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On Riesz Mean Inequalities for Subelliptic Laplacian

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Abstract

In this paper, we mainly focus on the Riesz means of eigenvalues of the subelliptic Laplacian on the Heisenberg group H^n. We establish a trace formula of associated eigenvalues, then we prove differential inequalities, difference inequalities and monotonicity formulas for the Riesz means of eigenvalues of the subelliptic Laplacian.

Keywords: Heisenberg Group, Riesz Mean, Subelliptic Laplacian

1. Introduction

Until now, the eigenvalue estimations of Laplacian on the bounded Euclidean domain have been extensively studied (see [1-5]). In recent years, some academics have already started to pay attention to the Heisenberg group H^n, such as P. Levy-Bmhl [6], D. Müller [7], P. C. Niu [8], G. Jia [9] and so on.

The Heisenberg group plays an important role in several branches of mathematics such as representation theory, harmonic analysis, several complex variables, partial differential equations and quantum mechanics. In the past decades research on Heisenberg sub-Laplacian has achieved considerable progress. But the problem of the invariant differential operator eigenvalue for the Heisenberg group, did not be studied deeply.

In this paper, the Riesz mean inequalities of eigenvalues for the subelliptic Laplacian is treated. And some differential inequalities and difference inequalities are established.

The outline of the paper is as follows. In Section 2, we first recall some definitions and the lemmas that will be used in the following, and then establish the trace formula of eigenvalues. Main results and their proofs will be given in Section 3.

2. Preliminaries and Trace Formula

Let H^n denote Heisenberg group which is a Lie group that has algebra $g = R^{2n+1}$, with a nonabelian group law

$$\begin{aligned}&(x_1, y_1, t_1)\cdot(x_2, y_2, t_2)\\&=\left(x_1+x_2, y_1+y_2, t_1+t_2+2\left(y_2x_1-x_2y_1\right)\right)\end{aligned}\tag{2.1}$$

For every $u_1=(x_1,y_1,t_1), u_2=(x_2,y_2,t_2)\in H^n$

The Lie algebra is generated by the left invariant vector fields

$$\begin{aligned}&X_i=\frac{\partial}{\partial x_i}+2y_i\frac{\partial}{\partial t}\quad Y_i=\frac{\partial}{\partial y_i}-2x_i\frac{\partial}{\partial t},\\&i=1,2,\cdots,n,\end{aligned}\tag{2.2}$$

And $T=\frac{\partial}{\partial t}$. We set

$$\nabla_{H^n}u=\left(X_1(u),\cdots,X_n(u),Y_1(u),\cdots,Y_n(u)\right).$$

Remark 2.1

It is easy to see that X_i, Y_i, T are skew symmetric operators, and

$$\left[X_i,Y_j\right]=-4T\delta_{ij},\ \left[X_i,T\right]=\left[Y_i,T\right]=0,$$

where $[X,Y]$ denotes the standard commutator $XY-YX$.

Definition 2.1 [10].

The subelliptic Laplacian is defined as

$$\begin{aligned}\Delta_{H^n}&=\sum_{i=1}^{n}\left(X_i^2+Y_i^2\right)=\sum_{i=1}^{n}\left[\frac{\partial^2}{\partial x_i^2}+\frac{\partial^2}{\partial y_i^2}-4x_i\frac{\partial^2}{\partial y_i\partial t}\right.\\&\left.+4y_i\frac{\partial^2}{\partial x_i\partial t}+4\left(x_i^2+y_i^2\right)\frac{\partial^2}{\partial t^2}\right]\end{aligned}\tag{2.3}$$

By the definitions and properties of X_i and Y_i, it is easy to see that Δ_{H^n} is invariant with respect to left-translations.

Let us concern with the eigenvalue problem

$$\begin{cases}\Delta_{H^n}u=-\lambda u, & \text{in }\Omega,\\ u=0, & \text{on }\partial\Omega.\end{cases}\tag{2.4}$$

where Ω is a bounded domain of the Heisenberg group

H^n with smooth boundary. By [8], we see that the Dirichlet problem (2.4) has a discrete spectrum on a Hilbert space with Inner product denoted $\langle\cdot,\cdot\rangle$, and its eigenvalues by $0<\lambda_1\le\lambda_2\le\cdots\le\lambda_k\cdots$ with $\lim_{k\to\infty}\lambda_k=+\infty$ and orthonormalize its eigenfunctions $u_1,u_2,\cdots\in S_0^{1,2}(\Omega)$ so that

$$\langle u_i,u_j\rangle=\int_\Omega u_iu_j\mathrm{d}x\mathrm{d}y\mathrm{d}t=\delta_{ij},\forall i,j\ge1.$$

Here, $S^{1,2}(\Omega)$ denotes the Hilbert space of the functions $u\in L^2(\Omega)$ such that $X_i(u)$, $Y_i(u)\in L^2(\Omega)$, and $S_0^{1,2}$ denotes the closure of $C_0^\infty(\Omega)$.

For the sake of simplicity, let L be a form

$$L=-\Delta_{H^n}=-\sum_{\alpha=1}^n\left(X_\alpha^2+Y_\alpha^2\right).$$

There will be a distinguished subset $J_j=\{\lambda_1,\lambda_2,\cdots,\lambda_j\}$ of the spectrum of L, $J_j^c=\{\lambda_1,\cdots,\lambda_j,\cdots\}\setminus\{\lambda_1,\cdots,\lambda_j\}$ is the complement of J_j, and P_{J_j}, $P_{J_j^c}$ will be the corresponding spectral projections. We shall be interested in traces of $P_{J_j}f(L)$, where $f(\lambda_j)$ is any function defined on the spectrum of L.

Definition 2.2.

If $\{\lambda_k\}_{k=1}^\infty$ is an increasing sequence of real numbers, for $z\ge0$, the Riesz mean of order $\sigma>0$ of $\{\lambda_k\}$ can be defined as [11,12]

$$R_\sigma(z)=\sum_{k=1}^\infty(z-\lambda_k)_+^\sigma \tag{2.5}$$

where $(z-\lambda_k)_+:=\max(0,z-\lambda_k)$ is the ramp function.

Definition 2.3.

Two symmetric operators V_α,W_α are defined as $V_\alpha u=x_\alpha u,\ W_\alpha u=y_\alpha u,\ (\alpha=1,\cdots,n)$.

Remark 2.2.

$$[X_\alpha,V_\beta]=\begin{cases}I, & \alpha=\beta\\ 0, & \alpha\ne\beta\end{cases},$$

$$[Y_\alpha,W_\beta]=\begin{cases}I, & \alpha=\beta\\ 0, & \alpha\ne\beta\end{cases}$$

$$[X_\alpha,W_\beta]=0\quad[Y_\alpha,V_\beta]=0,$$

here I is an identity operator. In fact, we have

$$[X_\alpha,V_\alpha]u=\frac{\partial}{\partial x_\alpha}(x_\alpha u)+2y_\alpha\frac{\partial}{\partial t}(x_\alpha u)-x_\alpha\left(\frac{\partial u}{\partial x_\alpha}+2y_\alpha\frac{\partial u}{\partial t}\right)=u$$

i.e. $[X_\alpha,V_\alpha]=I$, similarly $[Y_\alpha,W_\alpha]=I$, and so on.

Theorem 2.1.

Let λ_j and u_j be eigenvalues and L^2-normalized eigenfunctions of the subelliptic Laplacian. Let

$$T_{\alpha jm}:=\left|\langle X_\alpha u_j,u_m\rangle\right|^2+\left|\langle Y_\alpha u_j,u_m\rangle\right|^2$$

for $j,m=1,2,\cdots$ and $\alpha=1,\cdots,n$. Then for each fixed α,

$$\sum_{\lambda_j\in J_j}f(\lambda_j)=-\sum_{\substack{\lambda_j,\lambda_m\in J_j\\ \lambda_j\ne\lambda_m}}\frac{f(\lambda_j)-f(\lambda_m)}{\lambda_j-\lambda_m}T_{\alpha jm}+2\sum_{\substack{\lambda_j\in J_j\\ \lambda_q\in J_j^c}}\frac{f(\lambda_j)}{\lambda_q-\lambda_j}T_{\alpha jq} \tag{2.6}$$

Specialized to $f(\lambda_j)=(z-\lambda_j)_+^\sigma$, we can obtain

$$R_\sigma(z)=\sum_{j,m:\lambda_j\ne\lambda_m}\frac{(z-\lambda_j)_+^\sigma-(z-\lambda_m)_+^\sigma}{\lambda_m-\lambda_j}T_{\alpha jm}+2\sum_{j,q:\lambda_j\le z<\lambda_q}\frac{(z-\lambda_j)_+^\sigma}{\lambda_q-\lambda_j}T_{\alpha jq} \tag{2.7}$$

To derive out Theorem 2.1, we need the following lemma.

Lemma 2.1 [6].

Let $0<x<y$ and $\sigma\ge0$. Then

$$\frac{y^\sigma-x^\sigma}{y-x}\le C_\sigma\left(y^{\sigma-1}+x^{\sigma-1}\right) \tag{2.8}$$

where

$$C_\sigma:=\begin{cases}\dfrac{\sigma}{2},\text{if } 0\le\sigma<1\text{ or }2\le\sigma<\infty\\ 1,\ \text{if } 1\le\sigma\le2\end{cases}.$$

Proof of Theorem 2.1.

Observe that because $\{u_j\}_{j=1}^\infty$ is a complete orthonormal system,

$$T_j=\sum_{\alpha=1}^n\sum_{m=1}^\infty T_{\alpha jm}=\sum_{\alpha=1}^n\sum_{m=1}^\infty\left|\langle X_\alpha u_j,u_m\rangle\right|^2+\left|\langle Y_\alpha u_j,u_m\rangle\right|^2$$

According to [10] the formal commutator identity $[A,BC]=B[A,C]+[A,B]C$, we have $[L,V_\alpha]=[-X_\alpha\cdot X_\alpha-Y_\alpha\cdot Y_\alpha,V_\alpha]=-2X_\alpha$. Similarly, we get $[L,W_\alpha]=-2Y_\alpha$.

Thus

$$T_j=\frac14\sum_{\alpha=1}^n\sum_{m=1}^\infty\left|\langle[L,V_\alpha]u_j,u_m\rangle\right|^2+\left|\langle[L,W_\alpha]u_j,u_m\rangle\right|^2=\int_\Omega\left|\nabla_{H^n}u_j\right|^2=\lambda_j \tag{2.9}$$

By [2], we obtain $tr\left(P_{J_j}f(L)\right)=tr\left(P_{J_j}f(L)[X_\alpha,V_\alpha]\right)$, and

$$tr\left(P_{J_j} f(L)\right) = -2 \sum_{\substack{\lambda_j,\lambda_m \in J_j \\ \lambda_j \neq \lambda_m}} \frac{f(\lambda_j) - f(\lambda_m)}{\lambda_j - \lambda_m} \left|\left\langle X_\alpha u_j, u_m \right\rangle\right|^2 + 4 \sum_{\substack{\lambda_j \in J_j \\ \lambda_q \in J_j^C}} \frac{f(\lambda_j)}{\lambda_q - \lambda_j} \left|\left\langle X_\alpha u_j, u_q \right\rangle\right|^2 \quad (2.10)$$

And similarly $tr\left(P_{J_j} f(L)\right) = tr\left(P_{J_j} f(L)[Y_\alpha, W_\alpha]\right)$, then

$$tr\left(P_{J_j} f(L)\right) = -2 \sum_{\substack{\lambda_j,\lambda_m \in J_j \\ \lambda_j \neq \lambda_m}} \frac{f(\lambda_j) - f(\lambda_m)}{\lambda_j - \lambda_m} \left|\left\langle Y_\alpha u_j, u_m \right\rangle\right|^2 + 4 \sum_{\substack{\lambda_j \in J_j \\ \lambda_q \in J_j^C}} \frac{f(\lambda_j)}{\lambda_q - \lambda_j} \left|\left\langle Y_\alpha u_j, u_q \right\rangle\right|^2 \quad (2.11)$$

Summing of the (2.10) and (2.11), we obtain

$$2tr\left(P_{J_j} f(L)\right) = -2 \sum_{\substack{\lambda_j,\lambda_m \in J_j \\ \lambda_j \neq \lambda_m}} \frac{f(\lambda_j) - f(\lambda_m)}{\lambda_j - \lambda_m} T_{\alpha jm} + 4 \sum_{\substack{\lambda_j \in J_j \\ \lambda_q \in J_j^C}} \frac{f(\lambda_j)}{\lambda_q - \lambda_j} T_{\alpha jq}$$

Since $\sum_{\lambda_j \in J_j} f(\lambda_j) = tr\left(P_{J_j} f(L)\right)$, we get (2.6), and the proof of the Theorem is completed.

3. Riesz Means Inequalities

In this section, we derive differential inequalities and difference inequalities for the Riesz means $R_\sigma(z) = \sum_{k=1}^{\infty} (z - \lambda_k)_+^\sigma$. Here $\{\lambda_k\}_{k=1}^{\infty}$ are ordered eigenvalues of the subelliptic Laplacian on a bounded domain.

Theorem 3.1.

For $0 < \sigma \le 2$ and $z \ge \lambda_1$, then we have

$$R_{\sigma-1}(z) \ge \left(1 + \frac{n}{2}\right) \frac{1}{z} R_\sigma(z), \quad (3.1)$$

$$R'_\sigma(z) \ge \left(1 + \frac{n}{2}\right) \frac{\sigma}{z} R_\sigma(z), \quad (3.2)$$

and $\dfrac{R_\sigma(z)}{z^{\sigma\left(1+\frac{n}{2}\right)}}$ is a nondecreasing function with respect to z.

For $\sigma \ge 2$ and $z \ge \lambda_1$, then we have

$$R_{\sigma-1}(z) \ge \left(1 + \frac{n}{\sigma}\right) \frac{1}{z} R_\sigma(z) \quad (3.3)$$

$$R'_\sigma(z) \ge (\sigma + n) \frac{1}{z} R_\sigma(z) \quad (3.4)$$

and $\dfrac{R_\sigma(z)}{z^{\sigma+n}}$ is a nondecreasing function with respect to z.

Proof. Let the first term on the right of (2.7) be

$$G(\sigma,\alpha,z) := \sum_{\substack{j,m:\lambda_j,\lambda_m \le z \\ \lambda_j \neq \lambda_m}} \frac{(z-\lambda_j)_+^\sigma - (z-\lambda_m)_+^\sigma}{\lambda_m - \lambda_j} T_{\alpha jm}$$

By Lemma 2.1, the expression can be simplified to

$$G(\sigma,\alpha,z) := \sum_{\substack{j,m:\lambda_j,\lambda_m \le z \\ \lambda_j \neq \lambda_m}} \frac{(z-\lambda_j)_+^\sigma - (z-\lambda_m)_+^\sigma}{\lambda_m - \lambda_j} T_{\alpha jm} \le 2C_\sigma \sum_{j,m:\lambda_j \le z} \sum_{m=1}^{\infty} (z-\lambda_j)_+^{\sigma-1} T_{\alpha jm} - 2C_\sigma \sum_{j,m:\lambda_j \le z < \lambda_m} (z-\lambda_j)_+^{\sigma-1} T_{\alpha jm}$$

By symmetry in $j \leftrightarrow m$, extending the sum to all m subtracting the same quantity from the final term in (2.7), we find

$$R_\sigma(z) \le 2C_\sigma \sum_{j,m:\lambda_j \le z} \sum_{m=1}^{\infty} (z-\lambda_j)_+^{\sigma-1} T_{\alpha jm} + 2R(\sigma,\alpha,z) \quad (3.5)$$

where

$$R(\sigma,\alpha,z) = \sum_{j,q:\lambda_j \le z < \lambda_q} T_{\alpha jq} (z-\lambda_j)_+^{\sigma-1} \frac{(z-\lambda_j) - C_\sigma(\lambda_q - \lambda_j)}{\lambda_q - \lambda_j}.$$

We average over $\alpha = 1,\cdots,n$ in (3.5),

$$nR_\sigma(z) \le 2C_\sigma \sum_{j=1}^{\infty} (z-\lambda_j)_+^{\sigma-1} \lambda_j + 2\sum_{\alpha=1}^{n} R(\sigma,\alpha,z) \quad (3.6)$$

Since $R_\sigma(z) = \sum_{k=1}^{\infty} (z-\lambda_k)_+^\sigma$, and $R'_\sigma(z) = \sigma R_{\sigma-1}(z)$,

$$\sum_{j=1}^{\infty} (z-\lambda_j)_+^{\sigma-1} \lambda_j = zR_{\sigma-1}(z) - R_\sigma(z) \quad (3.7)$$

By (3.6) and (3.7),

$$R_\sigma(z) \le \frac{2C_\sigma}{n}\left[zR_{\sigma-1}(z) - R_\sigma(z)\right] + \frac{2}{n}\sum_{\alpha=1}^{n} R(\sigma,\alpha,z)$$

and then

$$\left[\left(1+\frac{2C_\sigma}{n}\right)R_\sigma(z)-\frac{2C_\sigma z}{n}R_{\sigma-1}(z)\right]\leq\frac{2}{n}\sum_{\alpha=1}^{n}R(\sigma,\alpha,z)\quad(3.8)$$

Now we separate into three cases.

Case 1. $1\leq\sigma\leq 2$. In this case $C_\sigma=1$ and

$$R(\sigma,\alpha,z)$$
$$=\sum_{j,q:\lambda_j\leq z<\lambda_q}T_{\alpha jq}\left(z-\lambda_j\right)_+^{\sigma-1}\left(\frac{\left(z-\lambda_j\right)-\left(\lambda_q-\lambda_j\right)}{\lambda_q-\lambda_j}\right)\leq 0$$

then $\left(1+\frac{2}{n}\right)R_\sigma(z)-\frac{2z}{n}R_{\sigma-1}(z)\leq 0$ which is (3.1).

Since $R'_\sigma(z)=\sigma R_{\sigma-1}(z)$, substituting it to (3.1), we can get (3.2).

Case 2. $0<\sigma<1$. Since the sum defining $R(\sigma,\alpha,z)$ runs over $\lambda_j\leq z<\lambda_q$,

$$\frac{\left(z-\lambda_j\right)-C_\sigma\left(\lambda_q-\lambda_j\right)}{\lambda_q-\lambda_j}\leq\frac{\left(\lambda_q-\lambda_j\right)-C_\sigma\left(\lambda_q-\lambda_j\right)}{\lambda_q-\lambda_j}$$
$$=1-C_\sigma=1-\frac{\sigma}{2}>0$$

Therefore

$$\sum_{\alpha=1}^{n}R(\sigma,\alpha,z)\leq\left(1-C_\sigma\right)\sum_{j=1}^{\infty}\left(z-\lambda_j\right)_+^{\sigma-1}\lambda_j$$
$$=\left(1-C_\sigma\right)\left(zR_{\sigma-1}(z)-R_\sigma(z)\right)$$

Substituting this into (3.8), we have

$$\frac{2z}{n}R_{\sigma-1}(z)\geq\left(1+\frac{2}{n}\right)R_\sigma(z)$$

which is equivalent to (3.1), also we can get (3.2).

On the other hand

$$\left(\frac{R_\sigma(z)}{z^{\sigma\left(1+\frac{n}{2}\right)}}\right)'=\frac{R'_\sigma(z)z^{\sigma\left(1+\frac{n}{2}\right)}-R_\sigma(z)\sigma\left(1+\frac{n}{2}\right)z^{\sigma\left(1+\frac{n}{2}\right)-1}}{z^{2\sigma\left(1+\frac{n}{2}\right)}}$$
$$\geq 0$$

thus $\frac{R_\sigma(z)}{z^{\sigma\left(1+\frac{n}{2}\right)}}$ is a nondecreasing function with respect to z.

Case 3. $\sigma\geq 2$. Similar to case 2, we obtain

$$R(\sigma,\alpha,z)\leq\left(1-C_\sigma\right)\sum_{\lambda_j\leq z<\lambda_q}T_{\alpha jq}\left(z-\lambda_j\right)_+^{\sigma-1}$$

but now $1-C_\sigma=1-\frac{\sigma}{2}<0$, then $R(\sigma,\alpha,z)\leq 0$. Substituting it into (3.8),

$$\left(1+\frac{2C_\sigma}{n}\right)R_\sigma(z)-\frac{2C_\sigma z}{n}R_{\sigma-1}(z)\leq 0$$

Then we have $R_{\sigma-1}(z)\geq\left(1+\frac{n}{\sigma}\right)\frac{1}{z}R_\sigma(z)$, which is (3.3).

Since $R'_\sigma(z)=\sigma R_{\sigma-1}(z)$, we have $R'_\sigma(z)\geq(\sigma+n)\frac{1}{z}R_\sigma(z)$, which is (3.4).

Similarly

$$\left(\frac{R_\sigma(z)}{z^{\sigma+n}}\right)'=\frac{R'_\sigma(z)z^{\sigma+n}-R_\sigma(z)(\sigma+n)z^{\sigma+n-1}}{z^{2(\sigma+n)}}$$
$$\geq 0$$

thus $\frac{R_\sigma(z)}{z^{\sigma+n}}$ is a nondecreasing function with respect to z. This completes the proof of the theorem.

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5. References

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