Applied Mathematics, **2011, 2, 854-865**
doi:10.4236/am.2011.27115 Published Online July 2011 (http://www.SciRP.org/journal/am)

Some Applications of Optimal Control in Sustainable Fishing in the Baltic Sea

Dmitriy Stukalin, Werner H. Schmidt
Institut für Mathematik und Informatik, Greifswald, Germany
E-mail: *dmitriy_stukalin@uni-greifswald.de*
Received April 18, 2011; *revised May* 19, 2011; *accepted May* 22, 2011

Abstract

Issues related to the implementation of dynamic programming for optimal control of a three-dimensional dynamic model (the fish populations management problem) are presented. They belong to a class of models called Lotka-Volterra models. The existence of bionomic equilibria will be considered. The problem of optimal harvest policy is then solved for the control of various classes of its behaviour. Therefore the focus will be the optimality conditions by using the Bellman principle. Moreover, we consider a different form for the optimal value of the control vector, namely the feedback or closed-loop form of the control. Academic examples are studied in order to demonstrate the proposed methods.

Keywords: Optimal Control Problems, Maximum Principle, Piecewise Constant Optimal Control, Bellman Principle

1. The Problem

Currently the fish populations in the Baltic Sea have many problems, which are mainly caused by human influence. Some fish species are catched too much. The fundamental risk of overfishing is that a stock (occurrence of species in a given region) is so decimated that the natural regeneration ability is not given and at worst the species die out. The Living Planet Index for marine species of the WWF shows an average decrease of 14% between 1970 and 2005 (see Living Planet Report 2008). The over fishing is the main cause apart from possible environmental factors (climate change, pollutants, *etc.*).

Therefore, the goal of the Baltic Sea fishermen must be conscientious, by the policy prescribed regulations and the advance (such as from International Council for the Exploration of the Sea) to protect the Baltic Sea fauna deal. A responsible management must reduce the fishing effort to an environmentally acceptable level and call for the cooperation among the participating countries. This is of utmost importance, since the economic value of the catches depends on the stock and the biodiversity of the Baltic Sea.

Several interacting species are modeled, which inhabit in a common habitat with limited resources. So, a dynamic system is to be studied, which depends on several states and controls (e.g. the number of fishing boats). A typical question for such systems is to find a controller that regulates the system in a desired target. In many applications a cost functional is to be optimized, this is usually a functional of the state trajectory and the controls of the system. The profit of a sustainable fishing industry should be maximized without disappearance of the species.

In this paper necessary (and sometimes sufficient) optimality conditions are derived. Numerical methods are obtained from the optimality conditions in order to calculate (approximately) optimal controls.

2. Optimal Control Problems

Whenever a state function depending on the time is described by an ordinary differential equation which depends on the control variable, it is called a control system of ordinary differential equations. Optimal control is related to the development of space flight and military researches beginning from the 1950s. We can find the applications of the control theory in economics, in chemistry or even in population dynamics. The general task of optimal control is defined as follows:

Let $\Omega \subset \mathrm{R}^m$ be a nonempty (often convex and closed) control region. Let g, q, f be given smooth functions:

$$q : R^{n+1} \to R$$
$$f : R \times R^n \times \Omega \to R^n$$
$$g : R \times R^n \times \Omega \to R.$$

A continuous and piecewise continuously differentiable function $x(\cdot): \mathrm{R} \to \mathrm{R}^n$ (state function) as well as a piecewise continuous (or piecewise constant) function $u(\cdot): \mathrm{R} \to \Omega$ (control function) are called admissible, if the ODE

$$\dot{x}(t) = f(t, x(t), u(t)), \quad t_0 \le t \le T$$
$$x(t_0) = x_0$$

is valid. We are looking for admissible pairs $x(\cdot), u(\cdot)$ which maximize an objective (cost) functional of Bolza type:

$$J(u(\cdot)) = \int_{t_0}^{T} g(t, x(t), u(t))\mathrm{d}t + q(T, x(T)) \to \max_{u(\cdot)} \quad (1)$$

Often the optimal control can be calculated by methods using the Pontryagin maximum principle or by solving the Hamilton-Jacobi-Bellman equation.

3. Extended Lotka-Volterra Models with M Populations

A logistic model of development for a two-population system can be written in the following form [1,2]. Let be $\varepsilon_1, \varepsilon_2$ growth coefficients, γ_1, γ_2 the phagos coefficients and K_1, K_2 given numbers (capacities or logistical terms). We denote the population sizes as x_1 and x_2.

The differential equations for the development of the populations are

$$\begin{cases} \dot{x}_1(t) = x_1(t)\left[\varepsilon_1\left(1 - \dfrac{x_1(t)}{K_1}\right) - \gamma_1 x_2(t)\right] \\ \dot{x}_2(t) = -x_2(t)\left[\varepsilon_2\left(1 - \dfrac{x_2(t)}{K_2}\right) - \gamma_2 x_1(t)\right] \end{cases}$$

We denote generally:

ε_i are growth coefficients, γ_{ij} are the phagos coefficients of the population i with respect to the population j and K_i are logistical terms.

We denote the control of the fish populations $u_i(t)$ (it can be a regulation of the fishing, e.g. the number of the fishing boats if $u_i(t) \in N$), p_i are fish prices (per ton), r_i are catch proportionalities. Therefore, the development of m populations can be described by a generalized system

$$\dot{x}_i(t) = \varepsilon_i x_i(t)\left(1 - \frac{x_i(t)}{K_i}\right) - \sum_{j=1}^{m} \gamma_{ij} \frac{x_i(t)}{K_i}\frac{x_j(t)}{K_j}$$
$$-u_i(t) r_i d \cdot \frac{x_i(t)}{K_i}$$

where $x_i(0) = x_{i0}$ are given for $i = 1, \cdots, m.$

The objective function (the profit) is to be maximized

$$J(u) = \int_0^T \left\{\sum_{i=1}^{m} p_i u_i(t) r_i d \cdot \frac{x_i(t)}{K_i} - cd \cdot \sum_{i=1}^{m} u_i(t)\right\} e^{-\delta t} \to \max_{u(\cdot)}$$

under the restrictions

$$0 \le u_i(t) \le u^{\max}, i = 1, \cdots, m; 0 \le t \le T.$$

c are the cutter costs per day and d is the number of days in which we catch. If we calculate the present value of future profits, we consider a discount rate $e^{-\delta t}$. This plays an important role in economic models.

4. Bellman's Principle

A key aspect of dynamic programming is the Bellman principle. The basic idea is to calculate the optimal solutions of many small subproblems and then to compose these subsolutions to a suitable global optimal solution. It was formulated in 1957 by Bellman.

An optimal policy has the property that whatever the initial state and initial decision are, the remaining decisions must be an optimal policy with regard to the state resulting from the first decision [3].

This idea can be used to derive a necessary and sufficient condition. We consider here two forms of the optimal controls of (1), namely the open-loop form and the closed-loop form. The closed-loop form $\hat{u}(t, x)$ gives the optimal value of the control vector as a function of the time and the current state. The form of the optimal control vector derived via the necessary conditions is called open-loop. However, even though the closed-loop $\hat{u}(\cdot,\cdot)$ and open-loop $u^*(\cdot)$ controls differ in form, they yield identical values for the optimal control at each date of the planning horizon. It follows $\hat{u}(t, x^*(t)) = u^*(t)$.

The open-loop form gives the optimal value of the control vector as a function of the time and the initial values of the state vector. The closed-loop form of the optimal control is a decision rule, for it gives the optimal value of the control for any current period and an admissible state in the current period that may arise. In contrast, the open-loop form of the optimal control is a curve, for it gives the optimal values of the control as the independent variable time over the planning horizon.

We consider an optimal control problem (1) under the control condition:

$u(t) \in \Omega \subset R^m, \forall t \in [t_0, T]$, $u(\cdot)$ is piecewise continuous.

The cost function $V(t,x):[t_0,T]\times R^n \to R$ is defined as:

$$V(t,x)=\max_{u(\cdot)}\int_t^T g(\tau,x(\tau),u(\tau))\mathrm{d}\tau+q(T,x(T)), \quad (2)$$

where $u(\cdot):[t,T]\to R^m$ is admissible in $[t,T]$ and $x(\cdot)$ is the corresponding trajectory with $x(t)=x$.

$V(t,x)$ gives the optimal value of the objective function starting from the time $t\in[t_0,T]$ and the starting point *x*, following the ODE.

We define the Hamiltonian H as

$$H(t,x,u,V_x(t,x))=g(t,x,u)+V_x(t,x)\cdot f(t,x,u)$$

Necessary condition

Assume there exists the value function $V(t,x)$ to the problem (1) in $[t_0,T]\times R^n$ and this function is continuously differentiable.

Let $u^*(\cdot)$ be an open-loop optimal solution of (1). Then the corresponding closed-loop solution $\hat{u}(\cdot,\cdot)$ satisfies the condition

$$\hat{u}(t,x)=\arg\max_{u\in\Omega}H(t,x,u,V_x(t,x)),\forall x\in R^n$$

and $\forall t\in[t_0,T]$

and $V(t,x)$ is a solution of the PDE:

$$-V_t(t,x)=\max_{u\in\Omega}H(t,x,u,V_x(t,x)),t\in[t_0,T]$$

$$V(T,x(T))=q(T,x(T)).$$

Proof:

$V(t+\Delta t,x(t+\Delta t))$ is the cost function for the part of the solution, that begins at the time $t+\Delta t$ with state $x(t+\Delta t)$.

Then for $0<\Delta t<T-t$ it is:

$$V(t,x)=$$

$$\max_{u(\cdot)admissible}\left\{\int_t^{t+\Delta t}g(\tau,x(\tau),u(\tau))\mathrm{d}\tau+V(t+\Delta t,x(t+\Delta t))\right\}.$$

Since *V* is assumed to be continuous differentiable and *g* to be continuous, $\int_t^{t+\Delta t}g(\tau,x(\tau),u(\tau))\mathrm{d}\tau$ can be approximated for every continuity point *t* of $u(\cdot)$ as $g(t,x(t),u(t))\Delta t+o(\Delta t)$,where Δt is sufficiently small.

It follows:

$$V(t,x)=\max_{u(\cdot)admissible}\{g(t,x(t),u(t))\Delta t$$
$$+V(t+\Delta t,x(t+\Delta t))\}+((\Delta t),$$

where $o(\Delta t)$ represents the higher order terms, that means $\lim_{\Delta t\to 0}\frac{o(\Delta t)}{\Delta t}=0$.

According to Taylor's theorem it is:

$$V(t+\Delta t,x(t+\Delta t))=V(t,x(t))+V_t(t,x(t))\Delta t$$
$$+V_x(t,x(t))\dot{x}(t)\Delta t+o(\Delta t)$$

Substituting this result into the previous equation and using $\dot{x}=f(t,x,u)$, it follows for $\Delta t\to 0$ the partial differential equation

$$0=\max_{u\in\Omega}\{g(t,x(t),u)+V_t(t,x(t))$$
$$+V_x(t,x(t))f(t,x(t),u)\}.t\to 0 \quad (3)$$

We can write the PDE (3) as

$$-V_t(t,x)=\max_{u\in\Omega}H(t,x,u,V_x(t,x)), \quad (4)$$

because $V(t,x)$ does not depend on *u*. The boundary condition $V(T,x(T))=q(T,x(T))$ follows immediately.

The PDE (4) is the Hamilton-Jacobi-Bellman equation. It is an evolution equation with a final condition. The global solvability, assumed in the first definition, is not assured in general[1].

Sufficient condition

If it's given on $[t_0,T]\times R^n$ a real, continuously differentiable function $V(t,x)$, which satisfies the Hamilton-Jacobi-Bellman equation

$$-V_t(t,x)=\max_{u\in\Omega}H(t,x,u,V_x(t,x))$$
$$V(T,x(T))=q(T,x(T)) \quad (5)$$

and if the control

$$\hat{u}(t,x)=\arg\max_{u\in\Omega}H(t,x,u,V_x(t,x)) \quad (6)$$

(depending on *t* and *x*) is admissible, then the corresponding open-loop control $u^*(\cdot)$ with the corresponding state trajectory $x^*(\cdot)$ is an optimal solution of (1).

Proof:

Since the left-hand side is independent from *u*, (5) can be transformed into:

$$\max_{u\in\Omega}\left[V_t(t,x)+H(t,x,u,V_x(t,x))\right]=0. \quad (7)$$

We choose admissible open-loop controls $u^*(\cdot)$ and $u^-(\cdot)$ on $[t_0,T]$.

Let $x^*(\cdot)$ and $x^-(\cdot)$ be the unique state trajectory, which are generated by $u^*(\cdot)$ and $u^-(\cdot)$ in $[t_0,T]$, so that $x^*(t_0)=x^-(t_0)=x_0$. Then it follows from (6) and (7):

[1]The name refers to William Rowan Hamilton (1805-1865), who contributed to the development of the calculus of variations, to Carl Gustav Jacobi (1804-1851), who studied the theory of sufficient conditions in the calculus of variations, and to Richard Bellman (1920-1984), who brought the dynamic programming on the way. By the way, this equation comes from Constantin Carathéodory (1873-1950), whose name was not mentioned.

$$0 = V_t\left(t, x^*\right) + H\left(t, x^*(t), u^*(t), V_x\left(t, x^*\right)\right)$$
$$\geq V_t\left(t, x^-\right) + H\left(t, x^-(t), u^-(t), V_x\left(t, x^-\right)\right).$$

With the definition of the Hamiltonian $H = g + V_x \cdot f$ and taking into account

$$\frac{\mathrm{d}V(t,x)}{\mathrm{d}t} = \frac{\partial V(t,x)}{\partial t} + \frac{\partial V(t,x)}{\partial x} f(t,x,u)$$

that this inequality can be written as

$$0 = g\left(t, x^*(t), u^*(t)\right) + \frac{\mathrm{d}V\left(t, x^*\right)}{\mathrm{d}t}$$
$$\geq g\left(t, x^-(t), u^-(t)\right) + \frac{\mathrm{d}V\left(t, x^-\right)}{\mathrm{d}t}.$$

We integrate this inequality over the interval $[t_0, T]$ and obtain

$$V(t_0, x_0) = \int_{t_0}^{T} g\left(t, x^*(t), u^*(t)\right)\mathrm{d}t + q\left(T, x^*(T)\right)$$
$$\geq \int_{t_0}^{T} g\left(t, x^-(t), u^-(t)\right)\mathrm{d}t + q\left(T, x^-(T)\right),$$

by using

$$\int_{t_0}^{T} \frac{\mathrm{d}V}{\mathrm{d}t} = V\left(T, x(T)\right) - V(t_0, x_0), V\left(T, x(T)\right) = q\left(T, x(T)\right).$$

$V(t_0, x_0)$ was added on both sides. Since $u^-(\cdot)$ is arbitrary the value of the objective functional is them maximized by the control $u^*(\cdot)$ [4,5].

5. Algorithm

Now we can use this theorem and formulate a constructive algorithm:

1) Identify $f(t,x,u), g(t,x,u), q(x,t)$ with the functions of a specific problem.

2) Write down the corresponding Bellman equation.

3) Calculate $\hat{u}$ as function of $t, x, V_x : \hat{u}(t, x, V_x)$.

4) Add the maximizing value from $\hat{u}(t, x, V_x)$ in the right-hand side of the Bellman equation. (PDE)

5) Solve the Bellman equation. (analytically or numerically)

6) Compute $x^*(\cdot), u^*(\cdot)$ for $t_0 \leq t \leq T$ by using 3.

6. An Example. The Comparison with Methods Using the Maximum Principle

We want to compare this method with known methods based on the Pontryagin maximum principle.

Let us consider the problem:

$$\dot{x}(t) = -x(t) + u(t), \quad x(0) = x_0, \quad u(\cdot)$$

piecewise constant

$$J = \frac{1}{2}\int_0^1 u(t)^2 \mathrm{d}t + \frac{1}{2}x^2(1) \to \min \tag{8}$$

A necessary optimality condition for (8) is the maximum principle. The necessary conditions were developed by Pontryagin and his co-workers in Moscow in the 1950s. They introduced the idea of adjoint functions to append the differential equation to the objective functional [6].

Note that, the adjoint functions have a similar purpose as Lagrange multipliers in multivariate calculus, which append constraints to the functions of several variables to be maximized or minimized. Thus, one begins by finding appropriate conditions that the adjoint function should satisfy.

Let $x^*(\cdot), u^*(\cdot)$ be optimal, then there is a nontrivial solution of the adjoint equation $\dot{\lambda}(t) = -\frac{\partial}{\partial x} H\left(t, x^*(t), u^*(t), \lambda(t)\right)$, so that for almost all t

$$H\left(t, x^*(t), u^*(t), \lambda(t)\right) = \max_{u \in \Omega} H\left(t, x^*(t), u, \lambda(t)\right),$$

and the transversality condition $\lambda(t) = -\frac{\partial q}{\partial x}(T)$ is satisfied.

In our example it is

$$H(t, x, u, \lambda) = -\frac{1}{2}u^2 + \lambda(-x + u)$$

and consequently

$$\frac{\partial}{\partial u} H(t, x, u, \lambda) = 0 \Leftrightarrow u = \lambda.$$

That means $u^*(t) = \lambda(t)$ for almost all t.

Replacing this in the process equation, we obtain a two-point boundary value problem:

$$\dot{x}(t) = -x(t) + \lambda(t), \quad x(0) = x_0,$$

$$\dot{\lambda}(t) = -\frac{\partial}{\partial x} H(t, x, u, \lambda) = \lambda(t), \ \lambda(1) = -x(1).$$

The solutions of these equations are:

$$\lambda(t) = C_1 \cdot e^t, \ x(t) = C_2 e^{-t} + \frac{1}{2}C_1 e^t, \ 0 \leq t \leq 1.$$

The initial condition $C_2 + \frac{1}{2}C_1 = x_0$ and the final condition $-\lambda(1) = C_2 e^{-1} + \frac{1}{2}C_1 e = -C_1 e$ give the constants C_1, C_2:

$$C_1 = \frac{2x_0}{1 - 3e^2}, \ C_2 = -\frac{3x_0 e^2}{1 - 3e^2}.$$

Therefore, it follows the open-loop-solution

$$x^*(t)=\frac{x_0e^t-3x_0e^{2-t}}{1-3e^2},\ u^*(t)=\frac{2x_0e^t}{1-3e^2}. \tag{9}$$

The Bellman principle provides the same solution in another way:

Find the function $V(t,x)$, such that

$$V(T,x)=V(1,x)=-q(x(1))=-\frac{1}{2}x^2(1)$$
$$-V_t(t,x)=\max_{u\in\Omega}H(t,x,u,V_x(t,x)).$$

Due to $H(t,x,u,V_x)=-\frac{1}{2}u^2+V_x(-x+u)$ the necessary condition for a maximum of H in $u\in R$ is $\frac{\partial H}{\partial u}(t,x,u,V_x)=0.$

This is exactly satisfied when $u=V_x$, that means $\hat{u}(t,x,V_x)=V_x$. We have in mind $U=R$. Therefore, $\hat{H}=H(t,x,\hat{u},V_x)=-\frac{1}{2}V_x^2+V_x(-x+V_x)$.

The Hamilton-Jacobi-Bellman equation for this task has the form:

$$-V_t=-x\cdot V_x+\frac{1}{2}V_x^2.$$

We use the ansatz $V(t,x)=A(t)\cdot x^2$, because the objective function, as well as the process equation with respect to *u* are polynomial. Then it follows

$$V(t,x)=\frac{x^2}{1-3e^{2-2t}}.$$

Therefore,

$$\hat{u}(t,x)=V_x(t,x)=\frac{2x}{1-3e^{2-2t}}.$$

By using the differential equation the open-loop solution can be calculated. It is

$$\dot{x}(t)=-x(t)+\frac{2x(t)}{1-3e^{2-2t}},\ x(0)=x_0.$$

and this initial value problem has the solution $x^*(t)=\frac{x_0e^t-3x_0e^{2-t}}{1-3e^2}$.

With respect to $u^*(t)=\hat{u}(t,x^*)$ we obtain the optimal control $u^*(t)=\frac{2x_0e^t}{1-3e^2}$ as a func-tion of the time *t*.

7. Closed-Loop Optimality Conditions for Control Problems with Piecewise Constant Controls

Now we consider problem (1) with **piecewise constant controls**.

If the length of the interval is fixed, the Pontryagin maximum principle in the classical form is not applicable.

There is an alternate Pontryagin-like-way. Let $u^*(t)=u_k=u(t_k)$ be optimal on $[t_k,t_{k+1}]$. Then it follows:

$$\int_{t_k}^{t_{k+1}}H_u(t,x^*(t),u(t_k),\lambda(t))\mathrm{d}t=0, \tag{10}$$

Using the Bellman principle we can also win optimality conditions. Let be $t_0<t_1<\cdots<t_n$ predetermined time points and $x(\cdot)$ absolutely continuous.

The problem is now:

$$J(u)=\sum_{k=0}^{n-1}\left[\int_{t_k}^{t_{k+1}}g(t,x(t),u(t_k))dt\right]+q(t_n,x(t_n))\to\max_u \tag{11}$$

The process equation is:

$\dot{x}(t)=f(t,x(t),u(t_k))$, if $t\in T_k=[t_k,t_{k+1}), k=0,1,\cdots,n-1$. The optimal control $u^*(t)=u_k$ is to be found.

At first we consider the special case $n = 1$ (one control interval).

We denote $u(t_0)=v$ and define the new value function

$$W(t,x,v)=\int_t^{t_1}g(\tau,x(\tau),v)\mathrm{d}\tau+q(t_1,x(t_1))$$

for the process, which starts at time *t* with the vector $x(t)=x$ and is performed with the constant control $v\in\Omega$. This function *W* is continuously differentiable in *t* and *x*. It is not to be confused with the function *V* (chapter 4), since here is no maximum operator.

We can formulate new necessary conditions [7].

Necessary condition

Let $W(t,x,v)$ be continuously differentiable in *t* and *x*. Let $\hat{u}(t,x)$ be an optimal constant control that leads the process $\dot{x}(t)=f(t,x(t),u(t,x)),x(t)=x$ from *x* on $[t,t_1]$. The control $\hat{u}$ is constant also in $[t_0,t_1]$. Then this control $\hat{u}(t,x)$ satisfies for all $t\in[t_0,t_1]$ the condition

$$\hat{u}(t,x)=\hat{u}(t_0,x_0)=\arg\max_{v\in\Omega}W(t_0,x_0,v)$$

where $W(t,x,v)$ satisfies the partial differential equation:

$$-\frac{\partial W\ (t,x,v)}{\partial t}=\frac{\partial W\ (t,x,v)}{\partial x}f\ (t,x,v)+g\ (t,x,v),$$
$$\forall v\in\Omega.$$

in other words,

$$-\frac{\partial W(t,x,v)}{\partial t}=\sum_{i=1}^{n}\frac{\partial W(t,x,v)}{\partial x_i}f_i(t,x,v)+g(t,x,v),$$
$$\forall(t,x,v)\in[t_0,t_1)\times R^n\times\Omega$$

and

$$W(t_1,x,v)=q(t_1,x(t_1)),\quad \forall v\in\Omega.$$

Proof:

The proof is similar to the previous one. $W(t+\Delta t,x(t+\Delta t),v)$ is the cost function for the part of the solution, that starts at the time $t+\Delta t$ with state $x(t+\Delta t)$ under the influence of the control *v*.

It is obviously:

$$W(t,x,v)=\int_t^{t+\Delta t}g(\tau,x(\tau),v)\mathrm{d}\tau+W(t+\Delta t,x(t+\Delta t),v),$$
$$\forall v\in\Omega.$$

Since W is assumed to be continuous differentiable and g to be continuous, $\int_t^{t+\Delta t}g(\tau,x(\tau),v)\mathrm{d}\tau$ can be approximated as $g(t,x(t),v)\Delta t+o(\Delta t)$. It follows:

$$W(t,x,v)=\{g(t,x(t),v)\Delta t+W(t+\Delta t,x(t+\Delta t),v)\}+o(\Delta t).$$

$o(\Delta t)$ represents the higher order terms, that means $\lim_{\Delta t\to 0}\frac{o(\Delta t)}{\Delta t}=0$.

According to Taylor's theorem we obtain:

$$W(t+\Delta t,x(t+\Delta t),v)=W(t,x(t),v)+W_t(t,x(t),v)\Delta t+W_x(t,x(t),v)\dot{x}(t)\Delta t+o(\Delta t).$$

Substituting this result into the previous equation, it follows with $\dot{x}=f(t,x,v)$ the PDE

$$0=g(t,x(t),v)+W_t((t,x(t),v)+W_x(t,x(t),v)\cdot f(t,x(t),v),$$

or in another form

$$-\frac{\partial W(t,x,v)}{\partial t}=\frac{\partial W(t,x,v)}{\partial x}f(t,x,v)+g(t,x,v).$$

In the special problem it is $J(v)=W(t_0,x,v),v\in\Omega,$ so the optimal control vector can be obtained by

$$\hat{u}(t_0,x_0)=\arg\max_{v\in\Omega}W(t_0,x_0,v)$$

The boundary condition $W(t_1,x,v)=q(t_1,x(t_1))$ follows immediately.

In case of n control intervals the definition of necessary conditions is analogous. Let be $u(t)=\mathbf{v}=(v_0,v_1,\cdots,v_{n-1})$ with $v_k\in\Omega,k=0,\cdots,n-1.$ The function $W(t,x,v)$ is for all $t\in[t_{k-1},t_k],1<k<n$ defined as:

$$W(t,x,v)=\int_t^{t_k}g(\tau,x(\tau),v)\mathrm{d}\tau$$

and for all $t\in[t_{n-1},t_n]$ as:

$$W(t,x,v)=\int_t^{t_n}g(\tau,x(\tau),v)\mathrm{d}\tau+q(t_n,x(t_n)).$$

It follows for all $t\in[t_{k-1},t_k],1<k\le n$:

$$\hat{u}(t,x)=\hat{u}(t_{k-1},x(t_{k-1}))=\arg\max_{v\in\Omega}W(t_{k-1},x(t_{k-1}),v),$$
$$-\frac{\partial W(t,x,v)}{\partial t}=\frac{\partial W(t,x,v)}{\partial x}f(t,x,v)+g(t,x,v),$$
$$W(t_k-0,x,v)=W(t_k,x,\hat{u}(t_k,x)),1\le k<n,$$
$$W(t_n-0,x,v)=q(t_n,x),\forall v\in\Omega.$$

8. Open-Loop Optimality Conditions for Control Problems with Piecewise Constant Controls

We can also formulate the optimality conditions for the problem (11) in open-loop form. Let $x(t,v)$ be a solution of the process equation

$$\frac{\partial x(t,v)}{\partial t}=f(t,x(t,v),v),\forall(t,v)\in[t_k,t_{k+1})\times\Omega,$$

with $x(t_k,v)=x(t_k)$ for $k=0,\cdots,n-1$, for all $\forall v\in\Omega$ and $x(t_0)=x_0$.

The Hamiltonian is: $H(t,x,v,\lambda)=g(t,x,v)+\lambda\cdot f(t,x,v)$. We consider the special case n = 1 (one control interval).

Necessary condition in open-loop form

Let $x^*(\cdot),u^*(\cdot)$ be optimal with $u^*(t)=u^*(t_0)$, $x^*(t)=x(t,u^*(t_0))$, then it is

$$u^*(t_0)=\arg\max_{v\in\Omega}\{S(t_0,v)+\lambda(t_0,v)\cdot x(t_0)\},$$

where $\lambda(t,v)$ is a solution of

$$\dot{\lambda}(t,v)=-\frac{\partial}{\partial x}H(t,x(t,v),v,\lambda(t,v)),\quad \forall(t,v)\in[t_0,t_1)\times\Omega, \tag{12}$$

and $S(t,v)$ is a solution of

$$-\frac{\partial S(t,v)}{\partial t}=H(t,x(t,v),v,\lambda(t,v))-\frac{\partial H(t,x(t,v),v,\lambda(t,v))}{\partial x}x(t,v) \tag{13}$$

with $x(t_0,v)=x(t_0)=x_0$ and the transversality conditions

$$\lambda(t_1,v)=\frac{\partial q(t_1,x(t_1))}{\partial x}$$

and

$$S(t_1,v)=-\frac{\partial q(t_1,x(t_1))}{\partial x}\cdot x(t_1,v)+q(t_1,x(t_1))$$

are satisfied for all $\forall v\in\Omega$.

Proof:

We consider the equation (13):

$$-\frac{\partial S(t,v)}{\partial t}=H(t,x(t,v),v,\lambda(t,v))-\frac{\partial H(t,x(t,v),v,\lambda(t,v))}{\partial x}x(t,v)$$

is equivalent to

$$-\frac{\partial S(t,v)}{\partial t}=g(t,x,v)+\lambda(t,v)\cdot f(t,x,v)-\frac{\partial H(t,x(t,v),v,\lambda(t,v))}{\partial x}x(t,v).$$

It follows from (12)

$$-\frac{\partial S(t,v)}{\partial t}=g(t,x,v)+\lambda(t,v)\cdot\dot{x}(t,v)+\dot{\lambda}(t,v)\cdot x(t,v).$$

We integrate this equation over $[t,t_1]$:

$$-S(t_1,v)+S(t,v)=\int_t^{t_1}g(\tau,x(\tau),v)\mathrm{d}\tau+\lambda(t_1,v)\cdot x(t_1,v)-\lambda(t,v)\cdot x(t,v).$$

From the previous we obtain $\int^{t_1}g(\tau,x(\tau),v)\mathrm{d}\tau=W(t,x,v)-q(t_1,x(t_1))$ and from the transversality conditions

$$-S(t_1,v)=\frac{\partial q(t_1,x(t_1))}{\partial x}\cdot x(t_1,v)-q(t_1,x(t_1))=\lambda(t_1,v)\cdot x(t_1,v)-q(t_1,x(t_1)).$$

It follows:

$$S(t,v)=W(t,x,v)-\lambda(t,v)\cdot x(t,v)\Leftrightarrow W(t,x,v)=S(t,v)+\lambda(t,v)\cdot x(t,v).$$

As shown in the previous chapter it is

$$\hat{u}(t,x)=\hat{u}(t_0,x_0)=\arg\max_{v\in\Omega}W(t_0,x_0,v)$$

because $\hat{u}(t,x)$ is constant. It follows with

$$x(t_0,v)=x(t_0):$$
$$\hat{u}(t_0,x_0)=\arg\max_{v\in\Omega}\{S(t_0,v)+\lambda(t_0,v)\cdot x(t_0)\}.$$

The term on the right-hand side of the equation is the function of t_0. We obtain the open-loop form:

$$u^*(t_0)=\arg\max_{v\in\Omega}\{S(t_0,v)+\lambda(t_0,v)\cdot x(t_0)\}.$$

In case of n control intervals the definition of necessary conditions is analogous.

We have to maximize:

$$J(u)=\sum_{k=0}^{n-1}\left[\int_{t_k}^{t_{k+1}}g(t,x(t),u(t_k))\mathrm{d}t\right]+q(t_n,x(t_n))$$

under the constraint:

$$\dot{x}(t)=f(t,x(t),u(t_k)),t_k\le t\le t_{k+1},$$
$$x(t_0)=x_0,u(t_k)\in\Omega.$$

Let $x^*(\cdot),u^*(\cdot)$ be optimal with $u^*(t)=u^*(t_k)$, $x^*(t)=x(t,u^*(t_k))$, for $k=0,\cdots,n-1$, then it is

$$u^*(t_k)=\arg\max_{u\in\Omega}\{S(t_k,u)+\lambda(t_k,u)\cdot x(t_k)\},$$

where $S(t,v)$ is a solution of

$$-\frac{\partial S(t,v)}{\partial t}=H(t,x(t,v),v,\lambda(t,v))-\frac{\partial H(t,x(t,v),v,\lambda(t,v))}{\partial x}x(t,v) \tag{14}$$

and $\lambda(t,v)$ is a solution of

$$\dot{\lambda}(t,v)=-\frac{\partial}{\partial x}H(t,x(t,v),v,\lambda(t,v)),\quad\forall(t,v)\in[t_k,t_{k+1})\times\Omega, \tag{15}$$

with

$$x(t_0,v)=x(t_0)=x_0,$$
$$x(t_k,v)=x(t_k)=x(t_k-0,u^*(t_{k-1})),$$
$$\lambda(t_k-0,v)=\lambda(t_k,u^*(t_k)),$$
$$S(t_k-0,v)=S(t_k,u^*(t_k)),$$

and the transversality conditions

$$\lambda(t_n,v)=\frac{\partial q(t_n,x(t_n))}{\partial x},$$
$$S(t_n,v)=-\frac{\partial q(t_n,x(t_n))}{\partial x}\cdot x(t_n)+q(t_n,x(t_n))$$

are satisfied.

9. An Example. The Multistage Open-Loop Control

We want to solve a two stages-optimal control problem.

Let be $t\in[0,1]$ and $t_0=0,t_1=0.5$. We obtain two time intervals: $T_0=[0;0.5),T_1=[0.5;1)$. The process equation is:

$$\dot{x}(t)=-x(t)+u(t_k),k=0,1;x(0)=x_0, \quad (16)$$

$u(\cdot)$ piecewise constant:

$$u(t)=(u(t_0),u(t_1))=(v_0,v_1)=\boldsymbol{v},$$
$$u(t)=u(0),t\in T_0,$$
$$u(t)=u(0.5),t\in T_1, \quad (17)$$
$$J(\boldsymbol{v})=\frac{1}{2}\sum_{k=0}^{1}\int_{t_k}^{t_{k+1}}v_k^2\mathrm{d}t+\frac{1}{2}x^2(1)\to\min_{\boldsymbol{v}}$$
$$\Leftrightarrow J(\boldsymbol{v})=-\frac{1}{2}\sum_{k=0}^{1}\int_{t_k}^{t_{k+1}}v_k^2\mathrm{d}t-\frac{1}{2}x^2(1)\to\max_{\boldsymbol{v}}$$

The Hamiltonian is $H(t,x,v,\lambda)=\lambda\cdot(-x+v)-\frac{1}{2}v^2$.

The necessary conditions for the problem (17) are:

$$u^*(0.5)=u_1^* =\arg\max_{v\in\Omega}\{S(0.5,v)+\lambda(0.5,v)\cdot x(0.5,v)\}, \quad (18)$$

$$u^*(0)=u_0^*=\arg\max_{v\in\Omega}\{S(0,v)+\lambda(0,v)\cdot x_0\}, \quad (19)$$

and it is

$$\lambda(1,v_1)=-x(1,\boldsymbol{v}),\forall v_1\in\Omega$$
$$S(1,v_1)=\frac{1}{2}x^2(1,\boldsymbol{v}),\forall v_1\in\Omega$$

It follows for $i=0,1$:

$$\dot{\lambda}(t,v_i)=\lambda(t,v_i),$$
$$-\dot{S}(t,v_i)=\lambda(t,v_i)\cdot v_i-\frac{1}{2}v_i^2$$

and

$$\dot{x}(t,v_i)=-x(t,v_i)+v_i,$$
$$x(0,v_0)=x_0,x(0.5,v_1)=x(0.5-0,v_0) \quad (20)$$

The transition conditions are:

$$\lambda(0.5-0,v_0)=\lambda(0.5,u_1^*),$$
$$S(0.5-0,v_0)=S(0.5,u_1^*).$$

The solutions of the ODE $\dot{x}(t,v_i)=-x(t,v_i)+v_i$ $i=0,1$ are

$$x(t,v_0)=A_1\cdot e^{-t}+v_0,\quad t\in T_0,$$
$$x(t,v_1)=A_2\cdot e^{-t}+v_1,\quad t\in T_1.$$

According to $x(0)=x_0$ we obtain $A_1=x_0-v_0$ and with (20) $A_2=(x_0-v_0)+(v_0-v_1)e^{0.5}$. Therefore,

$$x(t,v_1)=((x_0-v_0)+(v_0-v_1)e^{0.5})\cdot e^{-t}+v_1,t\in T_1,$$
$$x(1,v_1)=(x_0-v_0)e^{-1}+(v_0-v_1)e^{-0.5}+v_1,$$
$$x(0.5,v_1)=(x_0-v_0)e^{-0.5}+v_0$$

and $\lambda(t,v_i)=C_1e^t,\quad i=0,1.$

The final condition for t = 1 delivers

$$\lambda(1,v_1)=C_1e=-x(1,\boldsymbol{v})$$
$$=-((x_0-v_0)e^{-1}+(v_0-v_1)e^{-0.5}+v_1)$$

and gives us the constant C_1. Therefore,

$$\lambda(t,v_i)=-((x_0-v_0)e^{-2}+(v_0-v_1)e^{-1.5}+v_1e^{-1})e^t,$$
$$i=0,1.$$

It follows for $t\in T_1$:

$$\dot{S}(t,v_1)=-\lambda(t,v_1)\cdot v_1+\frac{1}{2}v_1^2$$
$$=((x_0-v_0)e^{-2}+(v_0-v_1)e^{-1.5}+v_1e^{-1})v_1e^t+\frac{1}{2}v_1^2.$$

The solution of this equation is:

$$S(t,v_1)=((x_0-v_0)e^{-2}+(v_0-v_1)e^{-1.5}+v_1e^{-1})v_1e^t +\frac{1}{2}v_1^2t+C_2.$$

We obtain C_2 from the final condition:

$$S(1,v_1)=((x_0-v_0)e^{-1}+(v_0-v_1)e^{-0.5}+v_1)v_1+\frac{1}{2}v_1^2+C_2$$
$$=\frac{1}{2}x^2(1-0,\boldsymbol{v})=\frac{1}{2}\left[(x_0-v_0)e^{-1}+(v_0-v_1)e^{-0.5}+v_1\right]^2.$$

Due to (18):

$$u^*(0.5)=u_1^*=\arg\max_{v\in\Omega}\{S(0.5,v)+\lambda(0.5,v)\cdot x(0.5,v)\}$$
$$=\arg\max_{v\in\Omega}\Big\{(x_0-v_0)ve^{-1.5}+(v_0-v)ve^{-1}+v^2e^{-0.5}+\frac{1}{4}v^2$$
$$-\frac{1}{2}v^2+\frac{1}{2}\{(x_0-v_0)e^{-1}+(v_0-v)e^{-0.5}+v\}^2$$
$$-\{(x_0-v_0)ve^{-1}+(v_0-v)ve^{-0.5}+v^2\}$$
$$-\{(x_0-v_0)e^{-1.5}+(v_0-v)e^{-1}+ve^{-0.5}\}$$
$$\cdot\{(x_0-v_0)e^{-0.5}+v_0\}\Big\}$$

This is exactly satisfied when (with differentiation over v and substitution $v_0=u_0^*$:

$$2x_0(e^{0.5}-1)+2u_0^*(1-2e^{0.5}+e) =u_1^*(4e^1-2e^{0.5}-3e^{1.5}). \quad (21)$$

Analogically, we obtain for $t\in T_0$:

$$\dot{S}(t,v_0) = -\lambda(t,v_0)\cdot v_0 + \frac{1}{2}v_0^2$$
$$= \left((x_0 - v_0)e^{-2} + (v_0 - u_1^*)e^{-1.5} + u_1^* e^{-1}\right)v_0 e^t + \frac{1}{2}v_0^2.$$

That ODE has the following solution on T_0:

$$S(t,v_0) = \left((x_0 - v_0)e^{-2} + (v_0 - u_1^*)e^{-1.5} + u_1^* e^{-1}\right)v_0 e^t + \frac{1}{2}v_0^2 t + C_3.$$

We obtain C_3 from the final condition:

$$S(0.5-0, v_0) = \left((x_0 - v_0)v_0 e^{-1.5} + (v_0 - u_1^*)v_0 e^{-1} + u_1^* v_0 e^{-0.5}\right) + \frac{1}{4}v_0^2 + C_3$$
$$= S(0.5, u_1^*) = \left\{(x_0 - v_0)u_1^* e^{-1.5} + (v_0 - u_1^*)u_1^* e^{-1} + u_1^{*2} e^{-0.5} + \frac{1}{4}u_1^{*2} - \frac{1}{2}u_1^{*2} + \frac{1}{2}\left\{(x_0 - v_0)e^{-1} + (v_0 - u_1^*)e^{-0.5} + u_1^*\right\}^2 - \left\{(x_0 - v_0)u_1^* e^{-1} + (v_0 - u_1^*)u_1^* e^{-0.5} + u_1^{*2}\right\}\right\}$$

Therefore,

$$u^*(0) = u_0^* = \arg\max_{v\in\Omega}\left\{S(0,v) + \lambda(0,v)\cdot x(0)\right\}$$
$$= \arg\max_{v\in\Omega}\left\{(x_0 - v)ve^{-2} + (v - u_1^*)ve^{-1.5} + u_1^* ve^{-1} + (x_0 - v)u_1^* e^{-1.5} + (v - u_1^*)u_1^* e^{-1} + u_1^{*2}e^{-0.5} + \frac{1}{4}u_1^{*2} - \frac{1}{2}u_1^{*2} + \frac{1}{2}\left\{(x_0 - v)e^{-1} + (v - u_1^*)e^{-0.5} + u_1^*\right\}^2 - \left\{(x_0 - v)u_1^* e^{-1} + (v - u_1^*)u_1^* e^{-0.5} + u_1^{*2}\right\} - (x_0 - v)ve^{-1.5} - (v - u_1^*)ve^{-1} - u_1^* ve^{-0.5} - \frac{1}{4}v^2 - \left\{(x_0 - v)e^{-2} + (v - u_1^*)e^{-1.5} + u_1^* e^{-1}\right\}x_0\right\}$$

It follows:

$$2x_0\left(e^{0.5} - 1\right) = 2u_1^*\left(-e^{1.5} + 2e - e^{0.5}\right) + u_0^*\left(-e^2 + 4e^{0.5} - 2e - 2\right); \tag{22}$$

and from (21) and (22):

$$u^*(0) = u_0^* = \frac{2(1 - e^{0.5})}{3e^2 - 4e^{1.5} + 4e - 4e^{0.5} + 2}\cdot x_0,$$
$$u^*(0.5) = u_1^* = \frac{2(e^{0.5} - e)}{3e^2 - 4e^{1.5} + 4e - 4e^{0.5} + 2}\cdot x_0,$$

The optimal trajectory is:

$$x^*(t) = (x_0 - u_0^*)e^{-t} + u_0^*, t \in T_0,$$
$$x^*(t) = \left((x_0 - u_0^*) + (u_0^* - u_1^*)\cdot e^{0.5}\right)e^{-t} + u_1^*, t \in T_1,$$

and $\lambda(t) = -\left((x_0 - u_0^*)e^{-2} + (u_0^* - u_1^*)\cdot e^{-1.5} + u_1^* e^{-1}\right)e^t$.

This solution can be confirmed by substituting these values into the integral maximum principle. It is:

$$\int_0^{0.5} H_u\left(t, x^*(t), u_0^*, \lambda(t)\right)\mathrm{d}t = \int_0^{0.5}\left(\lambda(t) - u_0^*\right)\mathrm{d}t = 0$$
$$\int_{0.5}^{1} H_u\left(t, x^*(t), u_1^*, \lambda(t)\right)\mathrm{d}t = \int_{0.5}^{1}\left(\lambda(t) - u_1^*\right)\mathrm{d}t = 0$$

10. Various Types of Control Functions

Now we can compare the three types of tasks.

1) Piecewise continuous or measurable control functions: Here we can apply the Pontryagin maximum principle and the Bellman principle.

2) Piecewise constant functions and fixed t_k: In this case we can use the Bellman principle (in terms of [7]) and condition (10) [8]).

3) Integer valued control: $u(t_k) = u_k \in Z$: the Pontryagin maximum principle is not applicable.

In this case we can use the Bellman principle (in terms of [7]) and an additional constraint of the form $\prod_{i=1}^{k}(u - u_i) = 0.$

The following application areas are currently offered: (PMP is the Pontryagin's Maximum Principle)

Control functions	PMP	Bellman	Other methods
Piecewise continuous or measurable controls	classical form	applicable	reduction to “direct methods” [9]
Piecewise constant and fixed t_k	integral form [3]	applicable	reduction to “direct methods [9, 10]
Integer valued controls and fixed t_k	doesn't work	applicable	________

11. Numerical Solution Using Standard Software

For a concrete example of (cod-herring-sprat) we choose:

$$\dot{x}_1(t) = 0.4x_1(t)\cdot(1 - x_1(t)) - 1.5\cdot 250\cdot u_1(t)\cdot\frac{x_1(t)}{10^6} - 0.02\cdot\frac{x_1(t)x_2(t)}{1.2} - 0.02\cdot\frac{x_1(t)x_3(t)}{1.3};$$
$$\dot{x}_2(t) = 0.6x_2(t)\cdot\left(1 - \frac{x_2(t)}{1.2}\right) - 6.4\cdot 250\cdot u_2(t)\cdot\frac{x_2(t)}{1.2\cdot 10^6} - 0.0125\cdot\frac{x_1(t)x_2(t)}{1.2} - 0.01\cdot\frac{x_1(t)x_3(t)}{1.56};$$
$$\dot{x}_3(t) = 0.6x_1(t)\cdot\left(1 - \frac{x_3(t)}{1.3}\right) - 6.4\cdot 250\cdot u_3(t)\cdot\frac{x_3(t)}{1.3\cdot 10^6} - 0.0125\cdot\frac{x_1(t)x_3(t)}{1.3} - 0.01\cdot\frac{x_2(t)x_3(t)}{1.56};$$

$$J(u)=\int_0^T\left\{1130\cdot u_1(t)\cdot 1.5\cdot 250\cdot\frac{x_1(t)}{1}\right.$$
$$+270\cdot u_2(t)\cdot 6.4\cdot 250\cdot\frac{x_2(t)}{1.2}$$
$$+460\cdot u_3(t)\cdot 6.4\cdot 250\cdot\frac{x_2(t)}{1.2}$$
$$\left.+500\cdot 250\cdot\sum_{i=1}^{3}u_i(t)\right\}e^{-0.06t}\mathrm{d}t\to\max_u$$
$$0\le u_1(t)+u_2(t)+u_3(t)\le 1900,\quad 0\le t\le 20$$

The growth rates and interaction coefficients are arbitrarily choosen. It is assumed that the fishing cannot be reduced to zero. The only exception is the cod fishery in the early stages. Since their population in the Baltic Sea is currently too low, it is proposed in this strategy to fish for cod only after 3 years.

Matthias Gerdts developed the Fortran 77 package OC-ODE (Optimal Control of Ordinary Differential Equations) for the numerical solution of optimal control problems [9]. The program is a direct discretization and provides a numerical estimation of the controls. The controls are declared as piecewise continuous or piecewise constant functions.

The optimal strategy for catching a 3-population system cod (x_1) - herring(x_2) - sprat (x_3) for a time interval of 20 years was calculated with this software. u_1 is the number of cod cutters in any given year. u_2 and u_3 are the herring and sprat cutters. (**Table 1**). The data for the 0th year are based on the state of fish stocks in the Baltic Sea [6].

The system tends toward an equilibrium. The proposed fishing strategy achieves the largest profit with respect to sustainability. (**Table 2**). The fishing capacities for the Baltic Sea have been estimated from statistical data. A sustainable fishery can be achieved by converting the cod fishery on long lines [11,12].

The profit of the fishing industry in the beginning of the respective years are the following amounts (in million Euro) (**Figure 1**).

The number of fishing cutters that were used in the optimal case is certainly underestimated for the Baltic Sea. The maximum stock of herring and sprat in our model was taken far below the actual biomass (**Figures 2,3,4**).

Table 1. The optimal strategy for catching a 3-population system cod (x_1)-herring(x_2)- sprat (x_3) for a time interval of 20 years.

Year	x_1	x_2	x_3	u_1	u_2	u_3
0	0.2500000	0.8000000	1.0000000	0.000000	263.6494	531.3586
1	0.3244954	0.7006421	0.6495999	0.000000	154.1460	150.5292
2	0.4088918	0.7221604	0.7120884	0.000000	185.2492	240.4183
3	0.4974498	0.7117310	0.6920001	0.000000	180.4876	218.8285
4	0.5843451	0.7077763	0.6928961	417.7087	176.9903	220.8658
5	0.5778183	0.7075602	0.6918569	381.3450	176.7343	219.4632
10	0.5792128	0.7076102	0.6920774	389.0092	176.7938	219.7610
15	0.5792109	0.7076101	0.6920771	388.9998	176.7938	219.7610
20	0.5791999	0.7076000	0.6919999	389.0686	176.8090	219.8855

Table 2. The profit of the fishing industry.

Time(year)	Profit	Time(year)	Profit
1	206.10840	5	505.68780
2	260.51788	10	916.03505
3	338.06707	15	1220.4939
4	404.54111	20	1446.0516

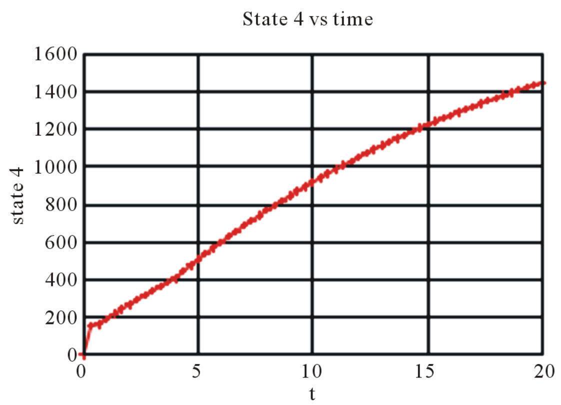

Figure 1. A potential profit of the fisheries of a 3-population system in million Euro.

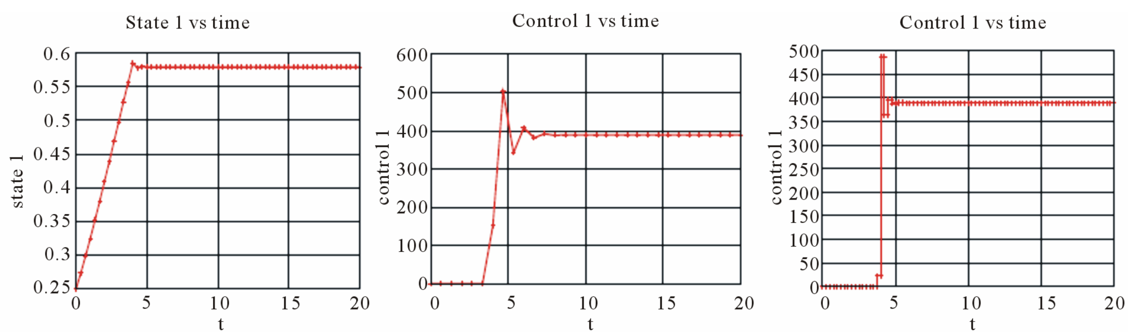

Figure 2. The 3-population system: Cod. Development of the population (left), piecewise continuous control (middle), piecewise constant control (right).

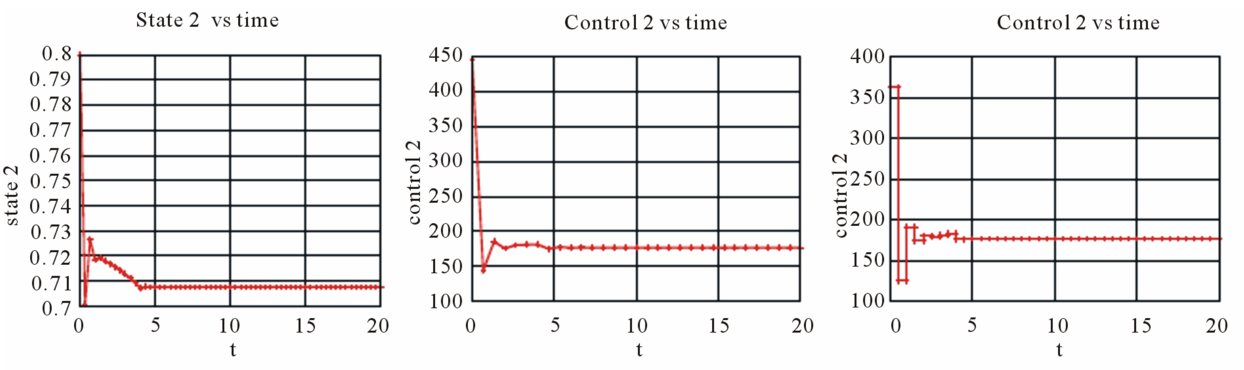

Figure 3. The 3-populationen-system: Herring. Development of the population (left), piecewise continuous control (middle), piecewise constant control (right).

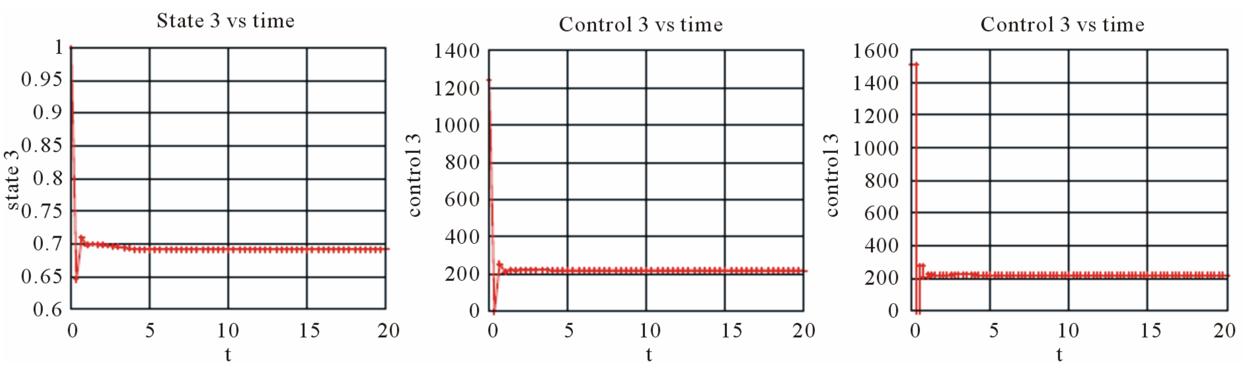

Figure 4. The 3-populationen-system: Sprat. Development of the population (left), piecewise continuous control (middle), piecewise constant control (right).

12. Comments

The value function $V(t,x)$ is globally continuously differentiable only in exceptional cases (for example, in the linear-quadratic problems).

The Bellman principle can operate even if the value function is only piecewise differentiable. This happens when the set of the points of discontinuity of V is composed of smooth surfaces.

Useful general principles that guarantee a C^1-solution of the HJB equation are not known. In general, the value function is not smooth. Even if the value function is smooth, then the solution can be not expressed in explicit formulas [13].

There is a possibility of introducing a generalized solution concept, which is also obtained in the case of non-differentiability of a value function.

This solution concept should be so general that it can also be applied when the derivative $Dv(x)$ does not exist for all $x \in R^n$. On the other hand, it should be taken so that onee does not get too many possible solutions of the HJB quation - in the ideal case, the optimal value function is the unique solution [14,15]. This generalized solution is called a viscosity solution.

11. References

[1] M. Begon and M. Mortimer, "Populationsökologie," Spektrum Akademischer Verlag, Heidelberg, 1997.

[2] A. J. Lotka,"Elements of Mathematical Biology," Dover, New York, 1956.

[3] M. Papageorgiou,"Optimierung," R. Oldenbourg Verlag, München, 1996.

[4] L. Grüne, "Modeling with Differential Equations," Lecture Notes, University of Bayreuth, 2003.

[5] T. Christiaans, "Neoklassische Wachstumstheorie," Books on Demand, Norderstedt, 2004.

[6] M. Brokate, "Control Theory," Institute of Informatics and Mathematics, Technical University of Munich, Munich, 1994.

[7] A. Pantelejew and A. Bortakovskij, "Control Theory in Examples and Practices," Vysshaya Shkola, Moscow, 2003.

[8] W. H. Schmidt, "Durch Integralgleichungen Beschriebene Optimale Prozesse mit Nebenbedingungen in Banachräumen-Notwendige Optimalit ätsbedingungen," *Journal of Applied Mathematics and Mechanics*, Vol. 62, No. 2, 1982, pp. 65-75. doi: 10.1002/zamm.19820620202

[9] M. Gerdts, "Optimal Control of Ordinary-Differential

Equations," University of Hamburg, Hamburg, 2006.

[10] W. Alt, "Nichtlineare Optimierung: Eine Einführung in Theorie, Verfahren und Anwendungen," Viewer and teubner Verlag, Wiesbaden, 2002.

[11] R. Döring, "Die Zukunft der Fischerei im Biosphären-reservat Südost- Rügen," Peter Lang GmbH, Frankfurt, 2001.

[12] P. Ernst and W. Müller, "Die Deutsche und Internationale Dorschfischerei in der Ostsee im jahr 1998," *Informationen für die Fischwirtschaft aus der Fischereiforschung*, Vol. 46, No. 3, 1999, pp. 32-35.

[13] M. Bardi, I. Capuzzo-Dolcetta, "Optimal Control and Viscosity Solutions of Hamilton-Jacobi-Bellman Equations," Birkhäuser, Boston, 1997. doi:10.1007/978-0-8176-4755-1

[14] L. Grüne, "Viskositätslösungen von Hamilton-Jacobi-Bellman Gleichungen: Eine Einführung," *Numerical Dynamics of Control Systems*, 2004.

[15] P. Lions, "On the Hamilton-Jacobi-Bellman Equations," *Acta Applicandae Mathematicae*, Vol. 1, No. 1, 1983, pp. 17-41. doi:10.1007/BF02433840

[16] O. Rechlin, "Fischbestände der Ostsee, ihre Entwicklung in den Jahren seit 1970 und Schlussfolgerungen," ihre Nutzung, Rostock, 1999.

