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Multiple Periodic Solutions for Some Classes of First-Order Hamiltonian Systems

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Abstract

Considering a decomposition $\mathbb{R}^2N = A \oplus B$ of $\mathbb{R}^2N$, we prove in this work, the existence of at least $(1+\dim A)$ geometrically distinct periodic solutions for the first-order Hamiltonian system $Jx'(t)+H'(t,x(t))+e(t)=0$ when the Hamiltonian $H(t,u+v)$ is periodic in (t,u) and its growth at infinity in v is at most like or faster than $|v|^a$, $0 \le a < 1$, and e is a forcing term. For the proof, we use the Least Action Principle and a Generalized Saddle Point Theorem.

Keywords: Hamiltonian Systems, Partial Nonlinearity, Multiple Periodic Solutions, Critical Point Theory

1. Introduction

Consider the nonautonomous first-order Hamiltonian system

$$Jx'(t)+H'(t,x(t))+e(t)=0$$

where $H:\mathbb{R}\times\mathbb{R}^{2N}\to\mathbb{R}$, $\mathbb{R},(t,x)\to H(t,x)$ is a continuous function, $T-$ periodic $(T>0)$ in the first variable and differentiable with respect to the second variable with continuous derivative $H'(t,x)=\frac{\partial H}{\partial x}(t,x)$, $e:\mathbb{R}\to\mathbb{R}^2N$ is a continuous $T-$ periodic function with mean value zero and J is the standard symplectic matrix:

$$J=\begin{pmatrix} 0 & -I_N \\ I_N & 0 \end{pmatrix}$$

I_N being the identity matrix of order N.

Using variational methods, there have been many papers devoted to the existence of periodic solutions for ($\mathcal{H}$), we refer the readers to [1-5] and the references therein. However, there are few papers discussing the multiplicity of periodic solutions for ($\mathcal{H}$) (see [6-9]). Under the assumptions that H is periodic in $x_1,\cdots,x_p$, where $1\le p\le 2N-1$, $x=(x_1,\cdots,x_{2N})$ and there exists $f\in L^2(0,T;\mathbb{R}^+)$ such that

$$|H'(t,x)|\le f(t), \forall x\in\mathbb{R}^{2N}, \quad a.e.t\in[0,1]. \tag{1.1}$$

$$\int_0^T H(t,x)\mathrm{d}t\to\pm\infty \; as|x|\to\infty, x\in\{0\}\times\mathbb{R}^{2N-p}, \tag{1.2}$$

the author has shown in [9] that system ($\mathcal{H}$) possesses at least $(p+1)$ geometrically distinct periodic solutions. The first goal of this note is to generalize the existence result of multiple periodic solutions obtained above to the sublinear case. Precisely, consider a decomposition $\mathbb{R}^{2N}=A\oplus B$ of $\mathbb{R}^{2N}$ with

$$A=\text{space}\{e_1,\cdots,e_{i_p}\}, B=\text{space}\{e_{i_{p+1}},\cdots,e_{i_{2N}}\}$$

where $0\le p\le 2N-1$ and $(e_i)_{1\le i\le 2N}$ is the standard basis of $\mathbb{R}^{2N}$ and let us denote P_A (resp. P_B) the projection of $\mathbb{R}^{2N}$ into A (resp. B). We obtain the following result

Theorem 1.1 Assume that *H* satisfies

(H_0) H is periodic in the variables. $x_{i_1},\cdots,x_{i_p}$;

(H_1) There exist α ∈ [0, 1] and two *T*–periodic functions $a\in L\frac{1}{1-\alpha}(0,T;\mathbb{R}^+)$ and $b\in L^2(0,T;\mathbb{R}^+)$ such that

$$|H'(t,x)|\le a(t)|P_B(x)|^\alpha+b(t), \forall x\in\mathbb{R}^{2N}, \quad a.e.t\in[0,1],$$

(H_2) Either

1) $$\frac{1}{|x|^{2\alpha}}\int_0^T H(t,x)\mathrm{d}t\to\infty \; as|x|\to\infty, x\in B$$

or

2) $$\frac{1}{|x|^{2\alpha}}\int_0^T H(t,x)\mathrm{d}t\to-\infty \; as|x|\to\infty, x\in B$$

Then the Hamiltonian system ($\mathcal{H}$) possesses at least $(p+1)T-$ periodic solutions geometrically distinct.

Example 1.1 Let $a:\mathbb{R}^{2N}\to\mathbb{R}$ be a periodic and continuously differentiable function. Consider the Hamiltonian:

$$H(t,r,p)=\left(\frac{1}{2}+\sin\left(\frac{2\pi}{T}t\right)\right)\left|p-a(r)\right|^{\frac{3}{2}} \qquad (1.3)$$

Then H satisfies conditions $(H_0)-(H_2)$ with $A=\mathbb{R}^N\times 0$ and $B=\{0\}\times\mathbb{R}^N$.

It is easy to see that conditions $(H_1),(H_2)$ don't cover some sublinear cases like

$$H(t,r,p)=\left(\frac{1}{2}+\cos\left(\frac{2\pi}{T}t\right)\right)\frac{\left|p-a(r)\right|^2}{\ln\left(2+\left|p-a(r)\right|^2\right)}, \qquad (1.4)$$

$\forall(t,r,p)\in\mathbb{R}\times\mathbb{R}^{2N}$

The second goal of this paper is to study the existence of multiple periodic solutions for ($\mathcal{H}$) when the Hamiltonian H satisfies a nonlinearity condition which covers the cases like (1.4). Precisely, we will require the nonlinearity to have a partial growth at infinity faster than $|x|^\alpha, 0\le\alpha<1$

Our second main result is:

Theorem 1.2 Consider a nonincreasing positive function $\omega\in C\left([0,+\infty[,\mathbb{R}^+\right)$ with the properties:

$$\liminf_s\to+\infty\frac{\omega(s)}{\omega(s)}>0,$$

$$\omega(s)\to 0,\omega(s)s\to+\infty \text{ as } s\to+\infty,$$

and assume that *H* atisfies (H_0) and the following assumptions

(H_3) There exist a positive constant *a* d a function $g\in L^2(0,T;\mathbb{R}^+)$ such that for all $\forall x\in\mathbb{R}^{2N}$ and $a.e.t\in[0,1]$

$$|H'(t,x)|\le a\omega(|P_B(x)|)|P_B(x)|+g(t),$$

(H_4) Either

1) $\frac{1}{\left[\omega(|x|)|x|\right]^2}\int_0^T H(t,x)\mathrm{d}t\to+\infty$ as $|x|\to\infty, x\in B$,

2) $\frac{1}{\left[\omega(|x|)|x|\right]^2}\int_0^T H(t,x)\mathrm{d}t\to-\infty$ as $|x|\to\infty, x\in B$.

Then the system ($\mathcal{H}$) possesses at least $(p+1)$ geo-metrically distinct $T-$periodic solutions.

Remark 1.1 The Hamiltonian H defined in (1.4) satisfies the conditions $(H_3),(H_4)$ introduced above with $\omega(s)=\frac{1}{\ln(2+s^2)}, s\ge 0$,

2. Preliminaries

Firstly, let us recall a critical point theorem due to G. Fournier, D. Lupo, M. Ramos and M. Willem [10]. Given a Banach space E and a complete connected Finsler manifold V of class C^2, we consider the space $X=ExV$. Let $E=W\oplus Z$ (topological direct sum) and ($E_n\oplus Z_n$) be a sequence of closed subspaces with $Z_n\subset Z$, $W_n\subset W$, $1\le\dim W_n<\infty$. Define $X_n=E_nxV$. For $f\in C^1(X,\mathbb{R})$, we denote by $f_n=f_{/X_n}$. Then we have $f_n\in C^1(X_n,\mathbb{R})$ for all $n\ge 1$.

Definition 2.1 Let $\in C^1(X,\mathbb{R})$. The function *f* satisfies the Palais-Smale condition with respect to (X_n) at a level $c\in\mathbb{R}$ if every sequence (X_n) satisfying

$$x_n\in X_n, f(x_n)\to c, f_n'(x_n)\to 0$$

has a subsequence which converges in X to a critical point of *f*. The above property will be referred as the $(PS)_c^*$ condition with respect to (X_n).

Theorem 2.1 (Generalized Saddle Point Theorem). Assume that there exist constants $r>0$ and $\alpha<\beta\le\gamma$ such that

1) f satisfies the $(PS)_c^*$ condition with respect to (X_n) for every $c\in[\beta,\gamma]$,

2) $f(w,v)\le\alpha$ for every $(w,v)\in W\times V$ such that $\|w\|=r$,

3) $f(z,v)\ge\beta$ for every $(z,v)\in Z\times V$,

4) $f(w,v)\le\gamma$ for every $(w,v)\in W\times V$ such that $\|w\|\le r$.

Then $f^{-1}([\beta,\gamma])$ contains at least cuplength $(V)+1$ critical points of *f*.

Consider the Hilbert space $E=H^{\frac{1}{2}}(S^1,\mathbb{R}^{2N})$ where $\mathbb{R}/(T\mathbb{Z})$ and the continuous quadratic form *Q* efined in *E* y

$$Q(x)=\frac{1}{2}\int_0^T Jx'(t).x(t)\mathrm{d}t$$

where x,y inside the sign integral is the inner product of $x,y\in\mathbb{R}^{2N}$. Let us denote by E^0, E^-, E^+ respectively the subspaces of E on which *Q* is null, negative definite and positive definite. It is well known that these sub-spaces are mutually orthogonal in $L^2(S^1,\mathbb{R}^{2N})$ and in E with respect to the bilinear form:

$$B(x,y)=\frac{1}{2}\int_0^T Jx'(t).y(t)\mathrm{d}t, \quad x,y\in E$$

associated to *Q*. If $x\in E^+$ and $y\in E^-$ then $B(x,y)=0$ and $Q(x+y)=Q(x)+Q(y)$.

For $x=x^++x^-+x^0\in E$, the expression

$$\|x\|=\left[Q(x^+)+Q(x^-)+|x^0|^2\right]^{\frac{1}{2}}$$

is an equivalent norm in E. Moreover, the space E is compactly embedded in $L^2(S^1,\mathbb{R}^{2N})$ for all $s\in[1,\infty]$. In particular for all $s\in[1,\infty]$, there exists a constant $\lambda_s>0$ such that for all $x\in E$,

$$\|x\|_{L^s} \le \lambda_s \|x\|. \tag{2.1}$$

3. Proof of the Theorems

Firstly, let us remark that if $x(t)$ is a periodic solution of ($\mathcal{H}$), then by replacing t by $-t$ in ($\mathcal{H}$) we obtain

$$Jx'(-t)+H'(-t,x(-t))+e(-t)=0.$$

So it is clear that the function $y(t)=x(-t)$ is a periodc solution of the system

$$Jy'(t)-H'(-t,y(-t))-e(-t)=0.$$

Moreover, $-H(-t,x)$ satisfies $(H_2)(i)$ (resp. $(H_4)(i)$) whenever $H(t,x)$ satisfies $(H_2)(ii)$ (resp. $(H_4)(ii)$). Hence, in the following, we will assume that H satisfies $(H_2)(i)$ in Theorem 1.1 and $(H_4)(i)$ in Theorem 1.2.

Associate to the system ($\mathcal{H}$) the functional φ defined on the space E, by:

$$\varphi(u)=\frac{1}{2}\int_0^T Ju'(t)\cdot u(t)\,\mathrm{d}t+\int_0^T \big(H(t,u)+e(t)\cdot u(t)\big)\mathrm{d}t .$$

It is well known that the functional φ is continuously differentiable in E and critical points of φ on E corres-pond to the $T-$ periodic solutions of the system ($\mathcal{H}$), moreover one has

$$\varphi'(u)v=\int_0^T\big[Ju'(t)+H'(t,u(t))+e(t)\big]\cdot v(t)\,\mathrm{d}t$$

for all $u,v\in E$. Consider the subspaces $W=E^-$, $Z=E^+\oplus B$ of E and the quotient space

$$V=A\big/\{x\sim x+e_i, i=i_1,\cdots,i_p\}$$

which is nothing but the torus T_p. We regard the functional φ as defined on the space $X=(W\oplus Z)\times V$ as follows

$$\varphi(u+v)=\frac{1}{2}\int_0^T Ju'(t)\cdot u(t)\,\mathrm{d}t+\int_0^T H(t,u(t)+v(t))\mathrm{d}t$$
$$+\int_0^T e(t)\cdot u(t)\,\mathrm{d}t$$

To find critical points of φ we will apply Theorem 2.1 to this functional with respect to the sequence of subspaces $X_n=E_n\times V$, where for $n\ge 0$

$$E_n=\left\{x\in E: x(t)=\sum_{|m|\le n}\exp\left(\frac{2\pi}{T}mtJ\right)\hat{u}_m \ a.e.\right\}.$$

Proof of the Theorem 1.1. Assume (H_0), (H_1) and $(H_2)(i)$ hold. Firstly, let us check the Palais-Smale condition.

Lemma 2.1. For all level $c\in\mathbb{R}$, the functional φ satisfies the $(PS)_c^*$ condition with respect to the sequence $(X_n)_{n\in N}$.

Proof. Let $c\in\mathbb{R}$ and let $(u_n,v_n)_{n\in\mathbb{N}}$ be a sequence of X such that for all $n\in\mathbb{N}$, $(u_n,v_n)\in X_n$ and

$$\varphi(u_n+v_n)\to c \ \text{ and } \ \varphi_n'(u_n+v_n)\to 0 \ \text{ as } \ n\to\infty, \tag{3.1}$$

where φ_n is the functional φ restricted to X_n. Set $u_n=u_n^++u_n^-+u_n^0$ with $u_n^+\in E^+$, $u_n^-\in E^+$, $u_n^0\in B$. We have the relation

$$\begin{aligned}&\varphi_n'(u_n+v_n)u_n^+\\&=\|u_n^+\|^2+\int_0^T\big[H'(t,u_n+v_n)+e(t)\big]\cdot u_n^+\mathrm{d}t\end{aligned} \tag{3.2}$$

Since $\varphi_n'(u_n+v_n)\to 0$ as $n\to\infty$, there exists a constant $c_1>0$ such that

$$\forall n\in N, \left|\varphi_n'(u_n+v_n)u_n^+\right|\le c_1\|u_n^+\|. \tag{3.3}$$

By assumption (H_1) and Hölder's inequality, with $p=\frac{1}{\alpha}$, $q=\frac{1}{1-\alpha}$, we have

$$\begin{aligned}&\left|\int_0^T H'(t,u_n+v_n)\cdot u_n^+\mathrm{d}t\right|\\&\le\int_0^T\Big[a(t)\left|P_B(u_n(t))\right|^\alpha+b(t)\Big]\left|u_n^+\right|\mathrm{d}t\\&\le\|u_n^+\|_{L^2}\left[\|a\|_{L^{\frac{2}{1-\alpha}}}\|P_B(u_n)\|_{L^2}^\alpha+\|b\|_{L^2}\right]\end{aligned} \tag{3.4}$$

Then by (3.2), (3.4) and (2.1), there exist two positive constants c_2, c_3 such that

$$\|u_n^+\|\le c_2\|P_B(u_n)\|^\alpha+c_3. \tag{3.5}$$

Observing that a similar result holds for (u_n^-):

$$\|u_n^-\|\le c_2\|P_B(u_n)\|^\alpha+c_3. \tag{3.6}$$

We conclude from (3.5) and (3.6) that the sequence (u_n) is bounded if and only if the sequence $(P_B(u_n))$ is bounded. Assume that $(P_B(u_n))$ is not bounded, we can assume, by going to a subsequence if necessary, that $\|P_B(u_n)\|\to\infty$ as $n\to\infty$. Since $0\le\alpha<1$, we conclude by (3.5) and (3.6) that

$$\frac{u_n^+}{\|P_B(u_n)\|}\to 0, \frac{u_n^-}{\|P_B(u_n)\|}\to 0 \text{ as } n\to\infty. \tag{3.7}$$

Therefore, we have

$$y_n=\frac{u_n}{\|P_B(u_n)\|}\to y\in B, y\in B, |y|=1 \text{ as } n\to\infty. \tag{3.8}$$

It follows that

$$\frac{|u_n^0|}{\|P_B(u_n)\|}\to 1 \text{ as } n\to\infty. \tag{3.9}$$

Consequently, by (3.5), (3.6) and (3.9), we can find a positive constant c_4 such that

$$\left\|u_n^i\right\| \le c_4 \left|u_n^0\right|^{\alpha}, i=+,-. \tag{3.10}$$

Now, we apply the fact that $\left(\varphi\left(u_n+v_n\right)\right)$ is bounded to get

$$\frac{\left\|u_n^+\right\|^2-\left\|u_n^-\right\|^2}{\left|u_n^0\right|^{2\alpha}} \int_0^T \frac{H\left(t,u_n+v_n\right)}{\left|u_n^0\right|^{2\alpha}}\mathrm{d}t + \int_0^T \frac{e(t).\left(u_n^+-u_n^-\right)}{\left|u_n^0\right|^{2\alpha}}\mathrm{d}t \le \frac{c_5}{\left|u_n^0\right|^{2\alpha}} \tag{3.11}$$

where c_3 is a positive constant. Using (3.10) and (3.11), we can find a constant c_6 satisfying

$$\begin{aligned}\int_0^T \frac{H\left(t,u_n^0\right)}{\left|u_n^0\right|^{2\alpha}}\mathrm{d}t &= \int_0^T \frac{H\left(t,u_n+v_n\right)}{\left|u_n^0\right|^{2\alpha}}\mathrm{d}t + \int_0^T \frac{H\left(t,u_n^0\right)-H\left(t,u_n+v_n\right)}{\left|u_n^0\right|^{2\alpha}}\mathrm{d}t \qquad (3.12). \\ &\le c_6 + \int_0^T \frac{H\left(t,u_n^0\right)-H\left(t,u_n+v_n\right)}{\left|u_n^0\right|^{2\alpha}}\mathrm{d}t\end{aligned}$$

On the other hand, by the Mean Value Theorem and assumption $\left(H_1\right)$, we have

$$\begin{aligned}&\int_0^T \left[H\left(t,u_n^0\right)-H\left(u_n+v_n\right)\right]\mathrm{d}t \\ &= -\int_0^T H'\left(t,u_n^0+\theta\left(u_n^+ + u_n^- + v_n\right)\right).\left(u_n^+ + u_n^- + v_n\right) \\ &\le \int_0^T \left[a(t)\left|P_B\left(u_n^0+\theta\left(u_n^+ + u_n^-\right)\right)\right|^{\alpha} + b(t)\right] \\ &\times \left|u_n^+ + u_n^- + v_n\right|\mathrm{d}t \\ &\le \left[\left\|a\right\|_{L^{\frac{2}{1-\alpha}}} \left\|P_B\left(u_n^0+\theta\left(u_n^+ + u_n^-\right)\right)\right\|_{L^2}^{\alpha} + \left\|b\right\|_{L^2}\right]\left\|u_n^+ + u_n^- + v_n\right\|_{L^2}.\end{aligned} \tag{3.13}$$

By considering (3.13) and Sobolev's embedding $E \hookrightarrow L^2\left(0,T;\mathbb{R}^{2N}\right)$ we can find a constant $c_7>0$ such that

$$\begin{aligned}&\int_0^T \left[H\left(t,u_n^0\right)-H\left(u_n+v_n\right)\right]\mathrm{d}t \\ &\le c_7\left[\left|u_n^0\right|^{\alpha} + \left\|u_n^+\right\|^{\alpha} + \left\|u_n^-\right\|^{\alpha}\right]\left[\left\|u_n^+\right\| + \left\|u_n^-\right\| + 1\right]\end{aligned} \tag{3.14}$$

After combining (3.10), (3.12) and (3.14), we get

$$\int_0^T \frac{H\left(t,u_n^0\right)}{\left|u_n^0\right|^{2\alpha}}\mathrm{d}t \le c_8 \tag{3.15}$$

for some positive constant c_8. However, the condition (3.15) contradicts $\left(H_2\right)(i)$ because $\left|u_n^0\right| \to \infty$ as $n\to\infty$. Consequently, $\left(u_n\right)$ is bounded in X. Going if necessary to a subsequence, we can assume that $u_n \rightharpoonup u$, $u_n^0 \to u^0$ and $v_n \to v$. Notice that

$$\begin{aligned}Q\left(u_n^+ - u^+\right) &= \left(\varphi_n'\left(u_n+v_n\right)-\varphi_n'\left(u+v\right)\right)\left(u_n^+ - u^+\right) \\ &-\int_0^T \left[H'\left(t,u_n+v_n\right)-H'\left(t,u+v\right)+e(t)\right] \\ &\cdot\left(u_n^+ - u^+\right)\mathrm{d}t\end{aligned} \tag{3.16}$$

which implies that $u_n^+ \to u^+$ in E. Similarly, $u_n^- \to u^-$ in E. It follows that $\left(u_n,v_n\right)\to\left(u,v\right)$ in X and $\varphi'\left(u+v\right)=0$. So φ satisfies the $(PS)_c^*$ condition for all $c\in\mathbb{R}$. The Lemma 3.1 is proved.

Now, let us prove that the functional φ satisfies the conditions a), b) and c) of Theorem 2.1.

a) Let $(u,v)\in W\times V$. By using the Mean Value Theorem, assumptions $\left(H_0\right)$, $\left(H_1\right)$ and (2.1), we have

$$\begin{aligned}\varphi(u+v) &= -\|u\|^2 + \int_0^T H(t,u+v)\mathrm{d}t + \int_0^T e(t)\cdot u\mathrm{d}t \\ &= -\|u\|^2 + \int_0^T H(t,v)\mathrm{d}t \\ &+ \int_0^T H'(t,v+\theta u).u\mathrm{d}t + \int_0^T e(t)\cdot u\mathrm{d}t \\ &\le -\|u\|^2 + \int_0^T H(t,v)\mathrm{d}t + \int_0^T \left[a(t)\left|P_B(u)\right|^{\alpha} + b(t)\right]|u|\mathrm{d}t + \int_0^T e(t)\cdot u\mathrm{d}t \\ &\le -\|u\|^2 + \int_0^T H(t,v)\mathrm{d}t + \int_0^T \left[a(t)\left\|P_B(u)\right\|^{\alpha} + b(t)\right]|u|\mathrm{d}t + \int_0^T e(t)\cdot u\mathrm{d}t \\ &\le -\|u\|^2 + \int_0^T H(t,v)\mathrm{d}t + \|u\|_{L^2} \\ &\times\left[\left(\int_0^T a^2(t)|u|^{2\alpha}\mathrm{d}t\right)^{\frac{1}{2}} + \|b\|_{L^2}\right] + \int_0^T e(t)\cdot u\mathrm{d}t \\ &\le -\|u\|^2 + \|u\|\left[c_9\|u\|^{\alpha} + c_{10}\right] + c_{11}\end{aligned} \tag{3.17}$$

where c_9, c_{10}, c_{11} are three positive constants. Since $0\le\alpha<1$, then

$$\varphi(u+v)\to-\infty \text{ as } u\in W, \|u\|\to\infty \text{ uniformly in } v\in V. \tag{3.18}$$

b) Let $(u,v)\in Z\times V$, with $u=u^+ + u^0$. By using the Mean Value Theorem, we get

$$\begin{aligned}\varphi(u+v) &= \left\|u^+\right\|^2 + \int_0^T H\left(t,u^+ + u^0 + v\right)\mathrm{d}t \\ &+ \int_0^T e(t)\cdot u^+\mathrm{d}t \\ &= \|u^+\|^2 + \int_0^T H\left(t,u^0\right)\mathrm{d}t \\ &+ \int_0^T H'\left(t,u^0+\theta\left(u^+ + v\right)\right)\cdot\left(u^+ + v\right) + \int_0^T e(t)\cdot u^+\mathrm{d}t\end{aligned} \qquad (3.19).$$

By assumption (H_1) and (2.1), we can find a constant $c_{12} > 0$ such that

$$\begin{aligned}
&\left|\int_0^T H'\left(t,u^0+\theta\left(u^++v\right)\right)\cdot(u^++v)\mathrm{d}t+\int_0^T e(t)\cdot u^+\mathrm{d}t\right| \\
&\le \int_0^T\left[a(t)\left|P_B\left(t,u^0+\theta\left(u^++v\right)\right)\right|^\alpha+b(t)\right] \\
&\times\left|u^++v\right|\mathrm{d}t+\|e\|_{L^2}\left\|u^+\right\|_{L^2} \\
&\le \left\|u^++v\right\|_{L^+} \\
&\times\left(\left[\int_0^T a^2(t)\left(\left|u^0\right|+\left|u^+\right|\right)^{2\alpha}\right]^{\frac{1}{2}}+\|b\|_{L^2}+\|e\|_{L^2}\right) \\
&\le c_{12}\left(\left\|u^+\right\|+1\right)\left[\left|u^0\right|^\alpha+\left\|u^+\right\|^\alpha+1\right]
\end{aligned} \tag{3.20}$$

Therefore, by using (3.19) and (3.20) we obtain

$$\begin{aligned}
\varphi(u+v)\ge&\left\|u^+\right\|^2+\int_0^T H\left(t,u^0\right)\mathrm{d}t \\
&-c_{12}\left(\left\|u^+\right\|+1\right)\left[\left|u^0\right|^\alpha+\left\|u^+\right\|^\alpha+1\right].
\end{aligned} \tag{3.21}$$

Now let $d > \frac{c_{12}^2}{2}$. By assumption $(H_2)(i)$, there exists a constant $c_{13} > 0$ such that

$$\int_0^T H\left(t,u^0\right)\mathrm{d}t \ge \mathrm{d}\left|u^0\right|^{2\alpha}-c_{13} \tag{3.22}.$$

So by (3.21) and (3.22), we have

$$\begin{aligned}
\varphi(u+v)\ge&\left\|u^+\right\|^2+d\left|u^0\right|^{2\alpha}-c_{13} \\
&-c_{12}\left(\left\|u^+\right\|+1\right)\left[\left|u^0\right|^\alpha+\left\|u^+\right\|^\alpha+1\right] \\
\ge&\frac{1}{2}\left\|u^+\right\|^2-c_{12}\left[\left\|u^+\right\|^{\alpha+1}-\left\|u^+\right\|-\left\|u^+\right\|^\alpha\right] \\
&+\frac{1}{2}\left[\left\|u^+\right\|-c_{12}\left|u^0\right|^\alpha\right]^2+\left[d-\frac{c_{12}^2}{2}\right]\left|u^0\right|^{2\alpha} \\
&-c_{12}\left|u^0\right|^\alpha-c_{12}-c_{13}.
\end{aligned} \tag{3.23}$$

Since $d > \frac{c_{12}^2}{2}$ and $0 \le \alpha < 1$, then

$$\varphi(u+v)\to\infty \text{ as } u\in Z, \|u\|\to\infty, \text{ uniformly in } v\in V. \tag{3.24}$$

Hence by Lemma 3.1 and properties (3.18), (3.24), we deduce that the functional φ satisfies all the assumptions of Theorem 2.1. Therefore the Hamiltonian system $(\mathcal{H})$ possesses at least $(p+1)T-$ periodic solutions geometrically distinct. The proof of Theorem 1.1 is complete.

Proof of Theorem 1.2. Assume (H_0), (H_1) and $(H_4)(i)$ hold. The following lemma will be needed for the study of the geometry of the functional φ.

Lemma 3.2. There exist a non-increasing positive function $\theta\in C(]0,\infty[,\mathbb{R}^+)$ and a positive constant c_0 satisfying the following conditions:

i) $\theta(s)\to 0, \theta(s)s\to+\infty$ as $s\to+\infty$,

ii) $\|H'(t,u)\|_{L^2}\le c_0\left[\theta\left(\|P_B(u)\|\right)\|P_B(u)\|+1\right], \forall u\in E,$

iii) $\frac{1}{\left[\theta\left(\left|u^0\right|\right)\left|u^0\right|\right]^2}\int_0^T H\left(t,u^0\right)\mathrm{d}t\to+\infty$ as $\left|u^0\right|\to+\infty.$

Proof : For $u\in E$, let

$$A=\left\{t\in[0,T]:\left|P_B(u)(t)\right|\ge\|P_B(u)\|^{\frac{1}{2}}\right\}.$$

By (H_3), we have

$$\begin{aligned}
&\|H'(t,u)\|_{L^2}\le\left[\int_0^T\left[a\omega\left(\left|P_B(u)(t)\right|\right)\left|P_B(u)(t)\right|+g(t)\right]^2\mathrm{d}t\right]^{\frac{1}{2}} \\
&\le\left[\int_0^T\left[a\omega\left(\left|P_B(u)(t)\right|\right)\left|P_B(u)(t)\right|\right]^2\mathrm{d}t\right]^{\frac{1}{2}}+\|g\|_{L^2} \\
&\le a\left[\int_A\omega^2\left(P_B(u)(t)\right)\left|P_B(u)(t)\right|^2\mathrm{d}t\right. \\
&\left.+\int_{[0,T]-A}\omega^2\left(P_B(u)(t)\right)\left|P_B(u)(t)\right|^2\mathrm{d}t\right]^{\frac{1}{2}}+\|g\|_{L^2} \\
&\le a\left[\int_A\omega^2\left(\|P_B(u)\|^{\frac{1}{2}}\right)\left|P_B(u)(t)\right|^2\mathrm{d}t\right. \\
&\left.+T\sup_{s\ge 0}\omega^2(s)\|P_B(u)\|\right]^{\frac{1}{2}}+\|g\|_{L^2}.
\end{aligned}$$

So, by (2.1) there exists a positive constant c_0 such that

$$\begin{aligned}
&\|H'(t,u)\|_{L^2} \\
&\le c_0\left(\left[\omega^2\left(\|P_B(u)\|^{\frac{1}{2}}\right)\|P_B(u)\|^2+\|P_B(u)\|\right]^{\frac{1}{2}}+1\right).
\end{aligned}$$

Take

$$\theta(s)=\left[\omega^2\left(s^{\frac{1}{2}}\right)+\frac{1}{s}\right]^{\frac{1}{2}}, s>0,$$

then θ satisfies (ii) and it is clear to see that θ satisfies (i).

Next, let us define

$$\rho=\liminf_{s\to\infty}\frac{\omega^2(s)}{\omega^2\left(s^{\frac{1}{2}}\right)}.$$

By $(H_4)(i)$, for any $\gamma > 0$, there exists a positive constant c_{14} such that

$$\int_0^T H(t,x)\,dt \ge \gamma\left[\omega(|x|)|x|\right]^2 - c_{14}. \tag{3.25}$$

which implies that for $u_0 \in B$, $u^0 \ne 0$,

$$\frac{\int_0^T H(t,u^0)\,dt}{\left[\theta(|u^0|)|u^0|\right]^2} \ge \frac{\gamma\left[\omega(|u^0|)|u^0|\right]^2 - c_{14}}{\omega^2\left(|u^0|^{\frac{1}{2}}\right)|u^0|^2 + |u^0|}. \tag{3.26}$$

By the definition of ρ, there exists $R > 0$ such that for all $s \ge R$

$$\frac{\omega^2(s)s^2}{\omega^2\left(s^{\frac{1}{2}}\right)s^2 + s} \ge \frac{\rho}{2}. \tag{3.27}$$

Therefore

$$\frac{\int_0^T H(t,u^0)\,dt}{\left[\theta(|u^0|)|u^0|\right]^2} \ge \frac{\gamma\rho}{2T} - \frac{c_{14}}{\omega^2\left(|u^0|^{\frac{1}{2}}\right)|u^0|^2 + |u^0|} \tag{3.28}$$

as $|u^0| \ge R$ and then

$$\lim_{|u^0|\to\infty} \frac{\int_0^T H(t,u^0)\,dt}{\left[\theta(|u^0|)|u^0|\right]^2} \ge \frac{\gamma\rho}{2T}. \tag{3.29}$$

Since γ is arbitrary choosen, condition (iii) holds.

Now, let us prove the Palais-Smale condition.

Lemma 3.3. For all level $c \in \mathbb{R}$, the functional φ satisfies the $(PS)_c^*$ condition with respect to the sequence $(X_n)_{n\in\mathbb{N}}$.

Proof. Let $(u_n, v_n)_{n\in\mathbb{N}}$ be a sequence in *X* such that for all $n \in \mathbb{N}, (u_n, v_n) \in X_n$ and

$$\varphi(u_n + v_n) \to c \text{ and } \varphi_n'(u_n + v_n) \to 0 \text{ as } n \to \infty. \tag{3.30}$$

Set $u_n = u_n^+ + u_n^- + u_n^0$ and $\tilde{u}_n = u_n^+ + u_n^-$. By Hölder's inequality, (2.1) and Lemma 2.2(ii), we get a positive constant c_{15} such that

$$\begin{aligned}&\left|\int_0^T \left(H'(t,u_n+v_n) + e(t)\right)\cdot\left(u_n^+ - u_n^-\right)dt\right| \\ &\le \left\|u_n^+ - u_n^-\right\|_{L^2}\left(\left[\int_0^T \left|H'(t,u_n+v_n)\right|^2\right]^{\frac{1}{2}} + \|e\|_{L^2}\right) \\ &\le c_{15}\|\tilde{u}_n\|\left[\theta\left(\|P_B(u_n)\|\right)\|P_B(u_n)\| + 1\right].\end{aligned} \tag{3.31}$$

Thus, for *n* large enough

$$\begin{aligned}\|\tilde{u}_n\| &\ge \varphi'(u_n+v_n)\left(u_n^+ - u_n^-\right) \\ &\ge 2\|\tilde{u}_n\|^2 - c_{15}\|\tilde{u}_n\|\left[\theta\left(\|P_B(u_n)\|\right)\|P_B(u_n)\| + 1\right].\end{aligned} \tag{3.32}$$

So there exists a positive constant c_{16} such that

$$\|\tilde{u}_n\| \le c_{16}\left[\theta\left(\|P_B(u_n)\|\right)\|P_B(u_n)\| + 1\right]. \tag{3.33}$$

By (3.33) and the properties (i) of θ, we deduce that $(\|u_n\|)$ is bounded if and only $(\|P_B(u_n)\|)$ is bounded.

Now, since θ is nonincreasing and $\|u\| \ge \|P_B(u_n)\| \ge \max\left(|u^0|, \|P_B(\tilde{u})\|\right)$, we get

$$\theta(\|u\|) \le \min\left(\theta(|u^0|), \theta(\|P_B(\tilde{u})\|)\right) \tag{3.34}$$

Combining (3.32) and (3.34), yields for n large enough

$$\begin{aligned}\|\tilde{u}_n\| &\ge 2\|\tilde{u}_n\|^2 \\ &-c_{15}\|\tilde{u}_n\|\left[\theta\left(\|P_B(\tilde{u}_n)\|\right)\|P_B(\tilde{u}_n)\| + \theta\left(|u_n^0|\right)|u_n^0| + 1\right]\end{aligned}$$

which implies

$$c_{15}\theta\left(|u_n^0|\right)|u_n^0| \ge \|\tilde{u}_n\|\left[2 - c_{15}\theta\left(\|P_B(\tilde{u}_n)\|\right)\right] - c_{15} - 1. \tag{3.35}$$

Assume that $(\|P_B(\tilde{u}_n)\|)$ is unbounded, then by going to a subsequence, if necessary, we can assume that $\|P_B(\tilde{u}_n)\| \to \infty$ as $n \to \infty$. Since $\theta(s) \to 0$ as $s \to \infty$, we deduce from (3.35) that there exists a positive constant c_{17} such that

$$\|\tilde{u}_n\| \le c_{17}\theta\left(|u_n^0|\right)|u_n^0| \tag{3.36}$$

for n large enough. Since the map $s \to \theta(s)s$ is continuous in $[0,\infty]$ and goes to $+\infty$ as $s \to \infty$, then $|u_n^0| \to \infty$ as $n \to \infty$.

Now, by the Mean Value Theorem, Hölder's inequality and Lemma 3.2(ii), we get

$$\begin{aligned}&\left|\int_0^T \left(H(t,u_n+v_n) - H(t,u_n^0)\right)dt\right| \\ &\left|\int_0^T\int_0^1 H'\left(t,u_n^0 + s(\tilde{u}_n+v_n)\right).(\tilde{u} = +v_n)\,ds\,dt\right| \\ &\le \|\tilde{u}_n+v_n\|_{L^2}\int_0^1\left(\int_0^T \left|H'\left(t,u_n^0 + s(\tilde{u}_n+v_n)\right)\right|^2 dt\right)^{\frac{1}{2}} ds \\ &\le c_0\|\tilde{u}_n+v_n\|_{L^2}\int_0^1\left[\theta\left(\|u_n^0 + sP_B(\tilde{u}_n)\|\right)\|u_n^0 + sP_B(\tilde{u}_n)\| + 1\right]ds.\end{aligned} \tag{3.37}$$

Since $\|u_n^0 + sP_B(\tilde{u}_n)\| \ge |u_n^0|$ for all $s \in [0,1]$, we dedu-ce from (2.1), (3.36) and (3.37) that there exists a positive constant c_{18} such that

$$\begin{aligned}&\left[\int_0^T \left(H(t,u_n+v_n) - H(t,u_n^0)\right)dt\right) \\ &\le c_0\,|\,\|\tilde{u}_n+v_n\|\,|_{L^2}\,[\theta\left(|u_n^0|\right)|u_n^0| + \theta\left(|u_n^0|\right)\|P_B(\tilde{u}_n)\| + 1] \\ &\le c_{18}\left[\left[\theta\left(|u_n^0|\right)|u_n^0|\right]^2 + \theta\left(|u_n^0|\right)\left[\theta\left(|u_n^0|\right)|u_n^0|\right]^2\right. \\ &\left.+\theta\left(|u_n^0|\right)|u_n^0| + 1\right],\end{aligned} \tag{3.38}$$

which with (2.1) and (3.36) imply that there exists a positive constant c_{19} such that

$$\begin{aligned}\varphi(u_n+v_n)&=\|u_n^+\|^2-\|u_n^-\|^2+\int_0^T H(t,u_n^0)\mathrm{d}t\\&+\int_0^T\left(H(t,u_n+v_n)-H(t,u_n^0)\right)\mathrm{d}t+\int_0^T e(t).\tilde{u}_n\mathrm{d}t\\&\geq+\int_0^T H(t,u_n^0)\mathrm{d}t-c_{19}\Big[\left[\theta(|u_n^0|)|u_n^0|\right]^2\\&+\theta(|u_n^0|)\left[\theta(|u_n^0|)|u_n^0|\right]^2+\theta(|u_n^0|)|u_n^0|+1\Big]\\&\geq c_{19}\left[\theta(|u_n^0|)|u_n^0|\right]^2\Bigg(-1-\theta(|u_n^0|)-\frac{1}{\theta(|u_n^0|)|u_n^0|}\\&-\frac{1}{\left[\theta(|u_n^0|)|u_n^0|\right]^2}+\frac{\int_0^T H(t,u_n^0)\mathrm{d}t}{c_{19}\left[\theta(|u_n^0|)|u_n^0|\right]^2}\Bigg)\end{aligned}\tag{3.39}$$

which, with Lemma 3.2 (iii), imply that $\varphi(u_n+v_n)\to\infty$ as $n\to\infty$. This contradicts the boundedness of $(\varphi(u_n+v_n))$. So $(\|P_B(\tilde{u}_n)\|)$ is bounded.

Assume that $(|u_n^0|)$ is unbounded, then up to a subsequence, if necessary, we can assume that $|u_n^0|\to\infty$ as $n\to\infty$. As in (3.38), and using (2.1), (3.34) and the fact that $\theta(s)\to 0$ as $s\to\infty$, we can find a constant $c_{21}>0$ such that

$$\begin{aligned}&\left|\int_0^T\left(H(t,u_n+v_n)-H(t,u_n^0)\right)\mathrm{d}t\right|\\&\leq c_0\|\tilde{u}_n+v_n\|_{L^2}\left[\theta(|u_n^0|)|u_n^0|+\theta(|u_n^0|)\|P_B(\tilde{u}_n)\|+1\right]\\&\leq c_{21}\|\tilde{u}_n+v_n\|\left[\theta(|u_n^0|)|u_n^0|+1\right].\end{aligned}\tag{3.40}$$

Now, since $\theta(s)\to 0$ as $s\to\infty$, then combining (3.33) and (3.34) yields

$$\begin{aligned}\|\tilde{u}_n+v_n\|&\leq\left[\theta(|u_n^0|)|u_n^0|+\theta(|u_n^0|)\|P_B(\tilde{u}_n)\|+1\right]+|v_n|\\&\leq c_{22}\left[\theta(|u_n^0|)|u_n^0|+1\right]\end{aligned}\tag{3.41}$$

for a positive constant c_{22}. Therefore there exists a positive constant c_{23} such that

$$\begin{aligned}&\left|\int_0^T\left(H(t,u_n+v_n)-H(t,u_n^0)+e(t).\tilde{u}_n\right)\mathrm{d}t\right|\\&\leq c_{23}\left[\theta(|u_n^0|)|u_n^0|+1\right]^2.\end{aligned}\tag{3.42}$$

We deduce from (3.41) and (3.42) that there exists a constant $c_{24}>0$ such that

$$\begin{aligned}\varphi(u_n+v_n)&\geq -c_{24}\left[\theta(|u_n^0|)|u_n^0|+1\right]^2+\int_0^T H(t,u_n^0)\mathrm{d}t\\&\geq\left[\theta(|u_n^0|)|u_n^0|+1\right]^2\left[-c_{24}+\frac{\int_0^T H(t,u_n^0)\mathrm{d}t}{\left[\theta(|u_n^0|)|u_n^0|+1\right]^2}\right]\end{aligned}$$

which implies by Lemma 3.2 (iii) that $\varphi(u_n+v_n)\to+\infty$ as $n\to\infty$. This contradicts the boundedness of $(\varphi(u_n+v_n))$. Then $(|u_n^0|)$ is also bounded and therefore $(\|u_n\|)$ is bounded. By a standard argument, we con-clude that (u_n) possesses a convergent subsequence. The proof of Lemma 3.3 is complete.

Now, let $(u,v)=(u^0+u^+,v)\in Z\times V$, then as in (3.38) there exists a positive constant c_{25} such that

$$\begin{aligned}&\left|\int_0^T(H(t,u_n+v_n)-H(t,u_n^0)+e(t).\tilde{u}_n)\mathrm{d}t\right|\\&\leq c_{25}\left[\|u^+\|+1\right]\left[\theta(|u^0|)|u^0|+\theta(|u^0|)\|P_B(u^+)\|+1\right].\end{aligned}\tag{3.43}$$

So, we have for a positive constant c_{26}

$$\begin{aligned}\varphi(u+v)\geq\|u^+\|^2-c_{26}\|u^+\|\Big[\theta(|u^0|)|u^0|\\+\theta(|u^0|)\|u^+\|+1\Big]-c_{26}+\int_0^T H(t,u_n^0)\mathrm{d}t.\end{aligned}\tag{3.44}$$

Let $0<\epsilon<1$, we have

$$c_{26}\theta(|u^0|)|u^0|\|u^+\|\leq\frac{c_{26}^2}{\epsilon^2}\left[\theta(|u_n^0|)|u_n^0|\right]^2+\epsilon^2\|u^+\|^2.\tag{3.45}$$

By combining (3.44) and (3.45), we get

$$\begin{aligned}\varphi(u+v)&\geq\left[1-\epsilon^2-c_{26}\theta(|u^0|)\right]\|u^+\|^2-c_{26}\|u^+\|\\&+\left[\theta(|u_n^0|)|u_n^0|\right]^2\left[-\frac{c_{26}^2}{\epsilon^2}+\frac{\int_0^T H(t,u_n^0)\mathrm{d}t}{\left[\theta(|u_n^0|)|u_n^0|+1\right]^2}\right]\end{aligned}$$

which implies that

$$\varphi(u+v)\to+\infty \text{ as } u\in Z,\ \|u\|\to\infty, \text{uniformly in } v\in V.\tag{3.46}$$

On the other hand, let $b\in B$, $|b|>0$. By the Mean Value Theorem, we have for $u\in W=E^-$

$$\begin{aligned}&\left|\int_0^T\left(H(t,u+v)-H(t,b)\right)\mathrm{d}t\right|\\&=\left|\int_0^T\int_0^1 H'\left(t,b+s(u+v-b)\right).(u+v-b)\mathrm{d}s\mathrm{d}t\right|\\&\leq\|u+v-b\|_{L^2}\int_0^1\left[\int_0^T H'(t,b+s(u+v-b))|^2\,\mathrm{d}t\right]^{\frac{1}{2}}\mathrm{d}s\\&\leq\|u+v-b\|_{L^2}\int_0^1\Big[\int_0^T\Big(a\omega\left(\left|b+s(P_B(u)-b)\right|\right)\\&\times\left|b+s(P_B(u)-b)\right|+g(t)\Big)^2\mathrm{d}t\Big]^{\frac{1}{2}}\mathrm{d}s\\&\leq\|u+v-b\|_{L^2}\Bigg(a\int_0^1\Big[\int_0^T\left(\omega\left(\left|b+s(P_B(u)-b)\right|\right)\right]\\&\times\left|b+s(P_B(u)-b)\right|\Big)^2\mathrm{d}t\Big]^{\frac{1}{2}}\mathrm{d}s+\|g\|_{L^2}\Bigg)\end{aligned}\tag{3.47}$$

Take for $s \in [0,1]$,

$$A(s) = \{t \in [0,1] : |b + s(P_B(u) - b)| \geq |b|\}.$$

By a similar calculation as in the proof of Lemma 3.2, we get for some positive constants c_{27} and $c(b)$

$$\begin{aligned} &\left|\int_0^T \left(H(t,u+v) - H(t,b) + e(t)\cdot u\right)\mathrm{d}t\right| \\ &\leq c_{27}\omega(|b|)\|u\|^2 + c(b)(\|u\|+1) \end{aligned} \quad (3.48)$$

which implies that

$$\begin{aligned} \varphi(u+v) &\leq -\|u\|^2 + c_{27}\omega(|b|)\|u\|^2 \\ &\quad + c(b)(\|u\|+1) + \int_0^T H(t,b)\mathrm{d}t. \end{aligned} \quad (3.49)$$

Since $\theta(s) \to 0$as$s \to \infty$, there exists $|b| > 0$ such that $c_{27}\omega(|b|) \leq \frac{1}{2}$, which implies that

$$\varphi(u+v) \leq -\frac{1}{2}\|u\|^2 + c(b)(\|u\|+1) + \int_0^T H(t,b)\mathrm{d}t.$$

So we have

$\varphi(u+v) \to -\infty$ as $u \in W$, $\|u\| \to \infty$, uniformly in $v \in V$. (3.50)

Thus, Lemma 3.3 and properties (3.46), (3.50) imply that the functional φ satisfies all the assumptions of the Generalized Saddle Point Theorem. Therefore the Hamiltonian system ($\mathcal{H}$) possesses at least $(p+1)T-$ periodic solutions geometrically distinct. The proof of Theorem 1.2 is complete.

4. References

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