dfrac{\varepsilon_k u_k'(0)}{b_k(0)}\exp\left(-b_k(0)\varepsilon_k^{-1}x\right) & b_k \ge \beta_k \\ \dfrac{\varepsilon_k u_k'(1)}{b_k(1)}\exp\left(b_k(1)\varepsilon_k^{-1}(1-x)\right) & b_k \le -\beta_k, \end{cases}$$

then, for $x \in [0,1]$ and $n = 0,1$

$$\left|z_k^{(n)}(x)\right| \le C.$$

Remark 2.5 By lemma 2.4 and direct computation, without loss of generality, assume that $b_k \ge \beta_k$; hence, we have

$$\begin{aligned} L_k\mathbf{z} &= -\varepsilon_k z_k'' - b_k(x)z_k'(x) + \sum_{m=1}^{l} a_{km}z_m(x) \\ &= f_k - \sum_{m=1}^{l} a_{km}v_m(x) + \varepsilon_k^{-1}b_k(0)\left[b_k(0)-b_k(x)\right]v_k(x) \\ &= g_k(x) \end{aligned}$$

and for $n = 0,1$

$$\left|g_k^{(n)}(x)\right| \le C\left[1+\varepsilon_k^{-n}\exp\left(\frac{-\beta_k x}{\varepsilon_k}\right)\right], k = 1,\cdots,l$$

3. Discretization

In this section, we deal with the discretization of prob-em (1.1) by the Il'in scheme. We apply the Il'in scheme : Find $U \in \left(R_0^{N+1}\right)^l$ such that:

$$\begin{aligned} \left[L_kU\right]_j &:= -\varepsilon_k\sigma_{k;j}D^+D^-U_{kj} - b_{k;j}D^0U_{kj} \\ &+ \sum_{m=1}^{l} a_{km;j}U_{mj} = f_{k;j}, \end{aligned} \tag{3.1}$$

for $k = 1,\cdots,l$ and $j = 1,\cdots,N-1$, where $\sigma_k(x) = q_k(x)\coth q_k(x)$ with $q_k(x) = \dfrac{hb_k(x)}{2\varepsilon_k}$, U_{kj} is the approximate value for $u_k(x_j)$ that is obtained by the Il'in scheme, $b_{k;j} = b_k(x_j)$ and $a_{km;j} = a_{km}(x_j)$. Consider the diagonal matrices

$$Q(x) = \mathrm{diag}\left\{q_1(x),\cdots,q_l(x)\right\}$$

and

$$\coth Q(x) = \mathrm{diag}\left\{\coth q_1(x),\cdots,\coth q_l(x)\right\}.$$

We define

$$\Sigma = Q(x)\coth Q(x) = \frac{hE^{-1}B}{2}\coth\frac{hE^{-1}B}{2}.$$

We rewrite (1.1) in matrix form as follows:

$$\begin{aligned} &-E\Sigma_j D^+D^-U_j - B_jD^0U_j + A_jU_j = f_j \\ &\text{for } j = 1,\cdots N-1, \end{aligned} \tag{3.2}$$

where $U_j = \left(U_{1j},\cdots,U_{lj}\right)^T$, $D^+U_j = \left(D^+U_{1j},\cdots,D^+U_{lj}\right)^T$, $D^-U_j = \left(D^-U_{1j},\cdots,D^-U_{lj}\right)^T$ and $D^0U_j = \left(D^0U_{1j},\cdots,D^0U_{lj}\right)^T$ The corresponding coefficient matrix in (3.1) is:

$$A^h = \begin{pmatrix} D_{11} & D_{12} & \cdots & D_{1l} \\ D_{21} & D_{22} & \cdots & D_{2l} \\ \vdots & \ddots & & \vdots \\ D_{l1} & D_{l2} & \cdots & D_{ll} \end{pmatrix} \tag{3.3}$$

where blocks D_{ij} in A^h are $(N-1)\times(N-1)$ and diagonal blocks D_{kk} for $k = 1,\cdots,l$ are tridiagonal matrices as follws

$$D_{kk} = \begin{pmatrix} \dfrac{2\varepsilon_k\sigma_{k;1}}{h^2} + a_{kk;1} & -\dfrac{\varepsilon_k\sigma_{k;1}}{h^2} - \dfrac{b_{k;1}}{2h} & \cdots & 0 \\ -\dfrac{\varepsilon_k\sigma_{k;2}}{h^2} + \dfrac{b_{k;2}}{2h} & \dfrac{2\varepsilon_k\sigma_{k;2}}{h^2} + a_{kk;2} & -\dfrac{\varepsilon_k\sigma_{k;2}}{h^2} - \dfrac{b_{k;2}}{2h} & 0 \\ \vdots & \ddots & & \vdots \\ 0 & \cdots & -\dfrac{\varepsilon_k\sigma_{k;N-1}}{h^2} + \dfrac{b_{k;N-1}}{2h} & \dfrac{2\varepsilon_k\sigma_{k;N-1}}{h^2} + a_{kk;N-1} \end{pmatrix}$$

and D_{ij} for $i, j = 1, \cdots, l$ where $i \neq j$ are diagonal matrix as

$$D_{ij} = \operatorname{diag}\{a_{ij;1}, \cdots, a_{ij;N-1}\}.$$

Remark 3.1 When $B = \operatorname{diag}\{b_1, \cdots, b_l\}$ is constant and $A \equiv 0$, the system (1.1) is reduced to

$$-E\mathbf{u}'' - B\mathbf{u}' = \mathbf{f}.$$

Then its general solution for homogeneous equation, which is also boundary layer function, is

$$u_k(x) = C_k \exp\left(-\frac{b_k x}{\varepsilon_k}\right) \text{ for } b_k > 0 \text{ and}$$

$$u_k(x) = C_k \exp\left(-\frac{b_k(1-x)}{\varepsilon_k}\right) \text{ for } b_k < 0,$$

where C_k for $k = 1, \cdots, l$ are arbitrary constants. The difference operator (3.2) is exact for these boundary layer functions. *i.e.*,

$$\begin{aligned} &-E\Sigma_j D^+ D^- U_j - B_j D^0 U_j = -E\mathbf{u}_j'' - B_j \mathbf{u}_j' \\ &\text{for } j = 1, \cdots, N-1, \end{aligned} \tag{3.4}$$

where index j indicates the evaluation at point x_j, $\mathbf{u}_j = (u_{1;j}, \cdots, u_{l;j})^T$ and $u_{k;j} = u_k(x_j)$.

In what follows, we try to estimate the condition number of the discrete linear system associated with the Il'in scheme presented. Then we apply M-criterion [13], [14] to obtain the condition number.

Theorem 3.2 (M-criterion). Let a matrix A satisfy $a_{ij} \le 0$ for $i \neq j$. Then A is an M-matrix if and only if there exists vector $e > 0$ such that $Ae > 0$. Furthermore, we have

$$\left\|A^{-1}\right\| \le \frac{\|e\|}{\min_k (Ae)_k}. \tag{3.5}$$

Theorem 3.3 The corresponding coefficient matrix in (3.1) is an M-matrix and satisfies

$$\left\|A^h\right\| \le CN$$

and

$$\left\|\left(A^h\right)^{-1}\right\| \le \frac{2}{\beta}$$

where C is independent of N and ε_k for $k = 1, \cdots, l$ and $\beta = \min_{k=1,\cdots,l}(\beta_k)$

Proof. To evaluate the maximum norm of the matrix A^h for the i th row of the k th row of the block matrix A^h, where $i = 1, \cdots, N-1$ and $k = 1, \cdots, l$, we have

$$\begin{aligned} \sum_{j=1}^{l(N-1)} \left|a_{ij}^h\right| &\le \left|\frac{b_{k;i}}{2h}\left(1 - \coth q_{k;i}\right)\right| + \left|\frac{b_{k;i}}{h} \coth q_{k;i}\right| + \left|\frac{b_{k;i}}{2h}\left(1 + \coth q_{k;i}\right)\right| + \sum_{m=1}^{l} \left|a_{km;i}\right| \\ &= \frac{\varepsilon_k}{h^2}\left[\left|q_{k;i}\left(1 - \coth q_{k;i}\right)\right| + 2\left|q_{k;i} \coth q_{k;i}\right| + 2q_{k;i} \coth q_{k;i} + \left|q_{k;i}\left(1 + \coth q_{k;i}\right)\right|\right] + \sum_{m=1}^{l} \left|a_{km;i}\right| \\ &= \frac{\varepsilon_k}{h^2}\left[q_{k;i}\left(\coth q_{k;i} - 1\right) + q_{k;i}\left(1 + \coth q_{k;i}\right)\right] + \sum_{m=1}^{l} \left|a_{km;i}\right| = \frac{4\varepsilon_k}{h^2} q_{k;i} \coth q_{k;i} + \sum_{m=1}^{l} \left|a_{km;i}\right|. \end{aligned}$$

By

$$\left(q_{k;i} \coth q_{k;i} - 1\right) \le C \frac{q_{k;i}^2}{q_{k;i} + 1},$$

See [14], we conclude that

$$\begin{aligned} \sum_{j=1}^{l(N-1)} \left|a_{ij}^h\right| &\le \frac{4\varepsilon_k}{h^2}\left(q_{k;i} \coth q_{k;i} - 1 + 1\right) + \sum_{m=1}^{l} \left|a_{km;i}\right| \le C \frac{\varepsilon_k}{h^2}\left(\frac{q_{k;i}^2}{q_{k;i} + 1} + 1\right) \\ &= C\left[\frac{b_{k;i}^2}{2\left(b_{k;i} h + 2\varepsilon_k\right)} + \frac{\varepsilon_k}{h^2}\right] + \sum_{m=1}^{l} \left|a_{km;i}\right| \\ &= C \frac{b_{k;i}^2 h^2 + 2b_i h \varepsilon_k + 4\varepsilon_k^2}{2\left(b_{k;i} h + 2\varepsilon_k\right) h^2} + \sum_{m=1}^{l} \left|a_{km;i}\right| \le C\left(\frac{b_{k;i} h + 2\varepsilon_k}{2h^2}\right) + \sum_{m=1}^{l} \left|a_{km;i}\right|. \end{aligned}$$

Since in singularly perturbed equations , $\varepsilon \le h$ is considered, hence

$$\sum_{j=1}^{l(N-1)} \left|a_{ij}\right| \le C\left(\frac{b_i+2}{2h}\right) \le C\frac{1}{h} = CN.$$

Therefore,

$$\left\|A^h\right\| = \max_i \left\{\sum_{j=1}^{N-1} \left|a_{ij}\right|\right\} \le CN.$$

If we choose $\mathbf{e} = \left[e^1, \cdots, e^l\right]^T$ where $e_i^k = 1 + x_i$ for $b_k(x) < 0$ and $e_i^k = 1 - x_i$ for $b_k(x) > 0$, then $A^h\mathbf{e}$ for the i th row of the k th row of the block matrix A^h for case $b_k(x) < 0$ we have

$$\begin{aligned}\left(A^h\mathbf{e}\right)_i &= \left[-\frac{b_{k;i}}{2h}\coth\left(\frac{hb_{k;i}}{2\varepsilon_k}\right) + \frac{b_{k;i}}{2h}\right]\left(1 + x_{i-1}\right) \\ &+ \left[\frac{b_{k;i}}{h}\coth\left(\frac{hb_{k;i}}{2\varepsilon_k}\right) + a_{kk;i}\right]\left(1 + x_i\right) \\ &+ \left[-\frac{b_{k;i}}{2h}\coth\left(\frac{hb_{k;i}}{2\varepsilon_k}\right) - \frac{b_{k;i}}{2h}\right]\left(1 + x_{i+1}\right) \\ &+ \sum_{m=1, m\ne k}^{l} a_{km;i}e_i^m \ge -b_{k;i} > \beta_k > \beta\end{aligned}$$

and from (3.1), for $i \ne j$ we have $a_{ij}^h \le 0$. Hence the M-criterion yields that A^h is an M-matrix and

$$\left\|\left(A^h\right)^{-1}\right\| \le \frac{\|e\|}{\min_k\left(A^h e\right)_k} \le \frac{2}{\beta}.$$

A similar proof is for the case $b_k(x) > 0$.

Notation. Throughout the above theorem we let C denote positive constant that may take different values in different formulas but that are always independent of both the perturbation parameters ε_k and of N, the number of mesh intervals. By theorem 3.3, the condition number of A is of order $O(N)$. We recall that, in theorem 3.2 and theorem 3.3, $\|A\|$ is the usual maximum norm of matrices. This shows that the matrices arising from the Il'in scheme for discretization of a singularly perturbed differential equation are well conditioned respect to the upwind finite difference method applied on Shiskin or Bakhvalov meshes.

For arbitrary $U = \left(U_1, \cdots, U_l\right) \in \left(R_0^{N+1}\right)^l$, we define

$$\|U\|_{\infty,\omega} = \max_{i=1,\cdots,l} \left\|U_i\right\|_{\infty,\omega}.$$

A single equation of (3.1) can be written as follows:

$$\begin{aligned}\left[L_kU_k\right]_j &:= -\varepsilon_k\sigma_k\left(q_j\right)D^+D^-U_{kj} - b_{k;j}D^0U_{kj} \\ &+ a_{kk;j}U_{kj} = f_{k;j} - \sum_{m=1,\, m\ne k}^{l} a_{km;j}U_{mj}.\end{aligned} \tag{3.6}$$

Suppose U solves (3.1). We propose the following lemma to obtain a bound for U.

Lemma 3.4 Suppose A is a matrix such that $a_{ii} > 0$ and $a_{ij} \le 0$ for $(i \ne j)$ and $i, j = 1, \cdots, n.$ Also assume that $\sum_{k=1}^{l} a_{jk} \ge 1$ for $j = 1, \cdots, n.$ Then for every arbitrary vector $\eta = \left(\eta_1, \cdots, \eta_n\right)^T$ we have

$$\|\eta\|_{\infty,\omega} \le \|A\eta\|_{\infty,\omega}$$

Proof. Suppose for the element j of η, $\|\eta\|_{\infty,\omega} = \left|\eta_j\right|$. Without lose of generality, let $\left|\eta_j\right| = \eta_j$ (otherwise we consider $\|A\eta\| = \|-A\eta\|$).

$$\begin{aligned}\left(A\eta\right)_j &= \sum_{k=1}^{n} a_{jk}\eta_k = a_{jj}\eta_j + \sum_{k=1,k\ne j}^{n} a_{jk}\eta_k \\ &\ge a_{jj}\eta_j + \sum_{k=1,k\ne j}^{n} a_{jk}\left|\eta_k\right|,\end{aligned}$$

(since $a_{jk} \le 0$ for $j \ne k$). Therefore

$$\begin{aligned}\left(A\eta\right)_j - \eta_j &\ge a_{jj}\eta_j + \sum_{k=1,k\ne j}^{n} a_{jk}\left|\eta_k\right| - \eta_j \\ &\ge a_{jj}\eta_j + \sum_{k=1,k\ne j}^{n} a_{jk}\left|\eta_k\right| - \sum_{k=1}^{n} a_{jk}\eta_j \\ &= \sum_{k=1,k\ne j}^{n} a_{jk}\left(\eta_k - \eta_j\right) \ge 0.\end{aligned}$$

So

$\left(A\eta\right)_j \ge \eta_j > 0.$ Hence

$$\|A\eta\|_{\infty,\omega} = \max_{k=1,\ldots,n}\left|\left(A\eta\right)_k\right| \ge \left|\left(A\eta\right)_j\right| \ge \eta_j = \|\eta\|_{\infty,\omega}$$

Similar to theorem 2.2, we have the following theorem in the discrete case.

Theorem 3.5 Suppose U solves (3.1). Assume b_k and a_{ik} satisfy (1.2), (2.3) and (2.4). Then

$$\|U\|_{\infty,\omega} \le C\|f\|_{\infty,\omega}.$$

Proof. Dividing (3.6) by $a_{kk;j}$, we have

$$\begin{aligned}\left[\tilde{L}_kU_k\right]_j &= \left[L_kU_k\right]_j \big/ a_{kk;j} := \frac{f_{k;j}}{a_{kk;j}} \\ &- \sum_{m=1,m\ne k}^{l} \frac{a_{km;j}}{a_{kk;j}}U_{m;j}.\end{aligned} \tag{3.7}$$

The matrix associated with operator $\tilde{L}_k$ is a matrix that satisfy lemma 3.4. We have

$$\begin{aligned}\left\|U_k\right\|_{\infty,\omega} \le \left\|\tilde{L}_kU_k\right\|_{\infty,\omega} &\le \left\|\frac{f_k}{a_{kk}}\right\|_{\infty,\omega} \\ &+ \sum_{m=1,m\ne k}^{l}\left\|\frac{a_{km}}{a_{kk}}\right\|\left\|U_m\right\|_{\infty,\omega}.\end{aligned}$$

Therefore

$$\|U_k\|_{\infty,\omega} + \sum_{m=1,m\neq k}^{l} \gamma_{km} \|U_m\|_{\infty,\omega} \le \left\| \frac{f_k}{a_{kk}} \right\|.$$

By (2.3) and after some manipulation, we obtain

$$\|U\|_{\infty,\omega} \le C \|f\|_{\infty,\omega}.$$

Corollary. According to theorem 3.5, we have

$$\|\mathbf{u} - U\|_{\infty,\omega} \le C \|L(\mathbf{u} - U)\|_{\infty,\omega}. \tag{3.8}$$

4. Error Analysis

For the error analysis of Il'in scheme applied to the system of singularly perturbed differential equations (1.1), suppose that $b_i(x) > 0$ for $i = 1, \cdots, l$ and $\mathbf{u}$ satisfies (1.1). Similar to lemma 2.4, we split $\mathbf{u}(x) = \mathbf{v}(x) + \mathbf{z}(x)$, where $\mathbf{v} = (v_1(x), \cdots, v_l(x))^T$, $\mathbf{z}(x) = (z_1(x), \ldots, z_l(x))^T$. So we have $u_k(x) = z_k(x) + v_k(x)$, and for $n = 0,1$

$$\|z_k^{(n)}(x)\|_{\infty,\omega} \le C$$

and

$$v_k(x) = \frac{\varepsilon_k u_k'(0)}{b_k(0)} \exp\left(-b_k(0)\varepsilon_k^{-1} x\right).$$

First we analyze the truncation error in a single equation. We introduce the continuous and discrete operators

$$\begin{aligned}(\mathcal{A}_i \mathbf{v})(x) &= \varepsilon_i v_i'(x) + b_i v_i + \int_x^1 b_i'(s) v_i(s)\,\mathrm{d}s \\ &+ \int_x^1 \sum_{m=1}^{l} (a_{im} v_m)(s)\,\mathrm{d}s \quad x \in (0,1),\ i = 1, \cdots, l\end{aligned} \tag{4.1}$$

and

$$\begin{aligned}[A_i V]_j &= \varepsilon_i \tilde{D}^- V_{ij}(x) + b_{i;j} V_{ij} \\ &+ \sum_{\nu=j}^{N-1} h V_{i\nu+1} D^+ b_{i;\nu} + h \sum_{\nu=j}^{N-1} \sum_{m=1}^{l} a_{im;\nu} V_{m\nu},\end{aligned} \tag{4.2}$$

where

$$\tilde{D}^- v_i = \frac{v_i - v_{i-1}}{\tilde{h}} \quad \text{and} \quad \tilde{h} = \frac{h}{q_i\left(\coth(q_i) - 1\right)}.$$

We note that $\tilde{h} > h$. We introduce the continuous and discrete functions

$$\varsigma_i(x) = \int_x^1 g_i(s)\,\mathrm{d}s \quad i = 1, \cdots, l \tag{4.3}$$

and

$$G_{ij} = \sum_{\nu=j}^{N-1} h g_{i\nu} \ i = 1 \cdots, l,\ \ j = 1 \cdots, N-1, \tag{4.4}$$

where $g_i(x)$ for $i = 1, \cdots, l$ is defined in remark 2.5. By (4.1)-(4.4), we can establish the following theorem.

Theorem 4.1 Let $\mathbf{u}$ be the solution of (1.1) and U be the solution obtained by the Il'in scheme (3.1). Suppose the data $g_k, b_k, a_{km}, f_k \in C^1[0,1]$ for $k, m = 1, \cdots, l$ satisfy (1.2), (2.3) *and* (2.4). *Then*

$$\|U - \mathbf{u}\|_{\infty,\omega} \le Ch. \tag{4.5}$$

Proof. From lemma 2.4, we can split $\mathbf{u}(x)$ as $\mathbf{u}(x) = \mathbf{v}(x) + \mathbf{z}(x)$, where $\mathbf{v}(x)$ is a boundary layer function. Hence,

$$\|U - \mathbf{u}\|_{\infty,\omega} = \|Z - \mathbf{z} + V - \mathbf{v}\|_{\infty,\omega} \le \|Z - \mathbf{z}\|_{\infty,\omega} + \|V - \mathbf{v}\|_{\infty,\omega}.$$

First, consider $\|Z - \mathbf{z}\|_{\infty,\omega}$. From (4.1)-(4.4), we have

$$L_i \mathbf{z} = -(\mathcal{A}_i \mathbf{z})',\ g_i = -\varsigma_i' on\ (0,1)$$

and

$$(L_i Z)_j = -D^+ (A_i Z)_j \ \text{ and } \ g_{ij} = -D^+ G_{ij}$$

for $j = 0, \cdots, N$.

Thus, by $(L_i \mathbf{z})(x) = g_i(x)$ and $(L_i Z)_j = g_{ij}$,

$$\begin{aligned}&(\mathcal{A}_i \mathbf{z})(x) - \varsigma_i(x) \equiv \alpha \quad x \in (0,1) \text{ and} \\ &(A_i Z)_j - G_{ij} = \gamma,\ \ j = 0, \cdots, N\end{aligned} \tag{4.6}$$

with constants α and γ.

From (2.1),

$$\|L_i(\mathbf{z} - Z)\|_{W_{-1,\infty}^h} = \min_{c \in R} \|A_i(\mathbf{z} - Z) + c\|_{\infty,\omega}.$$

Taking $c = \gamma - \alpha$, from (4.6) we get

$$\|L_i(\mathbf{z} - Z)\|_{W_{-1,\infty}^h} \le \|A_i \mathbf{z} - \varsigma_i(x)_i\, \mathbf{z} - G_i + \varsigma_i(x)\|_{\omega,\infty}.$$

Furthermore,

$$\begin{aligned}(A_i \mathbf{z} - \mathcal{A}_i \mathbf{z} - G_i + \varsigma_i)_j &= \varepsilon_i \left(\tilde{D}^- z_i - z_i'\right)_j + \sum_{\nu=j}^{N-1} h \varphi_{i;\nu} \\ &- \int_{x_j}^{x_N} \varphi_i(s)\,\mathrm{d}s + \sum_{\nu=j}^{N-1} h z_{i;\nu+1} D^+ b_{i;\nu} - \int_{x_j}^{x_N} (b_i' z_i)(s)\,\mathrm{d}s,\end{aligned}$$

where

$$\varphi_i = \sum_{m=1}^{l} a_{im} z_m - g_i.$$

Applying Taylor expansion with the integral remainder form and using lemma 2.4 and remark 2.5, we have

$$\begin{aligned}\left| h\varphi_{i;\nu} - \int_{x_\nu}^{x_{\nu+1}} \varphi_i(t)\,\mathrm{d}t \right| &= \left| \int_{x_\nu}^{x_{\nu+1}} \int_{x_\nu}^{t} \varphi_i'(s)\,\mathrm{d}s\mathrm{d}t \right| \le \int_{x_\nu}^{x_{\nu+1}} \left| \int_{x_\nu}^{t} \varphi_i'(s) ds \right| \mathrm{d}t \le Ch \,|\, h z_{i;\nu+1} D^+ b_{i;\nu} - \int_{x_\nu}^{x_{\nu+1}} (b_i' z_i)(t)\,\mathrm{d}t \,| \\ &= \left| \int_{x_\nu}^{x_{\nu+1}} b_i'(t) \int_t^{x_{\nu+1}} z_i'(s)\,\mathrm{d}s\mathrm{d}t \right| \le \int_{x_\nu}^{x_{\nu+1}} \left| b_i'(t) \int_t^{x_{\nu+1}} z_i'(s)\,\mathrm{d}s \right| \mathrm{d}t \le Ch,\end{aligned}$$

and

$$\left|\varepsilon_i\left(\tilde{D}^{-}z_i - z_i'\right)_j\right| = \left|\frac{\varepsilon_i}{\tilde{h}}\left[\left(\tilde{h}-h\right)z' + \int_{x_{\nu-1}}^{x_\nu}\int_t^{x_\nu} z_i''(s)\,\mathrm{d}s\mathrm{d}t\right]\right|$$

$$\le \frac{\varepsilon_i}{\tilde{h}}\left(\tilde{h}-h\right)\left|z_i'\right| + \frac{\varepsilon_i}{\tilde{h}}\left|\int_{x_{\nu-1}}^{x_\nu}\int_t^{x_\nu} z_i''(s)\,\mathrm{d}s\mathrm{d}t\right|.$$

On the other hand, we have

$$\frac{\varepsilon_i}{\tilde{h}}\left(\tilde{h}-h\right) = \varepsilon_i\left(1-q_i\left(\coth\left(q_i\right)-1\right)\right)$$

$$\le \varepsilon_i q_i = \frac{b_i h}{2} \le Ch.$$

Hence

$$\left|\varepsilon_i\left(\tilde{D}^{-}z_i - z_i'\right)_j\right|$$

$$\le Ch + \frac{1}{h}\left[\int_{x_{\nu-1}}^{x_\nu}\int_{x_\nu}^{t}\left(g_i + b_i z_i' - \sum_{m=1}^{l} a_{im} z_m\right)\mathrm{d}s\mathrm{d}t\right) \le Ch.$$

Thus

$$\left\|A_i\mathbf{z} - \varsigma_i\mathbf{z} - G_i + \varsigma_i\right\|_{\infty,\omega} \le Ch.$$

In [15] it has been shown that

$$\left\|\left(Z-\mathbf{z}\right)_i\right\|_{\infty,\omega} \le \frac{2}{\beta_i}\left\|L_i\left(Z-\mathbf{z}\right)\right\|_{\omega_{-1,\infty}^h}.$$

Hence we have

$$\left\|Z-\mathbf{z}\right\|_{\infty,\omega} \le Ch. \tag{4.7}$$

To bound the boundary layer function $\mathbf{v}(x) = \left(v_1(x),\cdots,v_l(x)\right)^{\mathrm{T}}$, a direct computation gives

$$L\mathbf{v} = -E^{-1}B(0)\left[B(0)-B(x)\right]\mathbf{v} + A\mathbf{v},$$

and, in the grid points we can write

$$LV = \frac{2}{h}\sinh^{-1}\left(Q(x)\right)B(x)\sinh\left(Q(0)\right)$$

$$\sinh\left(Q(x)-Q(0)\right)V + AV.$$

By a technique used in [9], we can show that

$$\left\|\mathbf{v}-V\right\|_{\infty,\omega} \le Ch. \tag{4.8}$$

So by (4.7) and (4.8) we have

$$\left\|\mathbf{u}-U\right\|_{\infty,\omega} \le Ch.$$

This completes the proof of the theorem.

5. Numerical Experiments

In this section, we compare the Il'in scheme with the upwind finite difference scheme (see [8,15]) for the following two examples.

Example 1. Consider

$$-\varepsilon_1 u_1'' - u_1' + 2u_1 - u_2 = e^x \qquad u_1(0) = u_1(1) = 0,$$
$$-\varepsilon_2 u_2'' - 2u_2' - u_1 + 4u_2 = \cos(x) \quad u_2(0) = u_2(1) = 0,$$

In this example, we expect two layers at $x = 0$ which behave like $\exp\left(-\frac{x}{\varepsilon_1}\right)$ for $b_1 = 1$ and like $\exp\left(-\frac{2x}{\varepsilon_2}\right)$ for $b_2 = 2$. Let $\varepsilon_1 = 10^{-8}$ and $\varepsilon_2 = 10^{-6}$, which are sufficiently small values to bring out the singularly perturbed nature of the problem. The exact solution to the test problem is not available, so we estimate the accuracy of the numerical solution by comparing it to the numerical solution computed on the finer mesh. Let U^N be a numerical solution in N grid point. We estimate the error by

$$\left\|U^N - U^{2N}\right\|_{\infty,\omega}$$

The rates of convergence r^N are computed using the following formula:

$$r^N = \log_2\left(\frac{\left\|U^N - U^{2N}\right\|_{\infty,\omega}}{\left\|U^{2N} - U^{4N}\right\|_{\infty,\omega}}\right).$$

For the upwind scheme we use Shishkin mesh with $\tau_1 = \mathrm{m}$ and $\tau_2 = \min\left(\frac{1}{2}, \varepsilon_2 \log N\right)$. We divide the intervals $\left[0,\tau_1\right]$ and $\left[\tau_1,\tau_2\right]$ into $N/4$ subintervals and $\left[\tau_2,1\right]$ into $N/2$ subintervals of equal length. For Il'in scheme, we use the uniform mesh and divide the interval $[0,1]$ into N subintervals of equal length. Numerical results are contained in **Table 1**. From this Table, we observe that the Il'in scheme is a first order uniformly convergent method.

Example 2. Let

$$-\varepsilon_1 u_1'' - 3u_1' + u_1 - u_2 = e^x$$
$$-\varepsilon_2 u_2'' + 0.5u_2' + u_2 - u_3 = \cos(x)$$
$$-\varepsilon_3 u_{31}'' - (5-x)u_3' - u_2 + 2u_3 = 1 + x^2.$$

In this example, we expect layers $\exp\left(-\left(3x/\varepsilon_1\right)\right)$,

Table 1. Numerical results for example 1.

	upwind scheme		Il'in scheme	
N	Error	Rate (r^N)	Error	Rate (r^N)
128	4.62e-2	0.8019	5.00e - 3	1
256	2.65e-2	0.8461	2.50e - 3	1
512	1.48e-2	0.8945	1.25e - 3	1
1024	7.90e-3	0.9462	6.30e - 4	1
2048	4.10e-3	-	-	-

Table 2. Numerical results for example 2.

	Upwind scheme		Il'in scheme	
N	Error	Rate	Error	Rate
32	1.03e - 2	1.0429	9.6e - 3	0.9652
64	5.0e - 2	0.9685	4.9e - 3	0.9826
128	2.55e - 2	0.9439	2.5e - 3	0.9913
256	1.33e - 2	0.9382	1.3e - 3	0.9957
512	6.9e - 3	0.9384	6.0e - 4	1
1024	3.6e - 3	-	3.0e - 4	-

$\exp\left(-\frac{0.5(1-x)}{\varepsilon_2}\right), \exp\left(-\frac{5x}{\varepsilon_3}\right)$. Let $\varepsilon_1 = 10^{-8}$, $\varepsilon_2 = 10^{-7}$ and $\varepsilon_3 = 10^{-6}$. We use the upwind finite difference scheme on a Shishkin mesh(See [11]). Assume that,

$$\tau_1 = \min\left\{\frac{\tau_3}{2}, \frac{\tilde{\varepsilon}_1}{3}\ln N\right\}, \tau_2 = \min\left\{\frac{3}{4}, 1-\frac{\varepsilon_2}{0.5}\ln N\right\},$$

$$\tau_3 = \min\left\{\frac{1}{2}, \frac{\tilde{\varepsilon}_3}{5}\ln N\right\},$$

with $\tilde{\varepsilon}_1 = \min\{\varepsilon_1, \varepsilon_3\}$ and $\tilde{\varepsilon}_3 = \max\{\varepsilon_1, \varepsilon_3\}$. Then the mesh is obtained by dividing each of the intervals $[0,\tau_1], [\tau_1,\tau_2], [\tau_2,\tau_3]$ and $[\tau_3,1]$ into $N/4$ subintervals. Numerical results are shown in **Table 2.**

From **Table 1** and **Table 2** we see that the rate of convergence for Il'in scheme is close to 1, which agree with the convergence estimate of theorem 4.1. Numerical results confirm the theoretical results.

11. References

[1] T. Linss and N. Madden, "Accurate Solution of a System of Coupled Singularly Perturbed Reaction-Diffusion Equations," *Computing*, Vol. 73, No. 2, 2004, pp. 121-133. doi:10.1007/s00607-004-0065-3

[2] N. Madden and M. Stynes, "A Uniformly Convergent Numerical Method for a Coupled System of Two Singularly Perturbed Linear Reaction-Diffusion Problems," *IMA Journal of Numerical Analysis*, Vol. 23, No. 4, 2003, pp. 627-644. doi:10.1093/imanum/23.4.627

[3] J. L. Gracia and F. J. Lisbona, "A Uniformly Convergent Scheme for a System of Reaction-Diffusion Equations," *Journal of Computational and Applied Mathematics*, Vol. 206, No. 1, 2007, pp. 1-16. doi:10.1016/j.cam.2006.06.005

[4] S. Bellew and E. O'Riordan, "A Parameter-Robust Numerical Method for a System of Two Singularly Perturbed Convection-Diffusion Equations," *Applied Numerical Mathematics*, Vol. 51, No. 2-3, 2004, pp. 171-186. doi:10.1016/j.apnum.2004.05.006

[5] Z. Cen, "Parameter-Uniform Finite Difference Scheme for a System of Coupled Singularly Perturbed Convection-Diffusion Equations," *International Journal of Computer Mathematics*, Vol. 82, No. 2, 2005, pp. 177-192. doi:10.1080/0020716042000301798

[6] G. M. Amiraliyev, "The Convergence of a Finite Difference Method on Layer-Adapted Mesh for a Singularly Perturbed System," *Applied Mathematics and Computation*, Vol. 162 No. 3, 2005, pp. 1023-1024. doi:10.1016/j.amc.2004.01.015

[7] V. B. Andreev and N. Kopteva, "On the Convergence, Uniform with Respect to a Small Parameter of Monotone Three-Point Finite-Difference Approximations," *Journal of Difference Equations*, Vol. 34, 1998, pp. 921-929.

[8] T. Linß, "Analysis of an Upwind Finite-Difference Scheme for a System of Coupled Singularly Perturbed Convection-Diffusion Equations," *Computing*, Vol. 79, No. 1, 2007, pp. 23-32. doi:10.1007/s00607-006-0215-x

[9] H. G. Roos, "A Note on the Conditioning of Upwind Schemes on Shishkin Meshes," *IMA Journal of Numerical Analysis*, Vol. 16, No. 4, 1996, pp. 529-538. doi:10.1093/imanum/16.4.529

[10] A. M. Il'in, "A Difference Scheme for a Differential Equation with a Small Parameter Affecting the Highest Derivative," in Russian, *Matematicheskie Zametki*, Vol. 6, 1969, pp. 237-248.

[11] V. B. Andreev, "The Green Function and A Priori Estimates of Solution of Monotone Three Point Singularly Perturbed Finite-Difference Schemes," *Differrence Equations*, Vol. 37, No. 7, 2001, pp. 923-933. doi:10.1023/A:1011949419389

[12] R. B. Kellogg and A. Tsan, "Analysis of Some Difference Approximations for a Singular Perturbation Problem without Turning Points," *Mathematics of Computation*, Vol. 32, 1978, pp. 1025-1039. doi:10.1090/S0025-5718-1978-0483484-9

[13] O. Axelsson and L.Kolotilina, "Monotonicity and Discretization Error Estimates," *SIAM Journal on Numerical Analysis*, Vol. 27, No. 6, 1990, pp. 1591-1611. doi:10.1090/S0025-5718-1978-0483484-9

[14] H. G. Roos, M. Stynes and L. Tobiska, "Robust Methods for Singularly Perturbed Differential Equations," 2nd Edition, Springer Series in Computational Mathematics, Springer, Berlin, 2008.

[15] T. Linß, "Analysis of a System of Singularly Perturbed Covection-Diffusion Equations with Strong Coupling," *SIAM Journal on Numerical Analysis*, Vol. 47, No. 3, 2009, pp. 1847-1862.doi:10.1137/070683970