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A Characterization of the Members of a Subfamily of Power Series Distributions

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Abstract

This paper discusses a characterization of the members of a subfamily of power series distributions when their probability generating functions $f(s)$ satisfy the functional equation $(a+bs)f'(s)=cf(s)$ where *a*, *b* and *c* are constants and f' is the derivative of *f*.

Keywords: Galton-Watson Process, Probability Generating Function, Binomial, Poisson, Negative Binomial Distributions

1. Introduction

Let a population behave like a Galton-Watson process $\{X_n; n\geq 0, X_0=1\}$ with a known offspring distribution $\{p_k\}_{k=0}^{\infty}$. Suppose that the generation size $(X_n=k)$ is observed and *n*, the age in generations, is to be estimated. Such a problem arises in many situations. For example, one might be interested in the length of existence of a certain species in its present form or how long ago a mutation took place, etc. (See Stigler [1]).

When the generation size $(X_n=k)$ is observed and the offspring distribution is known, the likelihood function is given by

$$L(n)=P(X_n=k\mid X_n>0)$$
$$=\frac{f_n^{(k)}(0)}{k!\left[1-f_n(0)\right]},$$

where $f_n(s)$ is the n^{th} functional iteration of the offspring probability generating function (p.g.f.)

$f(s)=\sum_{k=0}^{\infty}p_k s^k$ with $0\leq s\leq 1$ and $f_n^{(k)}$ is the k^{th} derivative of $f_n(s)$ with respect to *s*. The maximum likelihood estimator of *n* can be obtained by the method of calculus if $f_n(s)$ has a closed form expression. When the offspring distribution is binomial, Poisson or negative binomial, $f_n(s)$ does not have a closed form expression. Ades *et al.* [2] have obtained a recurrence formula to compute $P(X_n=k), k=1,2,3,\cdots$ when the offspring p.g.f. satisfies the functional equation

$$(a+bs)f'(s)=cf(s) \tag{1.1}$$

where *a*, *b* and *c* are constants and *f'* is the derivative of *f*. We derive a characterization result using this differential equation.

2. Characterization

We establish the following theorem.

Theorem: Let *X* be a non-negative integer valued random variable with $P(X=k)=p_k$, $k=0,1,\cdots$ and $p_k>0$ at least for $k=0,1$. If the p.g.f. $f(s)=\sum_{k=0}^{\infty}p_k s^k$, $0\leq s\leq 1$, satisfies (1.1), then the distribution of *X* is Poisson, binomial, or negative binomial.

Proof: It is straight forward to verify that

1) when *X* has a Poisson distribution with mean λ, (1.1) holds with $a=1$, $b=0$ and $c=\lambda$.

2) when *X* has a binomial (*N*,*p*)-distribution, (1.1) holds with $a=q$, $b=p$ and $c=Np$ with $q=1-p$.

3) when *X* has a negative binomial (*α*,*p*)-distribution, (1.1) holds with $a=1$, $b=-q$ and $c=\alpha q$ where $q=1-p$.

Now let us have a close look at the possible values of the constants in (1.1).

1) If $c=0$, then (1.1) reduces to $(a+bs)f'(s)=0$ $\forall s\in[0,1]$. In particular, for $s=0$, this becomes $af'(s)=0$. Since $f'(0)=p_1>0$, $a=0$. But then (1.1) turns out to be $f'(s)=0, \forall s\in[0,1]$ which implies $b=0$ and then (1.1) has no meaning. Thus $c\neq 0$.

2) Let $c\neq 0$. If $a=0$, (1.1) reduces to $bsf'(s)=cf(s)$, $\forall s\in[0,1]$. Then for $s=0$, we get

$cf(0)=0$ and hence $c=0$ which is a contradiction. Therefore $a \neq 0$.

3) Let $c \neq 0, a \neq 0$. Suppose, if possible, $b=0$. Then (1.1) becomes $af'(s)=cf(s)$, $\forall s \in [0,1]$. Identifying this as a linear differential equation and solving, we get

$$\log f(s) = (c/a)s + k_1,$$

where k_1 is an arbitrary constant. Since $f(1)=1$ and $k_1 = -c/a$, the above solution reduces to

$$f(s) = \exp\left[\frac{c}{a}(s-1)\right], \forall s \in [0,1].$$

Note that c/a cannot be negative because if $c/a<0$, then $f(0)>1$ which is impossible. Thus $c/a>0$ and $f(s)$ is the p.g.f. of a Poisson distribution with mean c/a.

4) Let $c \neq 0, a \neq 0$ and $b \neq 0$. Then $\frac{f'(s)}{f(s)} = \frac{c}{a+bs}$. Solving this differential equation, we get $f(s) = k(a+bs)^{\frac{c}{b}}$, where k is a constant. Since $f(1)=1$, $k=(a+b)^{-\frac{c}{b}}$. Hence

$$f(s) = \left(\frac{a+bs}{a+b}\right)^{\frac{c}{b}}. \tag{2.1}$$

Note that if $a+b=0$, then $f(s)$ in (2.1) does not define a p.g.f.

Also, (2.1) can be expressed as

$$f(s) = \left(a^* + b^* s\right)^{\frac{c}{b}}, \tag{2.2}$$

where $a^* = \frac{a}{a+b}$, $b^* = \frac{b}{a+b}$, and $a^* + b^* = 1$.

Since $0 < f(0) = p_0 < 1$, $0 < a^* < 1$ and hence $0 < b^* < 1$. This also implies that $a, b > 0$. Thus, case (4) reduces to $c \neq 0, a>0$ and $b>0$.

4a) Let $c>0$. Then $c/b>0$. Suppose that $c=Nb$ where N is a positive integer. Then $f(s)$ in (2.2) is the p.g.f. of a binomial (N, b^*)-distribution.

4b) Let $c<0$. Then $c/b<0$. Suppose that $c=-Nb$. Then, $f(s)$ in (2.2) is the p.g.f. of a negative binomial (N, b^*)-distribution.

Now it remains to verify whether c/b can be a fraction with $c \neq 0$. Note that (2.2) can be rewritten as

$$f(s) = \left(a^*\right)^{\frac{c}{b}} \left(1 + \frac{b^*}{a^*} s\right)^{\frac{c}{b}}. \tag{2.3}$$

The expansion of the RHS of (2.3) is a power series in s with some coefficients being negative if c/b is a fraction, which is not permitted because the coefficients p_k in $f(s) = \sum_{k=0}^{\infty} p_k s^k$, being probabilities, are non-negative.

Now the proof of the theorem is complete.

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4. References

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