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Evolution of Weak Shock Waves in Perfectly Conducting Gases

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Abstract

This article aims at studying one dimensional unsteady planar and cylindrically symmetric flow involving shocks under the influence of magnetic field. The method of generalized wavefront expansion (GWE) is employed to derive a coupled system of nonlinear transport equations for the jump of field variables and of its spatial derivatives across the shock, which, in turn determine the evolution of wave amplitude and admit a solution that agrees with the classical decay laws of weak shocks. A closed form solution exhibiting the features of nonlinear steepening of the wave front. A general criterion for a compression wave to steepen into a shock is derived. An analytic expression elucidating how the shock formation distance is influenced by the magnetic field strength is obtained. Also, the effects of geometrical spreading and nonlinear convection on the distortion of the waveform are investigated in the presence of magnetic field.

Keywords: Magnetogasdynamic Flow, Weak Shock, Induced Discontinuity, Generalized Wavefrontexpansion (GWE)

1. Introduction

To study the propagation of nonlinear waves, such as shock waves and acceleration waves, which belong to the singular surface theory, have been, and remain, the topic of considerable interest in continuum mechanics [1-6]. This is primarily due to the fact that when the nonlinearities are present in the governing equations, these waves can manifest a wide range of behaviors, the most striking one is finite time blow up. Thomas [7] appears to have been one of the first investigator describing the evolutionary behavior of acceleration waves in flows of inviscid ideal gases. Later, Coleman and Gurtin [8] studied the problem of acceleration waves and higher order waves (or 'mild discontinuities' as they are named by the authors) in fluids that exhibit mechanical dissipation via the relaxation of internal state variables. However, these authors went on to conjecture that blow-up of acceleration wave implies that a shock wave, which is a propagating jump in at least one of the acoustic field variables themselves, had in fact formed. Since then extensive investigations on singular surfaces and the phenomena of finite time blowup of acceleration waves have been documented in the literature of continuum mechanics (see [9-13] and those cited therein). Recently, Christov *et al*. [14] have considered the nonlinear acoustic propagation in homentropic prefect gas flow and provided numerical support for the shock-conjecture of Coleman and Gurtin [8]. Still more recently, Shekhar and Sharma [15] have studied the propagation of weak discontinuities in shallow water and their subsequent culmination into shock waves.

In contrast to acceleration wave, which is defined as the propagating jump discontinuity in at least one of the first derivative of the field variables, with the field variables themselves being continuous, the shock waves are very difficult to analyze because their evolutionary behavior is always coupled with that of the higher order discontinuities that accompany them. Whereby, the evolutionary behavior of shock amplitude is governed by an equation that also involves the amplitude of the accompanying second order discontinuity. One can derive another evolution equation for the amplitude of the accompanying second order discontinuity, but this equation involves the amplitude of the accompanying third order discontinuity. This procedure could be carried out to higher order derivatives and thus one obtains infinite number of transport equations. It was Maslov [16] who proposed for the first time the idea of an infinite system of compatibility conditions and provided a rigorous mathematical approach to describe the kinematics of a weak shock wave propagating through an inviscid, isentropic gas; using the theory of generalized functions. He derived an infinite set of identities for the shock amplitude and higher order derivatives of field variables which hold along the rays. Maslov's work presents a clear understanding of the problem mathematically and can be regarded as a major breakthrough in approximate determination, at least in theory, of the shock position. Similar method has been developed by Grinfeld [17] to study weak shocks in elastic materials.

Another approximate analytical method for studying the kinematics of weak shock, called generalized wavefront expansion, has been proposed by Anile [18] and is based on an asymptotic expansion in a neighborhood of the wavefront. Russo [19] applied this method to a rather simple case of single wave equation in one space dimension and made a comparison with the shock fitting method given by Whitham [1]. Further, Anile and Russo [20] extended this method to higher order corrections and derived an infinite hierarchy of coupled transport equations along the wavefront (rays) for the shock amplitude and the jumps of the field gradients. In this context Madhumita and Sharma [21] employed a different approach to describe the kinematics of a shock wave of arbitrary strength by considering an infinite sequence of transport equations for the variation of jump in the field variable and their space derivative across the shock, and used a truncation procedure similar to that proposed by Maslov.

The present work, which deals with the unsteady planar and cylindrically symmetric flow of an inviscid gas under the influence of magnetic field, derives motivation from the study relating to the propagation of weak shock proposed by Anile [18]. The method of generalized wavefront expansion is used to analyze the main features of weakly nonlinear waves propagating in an electrically conducting gas permeated by a transverse magnetic field. It is assumed that the fluid ahead of the shock is at rest and the dissipative effects, except due to the magnetic field, are negligible.

A system of two transport equations, coupled through the amplitude of accompanying discontinuity, is derived along the rays of governing equations; these equations effecttively describe the evolutionary behavior of shock front. The location of shock formation, *i.e.* the point where the characteristics begin to coalesce, is determined. Also, the influence of the magnetic field on the nonlinear distortion of the wave form and the shock formation distance is assessed.

2. Formulation of the Problem

In carrying out the analytical part of our study, it is convenient to treat the wave phenomena as being kinematic Whitham [1], rather than dynamic. Mathematically, this means recasting the equations, governing a physical phenomenon, as a system consisting of a 'conservation/balance law' and a 'flux' relation. Omitting the details, it is not difficult to establish that the fundamental equations, describing the nonlinear wave process, for one dimensional planar or cylindrically symmetric motion of an ideal gas in the presence of magnetic field can be modeled as Sharma *et al.* [22]

$$\frac{\partial F(\boldsymbol{U})}{\partial t}+\frac{\partial G(\boldsymbol{U})}{\partial x}+f(\boldsymbol{U})=0, \tag{1}$$

where $\boldsymbol{U}\equiv\boldsymbol{U}^i, 1\le i\le 4$, is the column vector representing dependent field variables,

$$\boldsymbol{U}=(\rho,u,p,h)^{\mathrm{T}}\ F(\boldsymbol{U})=\left(\rho,\rho u,\frac{p}{\gamma-1}+\frac{\rho u^2}{2}+h,h^{1/2}\right)^{\mathrm{T}},$$

$$G(\boldsymbol{U})=\left(\rho u,\rho u^2+p+h,\frac{\gamma pu}{\gamma-1}+\frac{\rho u^3}{2}+2hu,h^{1/2}u\right)^{\mathrm{T}},$$

$$f(\mathrm{U})=$$

$$\left(\frac{m\rho u}{x},\frac{m\rho u^2}{x},\frac{\gamma mpu}{(\gamma-1)x}+\frac{m\rho u^3}{2x}+\frac{2mhu}{x},\frac{muh^{1/2}}{x}\right)^{\mathrm{T}}.$$

Here, it is assumed that the electrical conductivity of the medium is infinite and the direction of the magnetic field is orthogonal to the trajectories of the fluid particles. The field variables ρ, u and p denote, respectively, the fluid density, velocity and pressure; h is the magnetic pressure defined as $h=\mu H^2/2$ with μ as the magnetic permeability and H the transverse magnetic field. The variables x and t, respectively, are the space and time coordinates and γ is the adiabatic index. We consider that initially the wave propagation takes place into a uniform state characterized by a flow field at rest with constant density and pressure fields, namely $\boldsymbol{U}_+=(\rho_0,0,p_0,h_0)^{\mathrm{T}}$. Hereafter the subscript "0" refers to evaluation at the uniform state unless stated otherwise.

It is well known that a system of equations written in the form (1) admits a shock wave that may be initiated in the flow region, and once it is formed, it will propagate by separating the portions of continuous regions. Let $x=\chi(t)$ represents the location of the moving shock front at any time t across which the flow variables and their derivatives suffer finite jump discontinuities, then a shock wave solution of (1) may exist if across the surface of discontinuity $\chi(t)$, following Rankine-Hugoniot conditions are satisfied,

$$-V\left[F(\boldsymbol{U})\right]+\left[G(\boldsymbol{U})\right]=0 \tag{2}$$

where $V=\mathrm{d}\chi/\mathrm{d}t$ is the speed of propagation of the wavefront into the medium characterized by U_+. The square brackets enclosing an entity denote the amplitude of the jump in that entity across the shock front $\chi(t)$, defined as $[\boldsymbol{U}]=\boldsymbol{U}-\boldsymbol{U}_+$. In this context we usually call U_+ the unperturbed field and $\boldsymbol{U}$ the perturbed field, which correspond, respectively, the states just ahead and behind the shock front $\chi(t)$. If it is assumed that the discontinuity $[\boldsymbol{U}]$ across $\chi(t)$ is a '$k-shock$' (see Jeffrey [4]), then there exist an eigenvalue of (1), say $\lambda^{(k)}$, such that, $\lambda^{(k)}<V$ and $\lim_{V\to\lambda^{(k)}}\boldsymbol{U}=\boldsymbol{U}_+$.

Assume that, over some finite time interval $[t_0,t]$, the following asymptotic expansion hold (see Anile [18])

$$[\boldsymbol{U}]=\sum_{l=1}^{\infty}\varepsilon^l Y_l(t),\ \left[\frac{\partial^m \boldsymbol{U}}{\partial x^m}\right]=\sum_{j=0}^{\infty}\varepsilon^j Y_j^{(m)}(t), \tag{3}$$

where $m=1,2,3\cdots$.

Then, because of the analyticity of $F(\boldsymbol{U})$ and $G(\boldsymbol{U})$ they can be expanded, behind the shock, in terms of small parameter ε; therefore for any quantity $q(\boldsymbol{U})$ (either of $F(\boldsymbol{U})$ or $G(\boldsymbol{U})$), we have

$$q(U)=q_+ +\varepsilon(\nabla_U q)_+ Y_1+\varepsilon^2(\nabla_U q)_+ Y_2 + \frac{\varepsilon^2}{2}(\nabla_U(\nabla_U q))_+ Y_1 Y_1+O(\varepsilon^3) \tag{4}$$

3. Evolution Law for Weak Shocks

In this section, we employ the elegant theory of Generalized Wavefront Expansion (GWE) Anile [18], to determine how an initial jump discontinuity in flow variables propagate and evolves over time. Equation (1) can be recast into a quasilinear hyperbolic system of first order PDEs as

$$\frac{\partial \boldsymbol{U}}{\partial t}+\boldsymbol{A}(\boldsymbol{U})\frac{\partial \boldsymbol{U}}{\partial x}+\boldsymbol{B}(\boldsymbol{U})=0 \quad , \tag{5}$$

where, $\boldsymbol{A}\equiv \boldsymbol{A}_{ij}=\boldsymbol{M}^{-1}\boldsymbol{N}$ is a 4×4 matrix having the non zero components

$$A_{11}=A_{22}=A_{33}=A_{44}=u\,,\ A_{12}=\rho\,,$$

$$A_{23}=A_{24}=\rho^{-1},\ A_{32}=\rho a^2,\ A_{42}=2h\,,$$

and $\boldsymbol{B}\equiv \boldsymbol{M}^{-1}f(\boldsymbol{U})=\left(\dfrac{m\rho u}{x},0,\dfrac{\gamma mpu}{x},\dfrac{2muh}{x}\right)^{\mathrm{T}}$,

where, $\boldsymbol{M}$ and $\boldsymbol{N}$ are the Jacobian Matrices defined as, $\boldsymbol{M}=\nabla_U F$ and $\boldsymbol{N}=\nabla_U G$.

Clearly, eigenvalues of the coefficient matrix $\boldsymbol{A}(\boldsymbol{U})$ are $\lambda^{(1,2)}=u\pm c$ and $\lambda^{(3,4)}=u$. Among them two (*i.e.* $\lambda^{(1,2)}$) represent the waves propagating in $\pm x$ direction with the speed $u\pm c$, where $c=(a^2+b^2)^{1/2}$ represents the magneto-acoustic speed with $b=(2h/\rho)^{1/2}$ as the Alfvén velocity. The remaining two characteristics represent entropy waves or particle paths propagating with fluid velocity.

As stated earlier, let a '$k-shock$' corresponds to eigenvalue $\lambda^{(1)}$, then it is possible to write

$$V=\lambda^{(1)}+\varepsilon v \tag{6}$$

With the foregoing assumptions, we set ourselves for the task of determining the evolution law of weak shocks and assessing the magnetic field effects on the process of shock formation. The first step is to use Equations (3) and (4) in (2) and equating to zero the coefficients of like orders of ε, we obtain the following equation governing the variables of first order

$$(\boldsymbol{A}-\lambda^{(1)}\boldsymbol{I})\boldsymbol{Y}_1=0\,,$$

which implies that $\boldsymbol{Y}^1$ is an eigenvector of $\boldsymbol{A}(\boldsymbol{U})$ corresponding to the eigenvalue $\lambda^{(1)}$. If the left and right eigenvectors of $\boldsymbol{A}$ corresponding to the eigenvalue $\lambda^{(1)}$ are

$$\boldsymbol{L}=(0,c/2,1/2\rho,1/2\rho)$$

$$\boldsymbol{R}=\left(\rho/c^2,1/c,\rho a^2/c,\rho b^2/c\right)^{\mathrm{T}}.$$

Then it is possible to write

$$\boldsymbol{Y}_1=\pi(t)\boldsymbol{R}\quad , \tag{7}$$

with $\pi(t)$ as the amplitude of a right running shock front impinging on the state $\boldsymbol{U}_+$ with the speed $\lambda^{(1)}=c_0$ and to be determined later.

Also, from the jump conditions, to the second order of ε, we get,

$$v=\frac{\pi(t)}{2}\boldsymbol{L}\boldsymbol{M}^{-1}\left\{(\nabla_U \boldsymbol{N})\boldsymbol{R}\boldsymbol{R}-\lambda^{(1)}(\nabla_U \boldsymbol{M})\boldsymbol{R}\boldsymbol{R}\right\}. \tag{8}$$

From the kinematics of singular surfaces, it is known that the following compatibility relation must hold at the wavefront Achenbach [2], Thomas [6]

$$\frac{\delta}{\delta t}[f]=[\partial_t f]+V[\partial_x f] \tag{9}$$

where, $\delta/\delta t$ is the Thomas displacement derivative and provides the time rate of change measured by an observer travelling with $\chi(t)$. Taking the jump in the field Equation (5), which is permissible since it is assumed that the equations holds on both sides of $\chi(t)$, and using the compatibility relation (9) we get

$$\frac{\delta[\boldsymbol{U}]}{\delta t}+\left\{\boldsymbol{A}(\boldsymbol{U})-V\boldsymbol{I}\right\}[\partial_x \boldsymbol{U}]+[\boldsymbol{A}(\boldsymbol{U})](\partial_x \boldsymbol{U})_+ +[\boldsymbol{B}(\boldsymbol{U})]=0 \tag{10}$$

Introducing expansions (3-4) into (10) and equating to zero, the coefficients of the various powers of ε one obtains,

$$\varepsilon^0:\ (\boldsymbol{A}(\boldsymbol{U}_+)-\lambda^{(1)}\boldsymbol{I})\boldsymbol{Y}_0^{(1)}=0\,,$$

which implies,

$$\boldsymbol{Y}_0^{(1)} = \psi(t)\boldsymbol{R} \tag{11}$$

$$\varepsilon^1: \begin{array}{l} \dfrac{\mathrm{d}\boldsymbol{Y}_1}{\mathrm{d}t} + \left(\boldsymbol{A}(\boldsymbol{U}_+) - \lambda^{(1)}\boldsymbol{I}\right)\boldsymbol{Y}_1^{(1)} + \left((\nabla_U \boldsymbol{A})_+ \boldsymbol{Y}_1 - v\boldsymbol{I}\right)\boldsymbol{Y}_0^{(1)} \\ + (\nabla_U \boldsymbol{A})_+ \boldsymbol{Y}_1 (\partial_x \boldsymbol{U})_+ + (\nabla_U \boldsymbol{B})_+ \boldsymbol{Y}_1 = 0 \end{array}.$$

Here, $\mathrm{d}/\mathrm{d}t = \partial_t + \lambda^{(1)}\partial_x$ is the derivative taken along the rays, which implies that, $\delta/\delta t = \mathrm{d}/\mathrm{d}t + \varepsilon v \partial_x$.

The above equation, after left multiplication by L and using Equations (7), (8) and (11), yields the following ordinary differential equation

$$\frac{\mathrm{d}\pi}{\mathrm{d}t} + \Phi\pi + \frac{\Gamma}{2c_0}\psi\pi = 0 \tag{12}$$

where, $\Phi = mc_0/2x$ and $\Gamma = \left(3/2 + (\gamma-2)/2\alpha^2\right)$, with $\alpha = c_0/a_0$ as the Alfvén number.

In deriving Equation (12), we have made use of the relation, $L(\nabla_U A)R = \left(\nabla_U \lambda^{(1)}\right)_+$. It evidently follows from Equation (12) that the temporal evolution of shock amplitude at any time t depends not only on the shock strength, its curvature and the dissipation on account of applied magnetic field but also on the function ψ . Since the quantity ψ is still unknown, this equation is, however, unable to give an analytical description of the complete evolutionary behavior of the wave front. We, therefore, need to work out certain aspects in more detail.

4. Evolution of Accompanying Discontinuity

In order to proceed further, it is necessary to obtain a transport equation for ψ , which may be think of as the amplitude of jump in the slope of unsteady disturbance at the wave front. To achieve this goal, we proceed as follows:

We differentiate Equation (5) with respect to x ahead and behind the shock and subtract these equations written ahead and behind the shock. The resulting equation, after using equations the compatibility relation (9) gives

$$\begin{aligned} &\frac{\delta[\partial_x \boldsymbol{U}]}{\delta t} + \left(\boldsymbol{A}(\boldsymbol{U}) - VI\right)\left[\partial_x^2 \boldsymbol{U}\right] \\ &+ (\nabla_U \boldsymbol{A})[\partial_x \boldsymbol{U}][\partial_x \boldsymbol{U}] + (\nabla_U \boldsymbol{A})[\partial_x \boldsymbol{U}](\partial_x \boldsymbol{U})_+ \\ &+ (\nabla_U \boldsymbol{A})(\partial_x \boldsymbol{U})_+[\partial_x \boldsymbol{U}] + [\boldsymbol{A}(\boldsymbol{U})](\partial_x^2 \boldsymbol{U})_+ \\ &+ [\nabla_U \boldsymbol{A}](\partial_x \boldsymbol{U})_+ (\partial_x \boldsymbol{U})_+ + (\nabla_U \boldsymbol{B})[\partial_x \boldsymbol{U}] \\ &+ [\nabla_U \boldsymbol{B}](\partial_x \boldsymbol{U})_+ = 0. \end{aligned}$$

The above equation after using expansions (3-4) and left multiplying by L , while zeroth order terms only are retained, yields the following compatibility equation

$$\frac{\mathrm{d}\psi}{\mathrm{d}t} + \Phi\psi + \frac{\Gamma}{c_0}\psi^2 = 0 \tag{13}$$

It follows immediately that Equation (13) does not contain any unknown term and thus the system of compatibility equations for weak shocks is closed at the second compatibility equation. Also, it may be noticed that the evolution Equation (16) contains only the zeroth order terms therefore the derivation is valid only if the amplitude of accompanying discontinuities are of $O(1)$.

It is interesting to notice that Equation (13) is in the form of Bernoulli type equation which governs the evolutionary behavior of acceleration waves in nonlinear material media and elsewhere. In (13) the linear term $\Phi\psi$ depends upon the unperturbed conditions of the medium and it takes into account the gradient in the flow variables as well as the geometry of the problem. The negative value of ψ corresponds for compression waves and positive for expansion waves. The coefficient of nonlinear term, Γ , which is positive for most of the fluids, is responsible for the nonlinear steepening of the wave front. However, in past years, fluids with negative nonlinearity have been found Murlidharan and Sujith [23]. The present paper deals with positive values of Γ . Thus, the nonlinear term makes a negative value of ψ more negative and a positive value of ψ less positive, that is, the nonlinearity alone causes a compression wave to steepen and an expansion wave to relax.

5. Results and Discussion

As Equation (13) is of the form which describes the evolutionary behavior of wave amplitude in various gasdynamic regimes [9-13] and therefore the analysis of this equation concerning the local behavior of wave amplitude follows on parallel lines. However, to investigate the magnetic field effects on the process of nonlinear steepening of wave front we rewrite Equations (12) and (13) in non-dimensional form as

$$\frac{\mathrm{d}\tilde{\pi}}{\mathrm{d}\tilde{x}} + \tilde{\pi}\left(\tilde{\Phi} + \tilde{\Gamma}\tilde{\psi}/2\right) = 0 \tag{14}$$

$$\frac{\mathrm{d}\tilde{\psi}}{\mathrm{d}\tilde{x}} + \tilde{\Phi}\tilde{\psi} + \tilde{\Gamma}\tilde{\psi}^2 = 0 \tag{15}$$

where, $\tilde{\pi}$, $\tilde{\psi}$, $\tilde{\Phi}$, $\tilde{\Gamma}$ and $\tilde{x}$ are dimensionless quantities defined as $\tilde{\pi} = \pi/\pi_0$, $\tilde{\psi} = \psi/\psi_0$, $\tilde{x} = x/x_0$ $\tilde{\Phi} = m/2\tilde{x}$ and $\tilde{\Gamma} = \left(3/2 + (\gamma-2)/2\alpha^2\right)\alpha^{-2}\Theta$ with $\alpha = c_0/a_0$ as the Alfvén number and $\Theta = \psi_0 x_0/a_0^2$ as the dimensionless measure of the strength of the initial jump discontinuity in the field gradients. Indeed, these two transport equations govern the evolutionary behavior of the shock amplitude as well as the jump in the first order gradient of the field variables. In deriving Equations (14) and (15) we have made use of the characteristic relation

$$\frac{\mathrm{d}x}{\mathrm{d}t} = \lambda_0^{(1)} \tag{16}$$

Performing required integration subject to the initial conditions for $\tilde{\psi}$ and $\tilde{\pi}$ at $\tilde{x}=1$, say $\tilde{\psi}=1$ and $\tilde{\pi}=1$, the Equations (14) and (15) along with the characteristic relation (16) yield the following solutions.

Plane case $(m=0)$: Integrating Equations (14) and (15) subject to the above initial conditions yield

$$\tilde{\psi}=\left(1+\Gamma\Theta\left(\tilde{x}-1\right)\alpha^{-2}\right)^{-1}, \quad (17)$$

$$\tilde{\pi}=\left(1+\Gamma\Theta\left(\tilde{x}-1\right)\alpha^{-2}\right)^{-1/2}. \quad (18)$$

Cylindrical case $(m=1)$: In this case the solution of Equations (14) and (15) takes the following form

$$\tilde{\psi}=\tilde{x}^{-1/2}\left\{1+2\Gamma\Theta\alpha^{-2}\left(\tilde{x}^{1/2}-1\right)\right\}^{-1} \quad (19)$$

$$\tilde{\pi}=\left\{\tilde{x}\left(1+2\Gamma\Theta\alpha^{-2}\left(\tilde{x}^{1/2}-1\right)\right)\right\}^{-1/2} \quad (20)$$

5.1. Nonlinear Steepening of the Wave Front

Since Γ and α are positive quantities and $\left(\tilde{x}^{1/2}-1\right)$ is an increasing function of $\tilde{x}$, therefore it follows from (17)-(20) that the behavior of $\tilde{\pi}$ and $\tilde{\psi}$ will depend on sign of ψ_0 and hence that of Θ. It is evident from Equations (17)-(20) that if ψ_0 is positive (*i.e.* an expansion wave front with $\Theta>0$), $\tilde{\pi}$ as well as $\tilde{\psi}$ decreases as the expansion wave advances in x direction. While for negative values of ψ_0 (*i.e.* a compression wave with $\Theta<0$), $\tilde{\pi}$ and $\tilde{\psi}$ increases monotonically and the solution provided by (17)-(20) no longer remains valid and steepens into a shock wave after a finite running length $\tilde{x}_s$. In fact, the weak shock assumption breaks down before $\tilde{\pi}$ and $\tilde{\psi}$ approach to infinitely large values and existence of the distance $\tilde{x}_s$ may be regarded as an indication of this. A simple physical explanation of the appearance of shock wave may be that it is formed owing to the inertial overtaking of flow particles, that is when the first characteristic could intersect the successive one. Thus a shock can form only when initial disturbance is compressive and the corresponding shock formation distance $\tilde{x}_s$ in the above two cases (for $\Theta<0$) are given by

Plane case: $\tilde{x}_s=1-\alpha^2/\Gamma\Theta$,
Cylindrical case: $\tilde{x}_s^{1/2}=1-\alpha^2/2\Gamma\Theta$.

5.2. Comparison with Exact Results for Decay of Weak Shocks

For large values of $\tilde{x}$ the asymptotic behavior of plane and cylindrical shock waves is shown in the flollowing table.

From **Table 1** it is clear that for plane waves $\tilde{\psi}\propto\tilde{x}^{-1}$ and $\tilde{\pi}\propto\tilde{x}^{-1/2}$ therefore, width of the plane shock, *i.e.* the distance between shock front and tail of the rarefaction wave increases like $\tilde{x}^{1/2}$. Also, for cylindrical waves, $\tilde{\psi}\propto\tilde{x}^{-1}$ and $\tilde{\pi}\propto\tilde{x}^{-3/4}$ therefore, width of the cylindrical shock waves increase like $\tilde{x}^{1/4}$. These results are in closed agreement with the earlier results obtained by Whitham [1, pp. 312-322], Courant and Friedrichs [24, pp. 164-168] and by Landau [25].

Table 1. Decay behavior of weak shock waves and first order discontinuities.

	Shock strength	First order discontinuity	Shock width varies as
Plane case	$\tilde{\pi}\propto\tilde{x}^{-1/2}$	$\tilde{\psi}\propto\tilde{x}^{-1}$	$\tilde{x}^{1/2}$
Cylindrical case	$\tilde{\pi}\propto\tilde{x}^{-3/4}$	$\tilde{\psi}\propto\tilde{x}^{-1}$	$\tilde{x}^{1/4}$

Now our objective is to investigate how the nonlinear steepening or flattening of the wave form is influenced by the presence of magnetic field. For the sake of comparison, the integral curves for Equations (17)-(20) are sketched in the **Figures 1-9**. These curves help to illustrate the effect of magnetic field strength, which enters through an increase in the Alfvén number α, and the geometry of the problem on the nonlinear steepening or flattening of the wave.

Figures 1-4 illustrate the magnetic field effects on the flattening of expansion waves for both the cases, that is, weak shock ($\tilde{\pi}$) and correspnding accompanying discontinuity ($\tilde{\psi}$). It is evidently clear from **Figures 1** and **2** that a cylindrical wave attenuate faster than a plane wave as one would expect and the attenuation rate is decreased by the dissipative mechanism due to presence of magnetic field strength (α) as compared to what it would be in non-magnetic case $(\alpha=1)$. Also, from **Figures 3** and **4** we infer that the amplitude of the first order discontinuity $\tilde{\psi}$, which accompany the weak shock, decays more rapidly as compared to the shock amplitude $\tilde{\pi}$ itself.

As discussed earlier that only compressive waves can evolve into a shock and the corresponding situations are depicted in the **Figures 5-8**, showing thereby that, for compression waves, an increase in the magnetic field strength delays the onset of shock. Also, it is clear from these figures that amplitude of accompanying discontinuity ($\tilde{\psi}$) steepen more rapidly in comparision to the shock amplitude $\tilde{\pi}$ itself.

Figure 9 shows the variation of shock formation distance with changes in the value of the initial jump ψ_0. It may be noticed that an increase in the magnetic field strength enhances the shock formation distance. However, higher values of the initial jump ψ_0 and hence Θ lead to shorter shock formation distance.

6. Conclusions

In this article the interaction between gasdynamic motion

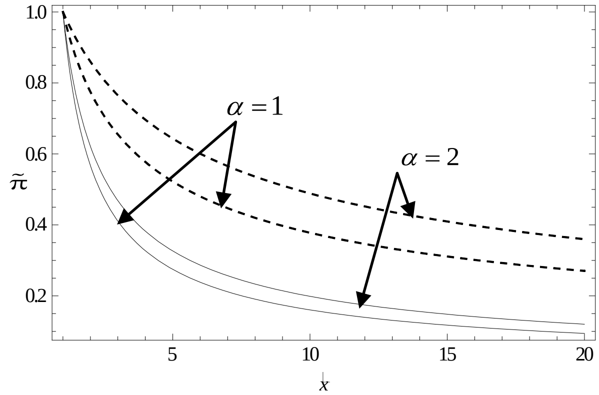

Figure 1. Effect of the magnetic field strength (α) on the flattening of expansion waves for $\gamma = 5/3$ and $\Theta = 0.5$; (a) $m = 0$ (Dashed lines), (b) $m = 1$ (smooth lines).

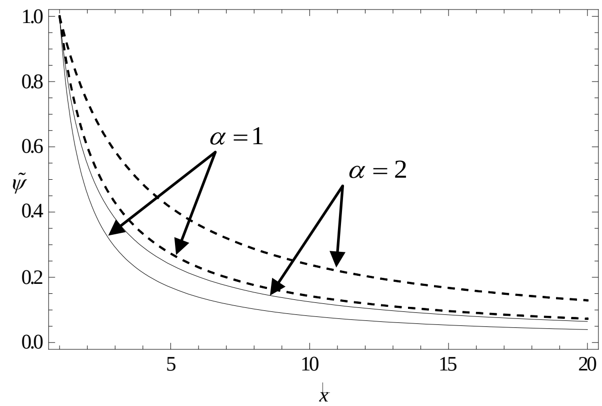

Figure 2. Effect of the magnetic field strength (α) on the flattening of accompanying discontinuity for $\gamma = 5/3$ and $\Theta = 0.5$ (a) $m = 0$ (Dashed lines), (b) $m = 1$ (smooth lines).

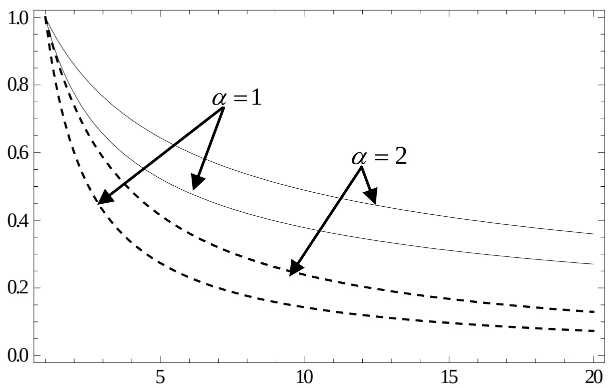

Figure 3. Effect of the magnetic field strength (α) on the flattening of expansion waves for m = 0, $\gamma = 5/3$ and $\Theta = 0.5$; (a) Dashed lines correspond to $\tilde{\psi}$ vs $\tilde{x}$, (b) Smooth lines correspond to $\tilde{\pi}$ vs $\tilde{x}$.

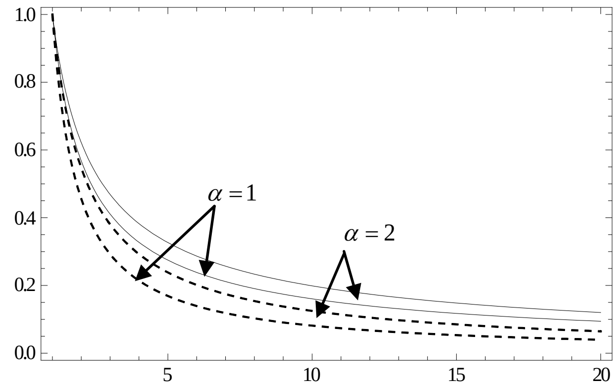

Figure 4. Effect of the magnetic field strength (α) on the flattening of expansion waves for $m = 1$, $\gamma = 5/3$ and $\Theta = 0.5$; (a) Dashed lines correspond to $\tilde{\psi}$ vs $\tilde{x}$, (b) Smooth lines correspond to $\tilde{\pi}$ vs $\tilde{x}$.

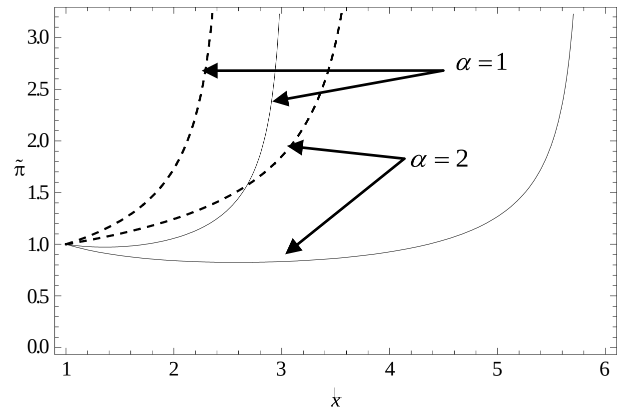

Figure 5. Effect of the magnetic field strength (α) on the growth of compression waves for $\gamma = 5/3$ and $\Theta = -0.5$; (a) $m = 0$ (Dashed lines), (b) $m = 1$ (smooth lines).

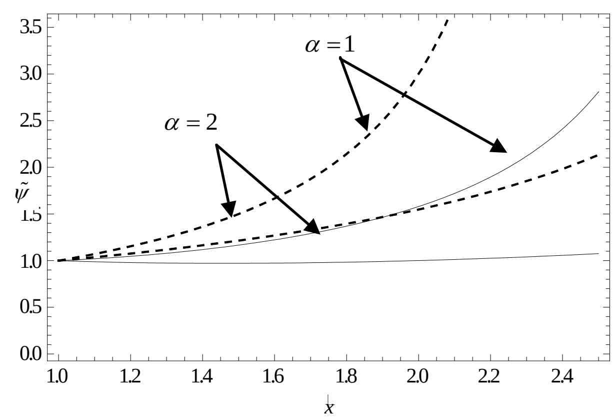

Figure 6. Effect of the magnetic field strength (α) on the steepening of compression waves $(\tilde{\psi})$ for $\gamma = 5/3$ and $\Theta = -0.5$; (a) $m = 0$ (Dashed lines), (b) $m = 1$ (smooth lines).

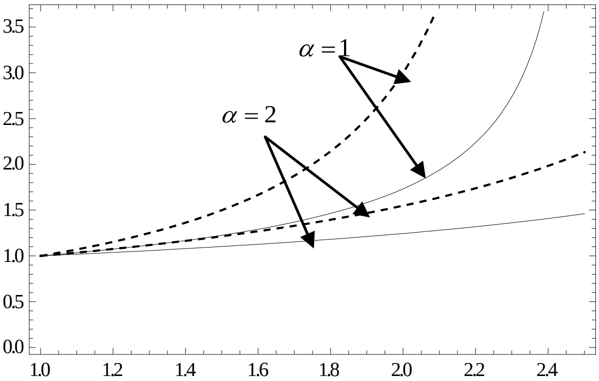

Figure 7. Effect of the magnetic field strength (α) on the growth of compression waves for $m=0$, $\gamma=5/3$ and $\Theta=-0.5$; (a) Dashed lines correspond to $\tilde{\psi}$ vs $\tilde{x}$, (b) Smooth lines correspond to $\tilde{\pi}$ vs $\tilde{x}$.

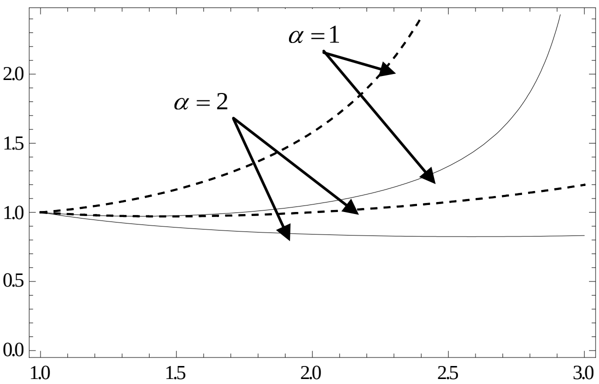

Figure 8. Effect of the magnetic field strength (α) on the growth of compression waves for $m=1$, $\gamma=5/3$ and $\Theta=-0.5$; (a) Dashed lines correspond to $\tilde{\psi}$ vs $\tilde{x}$, (b) Smooth lines correspond to $\tilde{\pi}$ vs $\tilde{x}$.

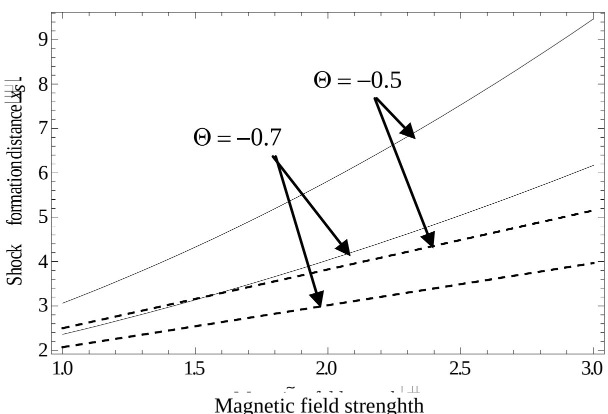

Figure 9. Effect of the magnetic field strength (α) and initial shock strength (Θ) on the shock formation distance ($\tilde{x}_s$) for $\gamma=5/3$; (a) $m=0$ (Dashed lines), (b) $m=1$ (smooth lines).

and magnetic field has been analyzed in detail for the classic problem of propagation of weak shocks in one-dimensional unsteady planar and cylindrically symmetric flows of an inviscid electrically conducting gas. It is assumed that the conductivity of the gas is infinite, and the direction of magnetic field is orthogonal to the trajectories of the fluid particles. Though the mathematics of the governing system of equations is quite complex, the qualitative physical results obtained are remarkably simple.

The method employed in this paper, for investigating general properties of propagating shock waves, is based on an expansion in a neighborhood of the wave front and in a subsequent expansion in terms of the shock amplitude (assumed to be small). To the first order this technique introduces the concept of rays and yields a coupled system of transport equations that hold along the rays of the governing equations. The solutions of this system efficiently describe shock motion and enable us to determine explicitly the position and time of shock formation which also serve as an important parameter in studying the effects of magnetic field strength and the wave front geometry on convective nonlinear steepening and dissipative flattening of the wave which is also illustrated through **Figures 1-9**. It may be noticed that the effects of dissipative mechanism due to the presence of magnetic field is to slow down the decaying process of expansion waves; whereas, it has stabilizing effect on shock formation in the sense that an increase in the magnetic field strength enhances the shock formation distance. Also, it is observed that the decaying of plane and cylindrical shocks varies according to $\tilde{x}^{-1/2}$ and $\tilde{x}^{-3/4}$, respectively; whereas the width of the shock for the above two cases increase like $\tilde{x}^{1/2}$ and $\tilde{x}^{1/4}$. These results are found to be in good agreement with earlier results investigated through various other approaches. We conclude this section with a remark regarding the mathematical structure noticed in Section 3, where the governing system of equations is used to derive transport equations for the jump in flow variables. In fact, in this development it is assumed that the flow on both sides of the shock is smooth. The case in which the flow behind a shock near the triple point is not smooth, yields many technical difficulties is postponed to a future work.

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