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A Primal-Dual Simplex Algorithm for Solving Linear Programming Problems with Symmetric Trapezoidal Fuzzy Numbers

Ali Ebrahimnejad
Department of Mathematics, Qaemshahr Branch, Islamic Azad University, Qaemshar, Iran
E-mail: aebrahimnejad@srbiau.ac.ir, aemarzoun@gmail.com
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Abstract

Two existing methods for solving a class of fuzzy linear programming (FLP) problems involving symmetric trapezoidal fuzzy numbers without converting them to crisp linear programming problems are the fuzzy primal simplex method proposed by Ganesan and Veeramani [1] and the fuzzy dual simplex method proposed by Ebrahimnejad and Nasseri [2]. The former method is not applicable when a primal basic feasible solution is not easily at hand and the later method needs to an initial dual basic feasible solution. In this paper, we develop a novel approach namely the primal-dual simplex algorithm to overcome mentioned shortcomings. A numerical example is given to illustrate the proposed approach.

Keywords: Fuzzy Linear Programming, Fuzzy Arithmetic, Fuzzy Orders, Primal-Dual Simplex Algorithm

1. Introduction

In optimizing real world systems, one usually ends up with a linear or nonlinear programming problem. For many cases, the coefficients involved in the objective and constraint functions are imprecise in nature and have to be interpreted as fuzzy numbers to reflect the real world situation. The resulting mathematical problem is therefore referred to as a fuzzy mathematical programming problem. After the pioneering work on fuzzy linear programming by Tanaka *et al.* [3,4] and Zimmermann [5], several kinds of fuzzy linear programming problems have appeared in the literature and different methods have been proposed to solve such problems [6-12]. One important class of these methods that has been highlighted by many researches is based on comparing of fuzzy numbers using ranking functions. Based on this idea, Maleki *et al.* [13] proposed a simple method for solving fuzzy number linear programming (FNLP) problems. They also applied an special kind of FNLP problems, involving fuzzy numbers only in objective function, as an auxiliary problem for solving fuzzy variable linear programming (FVLP) problems. Ebrahimnejad *et al.* [14] developed their method for solving bounded linear programming with fuzzy cost coefficients. Then Mahdavi-Amiri and Nasseri [15] used the certain linear ranking function to define the dual of FNLP problem as a similar problem that lead to an efficient algorithm called the dual simplex algorithm [16] for solving FNLP problems. Based on these algorithms, Ebrahimnejad [17] investigated the concept of sensitivity analysis in FNLP problems. Of course, Mahdavi-Amiri and Nasseri [18] and Mahdavi-Amiri *et al.* [19] proposed two efficient algorithms for solving FVLP problems directly without need of any auxiliary problem. Moreover, Nasseri and Ebrahimnejad [20] suggested the fuzzy primal simplex method to solve the flexible linear programming problems directly without solving any auxiliary problem. Then, Ebrahimnejad *et al.* [6] gave another efficient method namely primal-dual simplex method to obtain the fuzzy solution of FVLP problems. Ebrahimnejad and Nasseri [21] used the complementary slackness for solving both FNLP problem and FVLP problem. Hosseinzadeh Lotfi *et al.* [9] discussed full fuzzy linear programming (FFLP) problems of which all parameters and variable are triangular fuzzy numbers. They used the concept of the symmetric triangular fuzzy number and proposed an approach to defuzzify a general fuzzy quantity. After that Kumar *et al.* [11] proposed a new method to find the fuzzy optimal solution of same type of fuzzy linear programming problems.

Recently Ganesan and Veeramani [1] introduced a new method based on primal simplex algorithm for solving linear programming problem with symmetric trapezoidal fuzzy numbers without converting them to crisp linear programming problems. Ebrahimnejad *et al.* [7] extended their method for situations in which some or all variables are restricted to lie within fuzzy lower and fuzzy upper bounds. After that, Nasseri and Mahdavi-Amiri [22] and Nasseri *et al.* [23] developed the concept of duality of such problems that led to a new method based on dual simplex algorithm [2]. However, dual simplex algorithm begins with a basic (not necessarily feasible) dual solution and proceeds by pivoting through a series of dual basic fuzzy solution until the associated complementary primal basic solution is feasible. In this paper, we describe a new method for solving linear programming problem with symmetric trapezoidal fuzzy numbers, called the primal-dual algorithm, similar to the dual simplex method, which begins with dual feasibility and proceeds to obtain primal feasibility while maintaining complementary slackness. An important difference between the dual simplex method and the dual simplex method is that the primal-dual simplex method does not require a dual feasible solution to be basic.

This paper is organized as follows: In Section 2, we give some necessary concepts of fuzzy set theory. A review of linear programming problems with symmetric trapezoidal fuzzy numbers and two methods for solving such fuzzy problems are given in Section 3. We develop and present a fuzzy primal-dual algorithm to solve the fuzzy linear programming problems in Section 4 and explain it by an illustrative example. Finally, we conclude in Section 5.

2. Preliminaries

In this section, we review the fundamental notions of fuzzy set theory (see [1,7,24]).

Definition 2.1. A fuzzy number on R (real line) is said to be a symmetric trapezoidal fuzzy number if there exist real numbers a^L and a^U, $a^L \le a^U$ and $\alpha > 0$, such that

$$\tilde{a}(x) = \begin{cases} \dfrac{x}{\alpha} + \dfrac{\alpha - a^L}{\alpha}, & x \in \left[a^L - \alpha, a^L\right] \\ 1, & x \in \left[a^L, a^U\right] \\ \dfrac{-x}{\alpha} + \dfrac{a^U + \alpha}{\alpha}, & x \in \left[a^U, a^U + \alpha\right] \\ 0, & \text{otherwise.} \end{cases}$$

We denote a symmetric trapezoidal fuzzy number $\tilde{a}$ by $\tilde{a} = \left(a^L, a^U, \alpha, \alpha\right)$, where $\left(a^L - \alpha, a^U + \alpha\right)$ is the support of $\tilde{a}$ and $\left[a^L, a^U\right]$ its core, and the set of all symmetric trapezoidal fuzzy numbers by $F(R)$.

Let $\tilde{a} = \left(a^L, a^U, \alpha, \alpha\right)$ and $\tilde{b} = \left(b^L, b^U, \beta, \beta\right)$ be two symmetric trapezoidal fuzzy numbers. Then the arithmetic operations on $\tilde{a}$ and $\tilde{b}$ are given by (taken from [1]):

$$\tilde{a} + \tilde{b} = \left(a^L + b^L, a^U + b^U, \alpha + \beta, \alpha + \beta\right)$$

$$\tilde{a} - \tilde{b} = \left(a^L - b^U, a^U - b^L, \alpha + \beta, \alpha + \beta\right)$$

$$\tilde{a}\tilde{b} = \left(\left(\frac{a^L + a^U}{2}\right)\left(\frac{b^L + b^U}{2}\right) - \omega, \left(\frac{a^L + a^U}{2}\right)\left(\frac{b^L + b^U}{2}\right) + \omega, \left|a^U \beta + b^U \alpha\right|, \left|a^U \beta + b^U \alpha\right|\right),$$

where

$$\omega = \frac{t_2 - t_1}{2}, \quad t_1 = \min\left\{a^L b^L, a^L b^U, a^U b^L, a^U b^U\right\},$$

$$t_2 = \max\left\{a^L b^L, a^L b^U, a^U b^L, a^U b^U\right\}.$$

From the above definition it can be seen that

$$\lambda \ge 0,\ \lambda \in \mathbb{R};\ \lambda\tilde{a} = \left(\lambda a^L, \lambda a^U, \lambda\alpha, \lambda\alpha\right)$$

$$\lambda < 0,\ \lambda \in \mathbb{R};\ \lambda\tilde{a} = \left(\lambda a^U, \lambda a^L, -\lambda\alpha, -\lambda\alpha\right).$$

Note that depending upon the need, one can also use a smaller ω in the definition of multiplication involving symmetric trapezoidal fuzzy numbers.

Definition 2.2. Let $\tilde{a} = \left(a^L, a^U, \alpha, \alpha\right)$ and $\tilde{b} = \left(b^L, b^U, \beta, \beta\right)$ be two symmetric trapezoidal fuzzy numbers. Define the relations $\preceq$ and $\approx$ as given below: $\tilde{a} \preceq \tilde{b}$ (or $\tilde{b} \succeq \tilde{a}$) if and only if

1) $\dfrac{\left(a^L - \alpha\right) + \left(a^U + \alpha\right)}{2} < \dfrac{\left(b^L - \beta\right) + \left(b^U + \beta\right)}{2}$,

that is $\dfrac{a^L + a^U}{2} < \dfrac{b^L + b^U}{2}$ (in this case, we may write $\tilde{a} \prec \tilde{b}$),

2) or $\dfrac{a^L + a^U}{2} = \dfrac{b^L + b^U}{2}$, $b^L < a^L$ and $a^U < b^U$,

3) or $\dfrac{a^L + a^U}{2} = \dfrac{b^L + b^U}{2}$, $b^L = a^L, a^U = b^U$ and $\alpha \le \beta$.

Note that in cases (2) and (3), we also write $\tilde{a} \approx \tilde{b}$ and say that $\tilde{a}$ and $\tilde{b}$ are equivalent.

Remark 2.1. Two symmetric trapezoidal fuzzy numbers $\tilde{a} = \left(a^L, a^U, \alpha\right)$, $\tilde{b} = \left(b^L, b^U, \beta\right)$ are equivalent if and only if $\dfrac{a^L + a^U}{2} = \dfrac{b^L + b^U}{2}$.

Definition 2.3. For any trapezoidal fuzzy number $\tilde{a}$, we define $\tilde{a} \succeq \tilde{0}$, if there exist $\varepsilon \geq 0$ and $\alpha \geq 0$ such that $\tilde{a} \succeq (-\varepsilon, \varepsilon, \alpha)$. We also denote $(-\varepsilon, \varepsilon, \alpha)$ by $\tilde{0}$. Note that $\tilde{0}$ is equivalent to $(0,0,0) = 0$. Naturally, one may consider $\tilde{0} = (0,0,0)$ as the zero symmetric trapezoidal fuzzy number.

Remark 2.2. If $\tilde{x} \approx \tilde{0}$, then $\tilde{x}$ is said to be a zero symmetric trapezoidal fuzzy number. It is to be noted that if $\tilde{x} = \tilde{0}$, then $\tilde{x} \approx \tilde{0}$, but the converse need not be true. If $\tilde{x} \not\approx \tilde{0}$ (that is $\tilde{x}$ is not equivalent to $\tilde{0}$), then it is said to be a non-zero symmetric trapezoidal fuzzy number. It is to be noted that if $\tilde{x} \not\approx \tilde{0}$, then $\tilde{x} \neq \tilde{0}$, but the converse need not be true. If $\tilde{x} \succeq \tilde{0}$ $\left(\tilde{x} \preceq \tilde{0}\right)$ and $\tilde{x} \not\approx \tilde{0}$, then is said to be a positive (negative) symmetric trapezoidal fuzzy number and is denoted by $\tilde{x} \succ \tilde{0}$ $\left(\tilde{x} \prec \tilde{0}\right)$. Now if $\tilde{a}$, $\tilde{b} \in F(\mathbb{R})$, it is easy to show that if $\tilde{a} \succeq \tilde{b}$, then $\tilde{a} - \tilde{b} \succeq \tilde{0}$.

The following lemma immediately follows form Definition 2.1.

Lemma 2.1. If $\tilde{a}, \tilde{b} \in F(\mathbb{R})$, and $c \in \mathbb{R}$ such that $c \neq 0$, then

1) $\tilde{a}\tilde{b} \approx \tilde{b}\tilde{a}$.
2) $c(\tilde{a}\tilde{b}) \approx (c\tilde{a})\tilde{b} \approx \tilde{a}(c\tilde{b})$.

The two following results are taken from [1].

Lemma 2.2. For any symmetric trapezoidal fuzzy number $\tilde{a}, \tilde{b}$ and $\tilde{c}$, we have:

1) $\tilde{c}\left(\tilde{a} + \tilde{b}\right) \approx \left(\tilde{c}\tilde{a} + \tilde{c}\tilde{b}\right)$.
2) $\tilde{c}\left(\tilde{a} - \tilde{b}\right) \approx \left(\tilde{c}\tilde{a} - \tilde{c}\tilde{b}\right)$.

Lemma 2.3. If $\tilde{a}, \tilde{b} \in F(\mathbb{R})$, then

1) The relation $\preceq$ is a partial order relation on the set of symmetric trapezoidal fuzzy numbers.

2) The relation $\preceq$ is a linear order relation on the set of symmetric trapezoidal fuzzy numbers.

3) For any two symmetric trapezoidal fuzzy numbers $\tilde{a}$ and $\tilde{b}$, if $\tilde{a} \preceq \tilde{b}$ then $\tilde{a} \preceq (1-\lambda)\tilde{a} + \lambda\tilde{b} \preceq \tilde{b}$, for all λ, $0 \leq \lambda \leq 1$.

3. Fuzzy Linear Programming

Ganesan and Veeramani [1] introduced a new type of fuzzy arithmetic for symmetric trapezoidal fuzzy numbers. Here, we first review these new notions which are useful in our further consideration. After that we review the concept of duality for such problems proposed by Nasseri and Mahdavi-Amiri [22] and Nasseri *et al.* [23].

3.1. A Fuzzy Primal Simplex Algorithm

Definition 3.1. A linear programming problem with trapezoidal fuzzy number is defined as [1]:

$$\begin{aligned} \min \quad & \tilde{z} \approx \tilde{c}\tilde{x} \\ s.t. \quad & A\tilde{x} \succeq \tilde{b} \\ & \tilde{x} \succeq \tilde{0} \end{aligned} \tag{1}$$

where $\tilde{b} \in \left(F(\mathbb{R})\right)^m$, $\tilde{c}^T \in \left(F(\mathbb{R})\right)^n$, $A \in \mathbb{R}^{m\times n}$ are given and $\tilde{x} \in \left(F(\mathbb{R})\right)^n$ is to be determined.

Definition 3.2. We say that a fuzzy vector $\tilde{x} \in \left(F(\mathbb{R})\right)^n$ is a feasible solution to (1) if and only if x satisfies the constraints and non-negativity restrictions of the problem.

Definition 3.3. A fuzzy feasible solution $\tilde{x}_*$ is said to be a fuzzy optimal solution to (1), if for all fuzzy feasible solution $\tilde{x}$ for (1), we have $\tilde{c}\tilde{x}_* \preceq \tilde{c}\tilde{x}$.

Definition 3.4. Consider fuzzy linear programming problem (1) in its standard form as follows:

$$\begin{aligned} \min \quad & \tilde{z} \approx \tilde{c}\tilde{x} \\ s.t. \quad & A\tilde{x} \approx \tilde{b} \\ & \tilde{x} \succeq \tilde{0} \end{aligned} \tag{2}$$

where the parameters of the problem are as defined in (1). Let $A = \left[a_{ij}\right]_{m\times n}$. Assume $rank(A) = m$. Partition A as $\left[B \quad N\right]$ where B, $m\times m$, is nonsingular. It is obvious that $rank(B) = m$. Let y_j be the solution to $By = a_j$. It is apparent that the basic solution

$$\tilde{x}_B = \left(\tilde{x}_{B_1}, \cdots, \tilde{x}_{B_m}\right)^T \approx B^{-1}\tilde{b}, \tilde{x}_N \approx \tilde{0} \tag{3}$$

is a solution of $A\tilde{x} = \tilde{b}$. In fact, $\tilde{x} = \left(\tilde{x}_B{}^T, \tilde{x}_N{}^T\right)^T$. If $\tilde{x}_B \succeq \tilde{0}$, then the fuzzy basic solution is feasible and the corresponding fuzzy objective value is: $\tilde{z} \approx \tilde{c}_B\tilde{x}_B$, where $\tilde{c}_B = \left(\tilde{c}_{B_1}, \cdots, \tilde{c}_{B_m}\right)$. Now, corresponding to every nonbasic variable $x_j, 1 \leq j \leq n, j \neq B_i, i = 1, \cdots, m,$ define

$$\tilde{z}_j \approx \tilde{c}_B y_j \approx \tilde{c}_B B^{-1} a_j. \tag{4}$$

Below, we state some important results concerning to the optimality conditions improving a fuzzy feasible solution and unbounded criteria (taken from [1]).

Theorem 3.1. If we have a fuzzy basic feasible solution with fuzzy objective value $\tilde{z}$ such that $\tilde{z}_k \succ \tilde{c}_k$ for some nonbasic variable x_k, and $y_k \not\leq 0$, then it is possible to obtain a new basic feasible solution with new fuzzy objective value $\tilde{\tilde{z}}$, that satisfies $\tilde{\tilde{z}} \preceq \tilde{z}$.

Theorem 3.2. If we have a fuzzy basic feasible solution with $\tilde{z}_k \succ \tilde{c}_k$ for some nonbasic variable x_k, and $y_k \leq 0$, then the problem (2) has an unbounded solution.

Theorem 3.3. (Optimality conditions) If a fuzzy basic solution $\tilde{x}_B = B^{-1}\tilde{b}$, $\tilde{x}_N = \tilde{0}$ is feasible to (2) and $\tilde{z}_j \preceq \tilde{c}_j$ for all j, $1 \leq j \leq n$, then the fuzzy basic solution is a fuzzy optimal solution to (2).

Ganesan and Veeramani [1] based on these theorems proposed a new algorithm for solving problem (2). Here, we give a summary of their method in tableau format.

Algorithm 3.1. A fuzzy primal simplex method

Initialization step

Suppose an initial fuzzy basic feasible solution with basis B is at hand. Form the following initial **Table 1**.

Main step

1) Calculate $\tilde{z}_j - \tilde{c}_j$ for all nonbasic variables. Suppose $\tilde{z}_j - \tilde{c}_j = \left(h_j^L, h_j^U, h_j, h_j\right)$. Let

$$h_k^U + h_k^L = \max_{j \in T}\left\{h_j^U + h_j^L\right\}$$

where T is the index set of the current nonbasic variables.

If $h_k^U + h_k^L \le 0$, **then stop**; the current solution is optimal. **Otherwise**, go to (2).

2) Let $y_k = B^{-1}a_k$. **If** $y_k \le 0$, **then stop**; the problem is unbounded. **Otherwise**, suppose $\tilde{\bar{b}}_i = \left(\bar{b}_i^U, \bar{b}_i^L, \alpha_i, \alpha_i\right)$ and determine the index r as follows:

$$\frac{\bar{b}_r^U + \bar{b}_r^L}{y_{rk}} = \min_{1 \le i \le m}\left\{\frac{\bar{b}_i^U + \bar{b}_i^L}{y_{ik}} \middle| y_{ik} > 0\right\}.$$

3) Update the tableau by pivoting at y_{rk}. Update the fuzzy basic and nonbasic variables where $\tilde{x}_k$ enters the basis and $\tilde{x}_{B_r}$ leaves the basis, and go to (1).

3.2. A Fuzzy Dual Simplex Algorithm

Definition 3.5. Dual of the FLP problem (1) is defined as follows [22,23] :

$$\begin{aligned} \max \quad & \tilde{u} \approx \tilde{w}\tilde{b} \\ \text{s.t.} \quad & \tilde{w}A \preceq \tilde{c} \\ & \tilde{w} \succeq \tilde{0} \end{aligned} \tag{5}$$

where $\tilde{w} = (\tilde{w}_1, \cdots, \tilde{w}_m) \in F(\mathbb{R})^m$ is including the fuzzy variables corresponding to constraints of problem (1). In fact, $\tilde{w}_i$, $i = 1, \cdots, m$ is defined for the ith constraint of the problem (1). We name this problem as the DFLP problem.

We shall discuss here the relationships between the FLP problem and its corresponding dual and omit the proofs (taken from Nasseri and Mahdavi-Amiri [22] and Nasseri *et al.* [23]).

Theorem 3.4. (The weak duality property.) If $\tilde{x}_0$ and $\tilde{w}_0$ are fuzzy feasible solutions to FLP and DFLP problems, respectively, then $\tilde{c}\tilde{x}_0 \succeq \tilde{w}_0\tilde{b}$.

Corollary 3.1. If $\tilde{x}_0$ and $\tilde{w}_0$ are fuzzy feasible solutions to FLP and DFLP problems, respectively, and $\tilde{c}\tilde{x}_0 \approx \tilde{w}_0\tilde{b}$, then $\tilde{x}_0$ and $\tilde{w}_0$ are fuzzy optimal solutions to their respective problems.

Table 1. The initial primal simplex tableau.

Basis	$\tilde{x}_B$	$\tilde{x}_N$	$R.H.S.$
$\tilde{z}$	$\tilde{0}$	$\tilde{z}_N - \tilde{c}_N = \tilde{c}_B Y_N - \tilde{c}_N$	$\tilde{\bar{z}} = c_B B^{-1}\tilde{b}$
$\tilde{x}_B$	I	Y_N	$\tilde{\bar{b}} = B^{-1}\tilde{b}$

Corollary 3.2. If any one of the FLP or DFLP problem is unbounded, then the other problem has no fuzzy feasible solution.

Theorem 3.5. (Strong duality.) If any one of the FLP or DFLP problem has an fuzzy optimal solution, then both problems have fuzzy optimal solutions and the two optimal objective fuzzy values are equal. (In fact, if $\tilde{x}^*$ is fuzzy optimal solution of the primal problem then the fuzzy vector $\tilde{w}^* = \tilde{c}_B B^{-1}$, where B is the optimal basis, is a fuzzy optimal solution of the dual problem).

Theorem 3.6. (Complementary slackness). Suppose $\tilde{x}_*$ and $\tilde{w}_*$ are feasible solutions of the FLP problem and its corresponding dual, the DFLP problem, respectively. Then $\tilde{x}_*$ and $\tilde{w}_*$ are respectively optimal if and only if

$$\left(\tilde{w}_* A - \tilde{c}\right)\tilde{x}_* \approx \tilde{0}, \quad \tilde{w}_*\left(\tilde{b} - A\tilde{x}_*\right) \approx \tilde{0}.$$

Ebrahimnejad and Nasseri [2] using the above results, introduced a new fuzzy dual algorithm for solving problem (2).

Algorithm 3.2. A fuzzy dual simplex algorithm

Initialization step

Suppose that basis B be dual feasible for the problem (2). Form the Tableau 3.1 as an initial dual feasible simplex tableau. Suppose $\tilde{z}_j - \tilde{c}_j = \left(h_j^L, h_j^U, \alpha_j, \alpha_j\right)$, so $h_j^U + h_j^L \le 0$ for all j.

Main step

1) Suppose $\tilde{\bar{b}} = B^{-1}\tilde{b}$. **If** $\tilde{\bar{b}} \succeq \tilde{0}$, **then Stop**; the current fuzzy solution is optimal.

Else suppose $\tilde{\bar{b}}_i = \left(\bar{b}_i^L, \bar{b}_i^U, \alpha_i, \alpha_i\right)$ and let $\bar{b}_r^U + \bar{b}_r^L = \min_{1 \le i \le m}\left\{\bar{b}_i^U + \bar{b}_i^L\right\}$.

2) **If** $y_{rj} \ge 0$ for all j, **then Stop**; the problem (2) is infeasible.

Else select the pivot column k by the following test:

$$\frac{h_k^U + h_k^L}{y_{rk}} = \min_{1 \le j \le n}\left\{\frac{h_j^U + h_j^L}{y_{rj}} \middle| y_{rj} < 0\right\}.$$

3) Update the tableau by pivoting at y_{rk}. Update the fuzzy basic and nonbasic variables where x_k enters the basis and x_{B_r} leaves the basis, and go to (1).

4. A Fuzzy Primal-Dual Algorithm

We note that the method which is proposed by Ganesan and Veeramani in [1], starts with a fuzzy basic feasible solution for FLP and moves to an optimal basis by walking through a sequence of fuzzy feasible bases of FLP. All the bases with the possible exception of the optimal basis obtained in fuzzy primal simplex algorithm don't satisfy the optimality criteria for FLP or feasibility condition for DFLP. But their method has no efficient when a primal basic feasible solution is not at hand. So, Ebrahimnejad and Nasseri [2] developed a new dual simplex algorithm to overcome this shortcoming by using the duality results which have been proposed by Nasseri and Mahdavi-Amiri [22] and Nasseri *et al.* [23]. This algorithm starts with a dual basic feasible solution, but primal basic infeasible solution and walks to an optimal solution by moving among adjacent dual basic feasible solutions. However, the dual simplex method for solving FLP problem needs to an initial dual basic feasible solution. Here, we develop the fuzzified version of conventional primal-dual method of linear programming problems that any dual feasible solution, whether basic or not, is adequate to initiate this method.

Corollary 4.1. [22,23] The optimality criteria $\tilde{z}_j - \tilde{c}_j \preceq \tilde{0}$, for all j, for the FLP problem is equivalent to the feasibility condition for the DFLP problem.

To explain the main strategy employed by this method, we consider the following standard FLP:

$$\begin{aligned} \min \quad & \tilde{z} \approx \tilde{c}\tilde{x} \\ \text{s.t.} \quad & A\tilde{x} \approx \tilde{b} \\ & \tilde{x} \succeq \tilde{0} \end{aligned} \tag{6}$$

Let $\tilde{w} = (\tilde{w}_1, \tilde{w}_2, \cdots, \tilde{w}_m)$ be the row of dual vector variables. The Dual of FLP problem (6) is

$$\begin{aligned} \max \quad & \tilde{u} \approx \tilde{w}\tilde{b} \\ \text{s.t.} \quad & \tilde{w}A \succeq \tilde{c} \end{aligned} \tag{7}$$

The complementary slackness conditions are

$$\left(\tilde{c}_j - \tilde{w}a_j\right)\tilde{x}_j \approx \tilde{0} \quad j = 1, \cdots, n \tag{8}$$

Let $\tilde{\tilde{w}}$ be the initial fuzzy dual feasible solution. Suppose $\Omega = \left\{ j : \tilde{\tilde{w}}a_j - \tilde{c}_j \approx \tilde{0} \right\}$. Now consider the following problem known as the restricted fuzzy primal problem corresponding to $\tilde{\tilde{w}}$.

$$\begin{aligned} \min & \sum_{i \in \Omega} \tilde{0}\tilde{x}_j + 1\tilde{x}_a \\ \text{s.t.} & \sum_{i \in \Omega} a_j \tilde{x}_j + I\tilde{x}_a \approx \tilde{b} \\ & \tilde{x}_j \succeq \tilde{0}, \text{ for } j \in \Omega, \\ & \tilde{x}_a \succeq \tilde{0}, \end{aligned} \tag{9}$$

where $\tilde{x}_a = (\tilde{x}_{1a}, \cdots, \tilde{x}_{ma}) \in F(\mathbb{R})^m$ and $1 = ((1,1,0,0), \cdots, (1,1,0,0)) \in (F(\mathbb{R}))^m$.

The restricted fuzzy primal problem (9) is now solved by the method which is proposed by Ganesan and Veeramani in [1] beginning the fuzzy feasible basic solution $\tilde{x}_a$. In this process, once an artificial variable $\tilde{x}_{ja}$ drops out of the basic variables, discard it from the problem. Let $\tilde{\tilde{x}}$ and $\tilde{x}^0$ be the fuzzy primal solution and fuzzy objective value obtained at termination in this restricted fuzzy primal problem, respectively. If $\tilde{x}^0 \approx \tilde{0}$, $\tilde{\tilde{x}}$ and $\tilde{\tilde{w}}$ are optimal to (6) and (7), respectively. If $\tilde{x}^0 \succ \tilde{0}$, let $\tilde{\tilde{v}}$ be the optimal solution to the dual of the restricted fuzzy primal problem (9).

Now, we construct the new fuzzy dual solution for (6) such that all the basic primal variables in the restricted fuzzy primal problem remain in the new restricted fuzzy primal problem and in addition, at least one primal variables that did not belong to the set Ω would get passed to restricted fuzzy primal problem. Furthermore, this variable would reduced $\tilde{x}^0$ if introduced in the basic.

In order to construct such a dual fuzzy vector, consider the following fuzzy dual $\bar{\tilde{w}}$, where $\alpha > 0$:

$$\bar{\tilde{w}} = \tilde{\tilde{w}} + \alpha\tilde{\tilde{v}}$$

Then

$$\bar{\tilde{w}}a_j - \tilde{c}_j \simeq \left(\tilde{\tilde{w}} + \alpha\tilde{\tilde{v}}\right)a_j - \tilde{c}_j \simeq \left(\tilde{\tilde{w}}a_j - \tilde{c}_j\right) + \alpha\left(\tilde{\tilde{v}}a_j\right) \tag{10}$$

Now, if $\tilde{\tilde{x}}_j$ with $j \in \Omega$ is a basic variable in the restricted fuzzy primal problem, then from complementary slackness $\tilde{\tilde{v}}a_j \approx \tilde{0}$ and hence $\bar{\tilde{w}}a_j - \tilde{c}_j \approx \tilde{0}$, permitting j in the new restricted fuzzy primal problem. If $j \notin \Omega$ and $\tilde{\tilde{v}}a_j \preceq \tilde{0}$, then from (10) we have $\bar{\tilde{w}}a_j - \tilde{c}_j \prec \tilde{0}$. Finally, if $j \notin \Omega$ and $\tilde{\tilde{v}}a_j \succ \tilde{0}$, we show there is a $\alpha > 0$ such that $\bar{\tilde{w}}a_j - \tilde{c}_j \preceq \tilde{0}$ for $j \notin \Omega$ with at least one component equal to fuzzy zero.

First, we show in this case for each j there is an α_j such that $\bar{\tilde{w}}a_j - \tilde{c}_j \simeq \left(\tilde{\tilde{w}}a_j - \tilde{c}_j\right) + \alpha_j\left(\tilde{\tilde{v}}a_j\right) \approx \tilde{0}$. Let

$$\begin{aligned} \bar{\tilde{w}}a_j - \tilde{c}_j &= \left(\bar{w}_{1j}, \bar{w}_{2j}, l_j, l_j\right), \quad \bar{w}_{1j} \le \bar{w}_{2j} \\ \tilde{\tilde{w}}a_j - \tilde{c}_j &= \left(\hat{w}_{1j}, \hat{w}_{2j}, h_j, h_j\right) \prec \tilde{0}, \\ \hat{w}_{1j} &\le \hat{w}_{2j}, \quad \frac{\hat{w}_{1j} + \hat{w}_{2j}}{2} < 0 \end{aligned}$$

and

$$\tilde{\tilde{v}}a_j = \left(\hat{v}_{1j}, \hat{v}_{2j}, k_j, k_j\right) \succ \tilde{0}, \quad \hat{v}_{1j} \le \hat{v}_{2j}, \quad \frac{\hat{v}_{1j} + \hat{v}_{2j}}{2} > 0$$

Now $\bar{\tilde{w}}a_j - \tilde{c}_j \simeq \left(\tilde{\tilde{w}}a_j - \tilde{c}_j\right) + \alpha_j\left(\tilde{\tilde{v}}a_j\right) \approx \tilde{0}$, if and only if

$$\left(\hat{w}_{1j} + \alpha_j\hat{v}_{1j}, \hat{w}_{1j} + \alpha_j\hat{v}_{1j}, h_j + \alpha_j k_j, h_j + \alpha_j k_j\right) \approx \tilde{0}$$

if and only if

$$\hat{w}_{1j} + \alpha_j\hat{v}_{1j} = -\hat{w}_{2j} - \alpha_j\hat{v}_{2j}$$

if and only if

$$\alpha_j\left(\hat{v}_{1j} + \hat{v}_{2j}\right) = -\left(\hat{w}_{1j} + \hat{w}_{2j}\right)$$

if and only if

$$\alpha_j = -\frac{\hat{w}_{1j} + \hat{w}_{2j}}{\hat{v}_{1j} + \hat{v}_{2j}}.$$

Note that $\alpha_j > 0$, since $\hat{v}_{1j} + \hat{v}_{2j} > 0$ and $\hat{w}_{1j} + \hat{w}_{2j} < 0$. That is, if we choose α_j as above, then $\tilde{\bar{w}}a_j - \tilde{c}_j \simeq \tilde{0}$. Now, we define α as follows:

$$\alpha = \alpha_k = -\frac{\hat{w}_{1k} + \hat{w}_{2k}}{\hat{v}_{1k} + \hat{v}_{2k}} = \min_j \left\{ -\frac{\hat{w}_{1j} + \hat{w}_{2j}}{\hat{v}_{1j} + \hat{v}_{2j}} \middle| \tilde{\hat{v}}a_j \succ \tilde{0} \right\}.$$

By definition of α, we see $\tilde{\bar{w}}a_k - \tilde{c}_k \approx \tilde{0}$. In addition, we show $\tilde{\bar{w}}a_j - \tilde{c}_j \preceq \tilde{0}$ for each j with $\tilde{\hat{v}}a_j \succ \tilde{0}$. From definition of α, we have $\alpha \le \alpha_j$ for all j. Thus

$$\alpha\left(\hat{v}_{1j} + \hat{v}_{2j}\right) \le \alpha_j\left(\hat{v}_{1j} + \hat{v}_{2j}\right).$$

So, we have

$$\left(\hat{w}_{1j} + \hat{w}_{2j}\right) + \alpha\left(\hat{v}_{1j} + \hat{v}_{2j}\right) \le \left(\hat{w}_{1j} + \hat{w}_{2j}\right) + \alpha_j\left(\hat{v}_{1j} + \hat{v}_{2j}\right).$$

or

$$\frac{1}{2}\left(\left(\hat{w}_{1j} + \alpha\hat{v}_{1j}\right) + \left(\hat{w}_{2j} + \alpha\hat{v}_{2j}\right)\right) \le \frac{1}{2}\left(\left(\hat{w}_{1j} + \alpha_j\hat{v}_{1j}\right) + \left(\hat{w}_{2j} + \alpha_j\hat{v}_{2j}\right)\right)$$

that is

$$\left(\tilde{\hat{w}}a_j - \tilde{c}_j\right) + \alpha\left(\tilde{\hat{v}}a_j\right) \preceq \left(\tilde{\hat{w}}a_j - \tilde{c}_j\right) + \alpha_j\left(\tilde{\hat{v}}a_j\right)$$

But, we know $\left(\tilde{\hat{w}}a_j - \tilde{c}_j\right) + \alpha_j\left(\tilde{\hat{v}}a_j\right) \approx \tilde{0}$. Therefore $\tilde{\bar{w}}a_j - \tilde{c}_j \simeq \left(\tilde{\hat{w}}a_j - \tilde{c}_j\right) + \alpha\left(\tilde{\hat{v}}a_j\right) \preceq \tilde{0}$. Hence, modifying the dual fuzzy vector leads to a new feasible dual fuzzy solution. Also, all the variables that belonged to the restricted fuzzy primal problem basis are passed to the new basis. In addition, a new fuzzy variable $\tilde{x}_k$ that is a candidate to enter the basis, is passed to the restricted fuzzy primal problem. Thus, we continue from the present restricted fuzzy primal problem basis by entering $\tilde{x}_k$, which leads to a potential reduction in $\tilde{x}^0$. This process is continued until $\tilde{x}^0 \approx \tilde{0}$ in which case we have an optimal solution or else $\tilde{x}^0 \succ \tilde{0}$ and $\tilde{\hat{v}}a_j \preceq \tilde{0}$ for all $j \notin \Omega$. We explain this case as Theorem 4.1 as below.

Theorem 4.1. If at the end of the restricted fuzzy primal problem, we have $\tilde{x}^0 \succ \tilde{0}$ and $\tilde{\hat{v}}a_j \preceq \tilde{0}$ for all $j \notin \Omega$, then the FLP (6) has no solution.

Proof. In this case consider $\tilde{\bar{w}} = \tilde{\hat{w}} + \alpha\tilde{\hat{v}}$. Since $\tilde{\hat{w}}a_j - \tilde{c}_j \preceq \tilde{0}$, for all j and by assumption $\tilde{\hat{v}}a_j \preceq \tilde{0}$ for all $j \notin \Omega$, then from (10), $\tilde{\bar{w}}$ is a dual feasible fuzzy solution for all $\alpha > 0$. In addition, the dual objective fuzzy value is

$$\tilde{\bar{w}}\tilde{b} \approx \left(\tilde{\hat{w}} + \alpha\tilde{\hat{v}}\right)\tilde{b} \approx \tilde{\hat{w}}\tilde{b} + \alpha\tilde{\hat{v}}\tilde{b}$$

Since $\tilde{x}^0$ and $\tilde{\hat{v}}\tilde{b}$ are the optimal objective values for the restricted fuzzy primal problem and its dual, we have $\tilde{\hat{v}}\tilde{b} \approx \tilde{x}^0$. Also $\tilde{x}^0 \succ \tilde{0}$, so $\tilde{\bar{w}}\tilde{b}$ can be increased indefinitely by choosing α arbitrarily large. Therefore the DFLP is unbounded and hence from corollary 3.2 the FLP is infeasible.

Algorithm 4.1. A fuzzy primal-dual simplex algorithm

1) {dual feasibility} Choose a fuzzy vector $\tilde{w}$ such that $\tilde{z}_j - \tilde{c}_j \preceq \tilde{0}$ for all j.

2) $Q = \left\{ j : \tilde{w}a_j - \tilde{c}_j \approx \tilde{0} \right\}$ and solve the restricted fuzzy primal problem. **If** $\tilde{x}_0 \approx \tilde{0}$ **then Stop** (the current solution is optimal).

Else let $\tilde{\hat{v}}$ be the optimal dual fuzzy solution to the restricted fuzzy primal problem.

3) **If** $\tilde{\hat{v}}a_j \preceq \tilde{0}$, for all j **then stop** (the FLP problem is infeasible).

Else let

$$\alpha = \alpha_k = -\frac{\hat{w}_{1k} + \hat{w}_{2k}}{\hat{v}_{1k} + \hat{v}_{2k}} = \min_j \left\{ -\frac{\hat{w}_{1j} + \hat{w}_{2j}}{\hat{v}_{1j} + \hat{v}_{2j}} \middle| \tilde{\hat{v}}a_j \succ \tilde{0} \right\}.$$

and replace $\tilde{\hat{w}}$ by $\tilde{\hat{w}} + \alpha\tilde{\hat{v}}$ and go to (2).

For an illustration of the fuzzy primal-dual algorithm, we consider the following example.

Example 4.1. Consider the FLP problem: (See (11))

Thus, with introducing the slack variables x_6 and x_7, the above FLP problem reduces to the standard form: (See (12))

$$\begin{aligned} \min\quad & \tilde{z} \approx (1,5,1,1)\tilde{x}_1 + (2,6,1,1)\tilde{x}_2 + (5,7,2,2)\tilde{x}_3 + (6,8,1,1)\tilde{x}_4 + (0,2,1,1)\tilde{x}_5 \\ \text{s.t.}\quad & 2\tilde{x}_1 - \tilde{x}_2 + \tilde{x}_3 + 6\tilde{x}_4 - 5\tilde{x}_5 \succeq (4,8,2,2) \\ & \tilde{x}_1 + \tilde{x}_2 + 2\tilde{x}_3 + \tilde{x}_4 + 2\tilde{x}_5 \succeq (1,5,1,2) \\ & \tilde{x}_1, \tilde{x}_2, \tilde{x}_3, \tilde{x}_4, \tilde{x}_5, \succeq \tilde{0} \end{aligned} \tag{11}$$

$$\begin{aligned} \min\quad & \tilde{z} \approx (1,5,1,1)\tilde{x}_1 + (2,6,1,1)\tilde{x}_2 + (5,7,2,2)\tilde{x}_3 + (6,8,1,1)\tilde{x}_4 + (0,2,1,1)\tilde{x}_5 \\ \text{s.t.}\quad & 2\tilde{x}_1 - \tilde{x}_2 + \tilde{x}_3 + 6\tilde{x}_4 - 5\tilde{x}_5 - \tilde{x}_6 \approx (4,8,2,2) \\ & \tilde{x}_1 + \tilde{x}_2 + 2\tilde{x}_3 + \tilde{x}_4 + 2\tilde{x}_5 - \tilde{x}_7 \approx (1,5,1,2) \\ & \tilde{x}_1, \tilde{x}_2, \tilde{x}_3, \tilde{x}_4, \tilde{x}_5, \tilde{x}_6, \tilde{x}_7 \succeq \tilde{0} \end{aligned} \tag{12}$$

The dual problem is given by the following:

$$\begin{aligned} \max \quad & \tilde{w} \approx (4,8,2,2)\tilde{w}_1 + (1,5,1,1)\tilde{w}_2 \\ \text{s.t.} \quad & 2\tilde{w}_1 + \tilde{w}_2 \preceq (1,5,1,1) \\ & -\tilde{w}_1 + \tilde{w}_2 \preceq (2,6,2,2) \\ & \tilde{w}_1 + 2\tilde{w}_2 \preceq (5,7,2,2) \\ & 6\tilde{w}_1 + \tilde{w}_2 \preceq (6,8,1,1) \\ & -5\tilde{w}_1 + 2\tilde{w}_2 \preceq (0,2,1,1) \\ & \tilde{w}_1 \preceq \tilde{0} \\ & \tilde{w}_2 \preceq \tilde{0} \\ & \tilde{w}_1, \tilde{w}_2 \text{ unrestricted.} \end{aligned} \tag{13}$$

Now we solve the FLP problem (13) by the fuzzy primal-dual simplex algorithm.

Initialization step: The initial fuzzy dual solution is given by $\tilde{w} = (\tilde{w}_1, \tilde{w}_2) = (\tilde{0}, \tilde{0})$.

First iteration: Substituting $\tilde{w} = (\tilde{w}_1, \tilde{w}_2) = (\tilde{0}, \tilde{0})$ in each dual constraint, we find that the last two dual constraints hold at equality. Thus, $\Omega = \{6,7\}$. The restricted fuzzy primal problem becomes as follows:

$$\begin{aligned} \min \quad & (1,1,0,0)\tilde{x}_8 + (1,1,0,0)\tilde{x}_9 \\ \text{s.t.} \quad & -\tilde{x}_6 + \tilde{x}_8 \approx (4,8,2,2) \\ & -\tilde{x}_7 + \tilde{x}_9 \approx (1,5,1,1) \\ & \tilde{x}_6, \tilde{x}_7, \tilde{x}_8, \tilde{x}_9 \succeq \tilde{0} \end{aligned} \tag{14}$$

where $\tilde{x}_8$ and $\tilde{x}_9$ are the artificial fuzzy variables. Solving this problem by Ganesan's method [7] gives the optimal fuzzy solution and the optimal objective fuzzy value as follows:

$$\tilde{x}_6 \approx \tilde{0}, \tilde{x}_7 \approx \tilde{0}, \tilde{x}_8 \approx (4,8,2,2),$$
$$\tilde{x}_8 \approx (1,5,1,1), \tilde{x}^0 \approx (5,12,3,3).$$

Complementary slackness gives the dual solution as $\tilde{\tilde{v}} = (\tilde{\tilde{v}}_1, \tilde{\tilde{v}}_1) = ((1,1,0,0),(1,1,0,0))$. So, we have $\tilde{\hat{v}}a_1 \approx (3,3,0,0) \succ \tilde{0}$, $\tilde{\hat{v}}a_2 \approx \tilde{0}$, $\tilde{\hat{v}}a_3 \approx (3,3,0,0) \succ \tilde{0}$, $\tilde{\hat{v}}a_4 \approx (7,7,0,0) \succ \tilde{0}$ and $\tilde{\hat{v}}a_5 \approx (-5,-5,0,0) \prec \tilde{0}$. Thus, α is determined as follows:

$$\alpha = \min\left\{-\frac{-5-1}{3+3}, -\frac{-7-5}{3+3}, -\frac{-6-8}{7+7}\right\} = 1,$$

and we replace $\tilde{w}$ by

$$\begin{aligned} &((0,0,0,0),(0,0,0,0)) + 1((1,1,0,0),(1,1,0,0)) \\ &= ((1,1,0,0),(1,1,0,0)) \end{aligned}.$$

Second iteration: Recomputing of Ω with new dual solution $\tilde{w} \approx ((1,1,0,0),(1,1,0,0))$, gives $\Omega = \{1,4\}$ and the following new restricted fuzzy primal problem:

$$\begin{aligned} \min \quad & (1,1,0,0)\tilde{x}_8 + (1,1,0,0)\tilde{x}_9 \\ \text{s.t.} \quad & 2\tilde{x}_1 + 6\tilde{x}_4 + \tilde{x}_8 \approx (4,8,2,2) \\ & \tilde{x}_1 + \tilde{x}_4 + \tilde{x}_9 \approx (1,5,1,1) \\ & \tilde{x}_1, \tilde{x}_4, \tilde{x}_8, \tilde{x}_9 \succeq \tilde{0} \end{aligned} \tag{15}$$

The optimal fuzzy solution to this problem is given by

$$\tilde{x}_1 \approx (2,4,1,1),\ \tilde{x}_2 \approx \tilde{x}_8 \approx \tilde{x}_9 \approx \tilde{0}$$

with $\tilde{x}^0 \approx \tilde{0}$. Thus, we have an optimal solution to the main problems as follows:

$$\tilde{x}_1 \approx (2,4,1,1),\ \tilde{x}_2 \approx \tilde{x}_3 \approx \tilde{x}_4 \approx \tilde{x}_5 \approx \tilde{x}_6 \approx \tilde{x}_7 \approx \tilde{0}.$$

Remark 4.1. If we want to solve the problem (11) directly by use of Algorithm 3.1 proposed by Ganesan and Veeramani [1], we must first solve the following linear programming problem with introducing the slack variables x_6 and x_7 and the fuzzy artificial fuzzy variables x_8 and x_9, which minimize the sum of the artificial fuzzy variables to obtain a initial fuzzy basic feasible solution:

$$\begin{aligned} \min \quad & \tilde{z} \approx (1,1,0,0)\tilde{x}_8 + (1,1,0,0)\tilde{x}_9 \\ \text{s.t.} \quad & 2\tilde{x}_1 - \tilde{x}_2 + \tilde{x}_3 + 6\tilde{x}_4 - 5\tilde{x}_5 - \tilde{x}_6 + \tilde{x}_8 \approx (4,8,2,2) \\ & \tilde{x}_1 + \tilde{x}_2 + 2\tilde{x}_3 + \tilde{x}_4 + 2\tilde{x}_5 - \tilde{x}_7 + \tilde{x}_9 \approx (1,5,1,2) \\ & \tilde{x}_1, \tilde{x}_2, \tilde{x}_3, \tilde{x}_4, \tilde{x}_5, \tilde{x}_6, \tilde{x}_7, \tilde{x}_8, \tilde{x}_9 \succeq \tilde{0} \end{aligned} \tag{16}$$

After finding a initial fuzzy basic feasible solution by solving the problem (16), we must minimize the original objective function of the problem (11). So, this process is time consuming and has no efficiency computationally for solving such problems in which an initial fuzzy basic solution is not easily at hand.

5. Conclusions

Ganesan and Veeramani in [1] proposed a new approach based on primal simplex algorithm to obtain the fuzzy solution of fuzzy linear programming problem with symmetric trapezoidal fuzzy numbers without converting them to crisp linear programming problems. In this paper, we reviewed the dual of a linear programming problem with symmetric trapezoidal fuzzy numbers. Then, we introduced a fuzzy primal-dual algorithm for solving the FLP problems directly without converting them to crisp linear programming problems, based on the interesting results which have been established by Ganesan and Veeramani [1]. This approach can be expected to be efficient if an initial dual fuzzy solution can be computed readily. This algorithm is also useful specially for solving minimum cost flow problem with fuzzy parameters in which finding an initial dual feasible solution turns out to be a trivial task. However, development of network primal-dual simplex algorithm for solving such problem in fuzzy environment may also produce intersecting results.

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7. References

[1] K. Ganesan and P. Veeramani, "Fuzzy Linear Programming with Trapezoidal Fuzzy Numbers," *Annals of Operations Research*, Vol. 143, No. 1, 2006, pp. 305-315. doi:10.1007/s10479-006-7390-1

[2] A. Ebrahimnejad and S. H. Nasseri, "Linear Programs with Trapezoidal Fuzzy Numbers: A Duality Approach," *International Journal of Operations Research*, in Press.

[3] H. Tanaka and K. Asai, "Fuzzy Linear Programming Problems with Fuzzy Numbers," *Fuzzy Sets and Systems*, Vol. 13, No. 1, 1984, pp. 1-10. doi:10.1016/0165-0114(84)90022-8

[4] J. L. Verdegay, "A Dual Approach to Solve the Fuzzy Linear Programming Problems," *Fuzzy Sets and Systems*, Vol. 14, No. 2, 1984, pp. 131-141. doi:10.1016/0165-0114(84)90096-4

[5] H. J. Zimmermann, "Optimization in Fuzzy Environment," *Presented at XXI International TIMES and 46th ORSA Conference*, San Juan, 8-11 July 1974.

[6] A. Ebrahimnejad, S. H. Nasseri, F. H. Lotfi and M. Soltanifar, "A Primal-Dual Method for Linear Programming Problems with Fuzzy Variables," *European Journal of Industrial Engineering*, Vol. 4, No. 2, 2010, pp. 189-209. doi:10.1504/EJIE.2010.031077

[7] A. Ebrahimnejad, S. H. Nasseri and F. H. Lotfi, "Bounded Linear Programs with Trapezoidal Fuzzy Numbers," *International Journal of Uncertainty, Fuzziness and Knowledge-Based Systems*, Vol. 18, No. 3, 2010, pp. 269-286. doi:10.1142/S0218488510006532

[8] A. Ebrahimnejad and S. H. Nasseri, "A Dual Simplex Method for Bounded Linear Programmes with Fuzzy Numbers," *International Journal of Mathematics in Operational Research*, Vol. 2, No. 6, 2010, pp. 762-779. doi:10.1504/IJMOR.2010.035498

[9] F. H. Lotfi, T. Allahviranloo, M. A. Jondabeh and L. Alizadeh, "Solving a Full Fuzzy Linear Programming Using Lexicography Method and Fuzzy Approximate Solution," *Applied Mathematical Modelling*, Vol. 33, No. 7, 2009, pp. 3151-3156. doi:10.1016/j.apm.2008.10.020

[10] M. Inuiguchi and M. Sakawa, "Possible and Necessary Optimality Tests in Possibilistic Linear Programming Problems," *Fuzzy Sets and Systems*, Vol. 67, No. 1, 1994, pp. 29-46. doi:10.1016/0165-0114(94)90206-2

[11] A. Kumar, J. Kaur and P. Singh, "A New Method for Solving Fully Fuzzy Linear Programming Problems," *Applied Mathematical Modelling*, Vol. 35, No. 2, 2011, pp. 817-823. doi:10.1016/j.apm.2010.07.037

[12] J. L. Verdegay, "A Dual Approach to Solve the Fuzzy Linear Programming Problems," *Fuzzy Sets and Systems*, Vol. 14, No. 2, 1984, pp. 131-141. doi:10.1016/0165-0114(84)90096-4

[13] H. R. Maleki, M. Tata and M. Mashinchi, "Linear Programming with Fuzzy Variables," *Fuzzy Sets and Systems*, Vol. 109, No. 1, 2000, pp. 21-33. doi:10.1016/S0165-0114(98)00066-9

[14] A. Ebrahimnejad, S. H. Nasseri and S. M. Mansourzadeh, "Bounded Primal Simplex Algorithm for Bounded Linear Programming with Fuzzy Cost Coefficients," *International Journal of Operational Research and Information Systems*, Vol. 2, No. 1, 2011, pp. 96-120. doi:10.4018/IJORIS.2011010105

[15] N. Mahdavi-Amiri and S. H. Nasseri, "Duality in Fuzzy Number Linear Programming by Use of a Certain Linear Ranking Function," *Applied Mathematics and Computation*, Vol. 180, No. 1, 2006, pp. 206-216. doi:10.1016/j.amc.2005.11.161

[16] S. H. Nasseri and A. Ebrahimnejad, "A Fuzzy Dual Simplex Method for Fuzzy Number Linear Programming Problem," *Advances in Fuzzy Sets and Systems*, Vol. 5, No. 2, 2010, pp. 81-95.

[17] A. Ebrahimnejad, "Sensitivity Analysis on Fuzzy Number Linear Programming Problems," *Mathematical and Computer Modelling*, Vol. 53, No. 9-10, 2011, pp. 1878-1888. doi:10.1016/j.mcm.2011.01.013

[18] N. Mahdavi-Amiri and S. H. Nasseri, "Duality Results and a Dual Simplex Method for Linear Programming Problems with Trapezoidal Fuzzy Variables," *Fuzzy Sets and Systems*, Vol. 158, No. 17, 2007, pp. 1961-1978. doi:10.1016/j.fss.2007.05.005

[19] N. Mahdavi-Amiri, S. H. Nasseri and A. Yazdani, "Fuzzy Primal Simplex Algorithms for Solving Fuzzy Linear Programming Problems," *Iranian Journal of Operational Research*, Vol. 1, No. 2, 2009, pp. 68-84.

[20] S. H. Nasseri and A. Ebrahimnejad, "A Fuzzy Primal Simplex Algorithm and Its Application for Solving Flexible Linear Programming Problems," *European Journal of Industrial Engineering*, Vol. 4, No. 3, 2010, pp. 372-389. doi:10.1504/EJIE.2010.033336

[21] A. Ebrahimnejad and S. H. Nasseri, "Using Complementary Slackness Property to Solve Linear Programming with Fuzzy Parameters," *Fuzzy Information and Engineering*, Vol. 1, No. 3, 2009, pp. 233-245. doi:10.1007/s12543-009-0026-9

[22] S. H. Nasseri and N. Mahdavi-Amiri, "Some Duality Results on Linear Programming Problems with Symmetric Fuzzy Numbers," *Fuzzy Information and Engineering*, Vol. 1, No. 1, 2009, pp. 59-66. doi:10.1007/s12543-009-0004-2

[23] S. H. Nasseri, A. Ebrahimnejad and S. Mizuno, “Duality in Fuzzy Linear Programming with Symmetric Trapezoidal Numbers,” *Applications and Applied Mathematics*, Vol. 5, No. 10, 2010, pp.1467-1482.

[24] A. Zadeh, “Fuzzy Sets,” *Information and Control*, Vol. 8, No. 3, 1965, pp. 338-353. doi:10.1016/S0019-9958(65)90241-X