Applied Mathematics, 2011, 2, 646-652
doi:10.4236/am.2011.25085 Published Online May 2011 (http://www.SciRP.org/journal/am)

Numerical Solution of a Class of Nonlinear Optimal Control Problems Using Linearization and Discretization

Mohammad Hadi Noori Skandari, Emran Tohidi
Department of Applied Mathematics, School of Mathematical Sciences, Ferdowsi University of Mashhad, Mashhad, Iran
E-mail: {*hadinoori*344, *etohidi*110}*@yahoo.com*
Received March 13, 2011; *revised March* 30, 2011; *accepted April* 4, 2011

Abstract

In this paper, a new approach using linear combination property of intervals and discretization is proposed to solve a class of nonlinear optimal control problems, containing a nonlinear system and linear functional, in three phases. In the first phase, using linear combination property of intervals, changes nonlinear system to an equivalent linear system, in the second phase, using discretization method, the attained problem is converted to a linear programming problem, and in the third phase, the latter problem will be solved by linear programming methods. In addition, efficiency of our approach is confirmed by some numerical examples.

Keywords: Linear and Nonlinear Optimal Control, Linear Combination Property of Intervals, Linear Programming, Discretization, Dynamical Control Systems.

1. Introduction

Control problems for systems governed by ordinary (or partial) differential equations arise in many applications, e.g., in astronautics, aeronautics, robotics, and economics. Experimental studies of such problems go back recent years and computational approaches have been applied since the advent of computer age. Most of the efforts in the latter direction have employed elementary strategies, but more recently, there has been considerable practical and theoretical interest in the application of sophisticated optimal control strategies, e.g., multiple shooting methods [1-4], collocation methods [5,6], measure theoretical approaches [7-10], discretization methods [11,12], numerical methods and approximation theory techniques [13-16], neural networks methods [17-19], etc.

The optimal control problems we consider consist of

1) State variables, *i.e.*, variables that describe the system being modeled;

2) Control variables, *i.e.*, variables at our disposal that can be used to affect the state variables;

3) A state system, *i.e.*, ordinary differential equations relating the state and control variables;

4) A functional of the state variables whose minimization is the goal.

Then, the problems we consider consist of finding state and control variables that minimize the given functional subject to the state system being satisfied. Here, we restrict attention to nonlinear state systems and to linear functionals.

The approach we have described for finding approximate solutions of optimal control problems for ordinary diffrential equations is of the linearize-then-discretize-then-optimize type.

Now, consider the following subclass of nonlinear optimal control problems:

$$\min \int_{t_0}^{t_f} c(t)x(t)\,\mathrm{d}t \tag{1}$$

$$\text{subject to} \quad \begin{aligned} &\dot{x}(t) = A(t)x(t) + h(t,u(t)), \\ &u(t) \in U, t \in [t_0, t_f] \\ &x(t_0) = \alpha,\ x(t_f) = \eta, \end{aligned} \tag{2}$$

where $A(.) \in \mathbb{R}^{n\times n}$, $c(.)$, α and $\eta \in \mathbb{R}^n$ are known, $x(.) \in \mathbb{R}^n$ and $u(.) \in \mathbb{R}^m$ are the state and control variables respectively. It is assumed that U is a compact and connected subset of $\mathbb{R}^m$ and $h(.,.) \in \mathbb{R}^n$ is a smooth or non-smooth continuous function on $[t_0, t_f] \times U$. More-over, there exists a pair of state and control variables $(x(.), u(.))$ such that satisfies (2) and boundary conditions $x(t_0) = \alpha$ and $x(t_f) = \eta$. Here, we use the linear combination property of intervals to convert the nonlinear dynamical control system (2) to the equivalent linear system. The new optimal control problem with this linear dynamical control system is transformed to a discrete-time problem that could be solved by linear programming methods (e.g. simplex method).

There exist some systems containing non-smooth function $h(.,.)$ with regard to control variables. In such systems, multiple shooting methods [1-4] do not dealing with the problem in a correct way. Because, in these methods needing to computation of gradients and hessians of function $h(.,.)$ is necessary. However, considering of non-smoothness of function $h(.,.)$ could not make any difficulty in our approach. Moreover, in another approaches (see [11,12]), which discretization methods are the major basis of them, if a complicated function $h(.,.)$ is chosen, obtaining an optimal solution seems to be difficult. Here, we show that our strategy acquire better solutions, that attained in fewer time, than one of the above-mentioned methods through several simplistic examples, which comparison of the solutions is included in each example.

This paper is organized as follows. Section 2, transforms the nonlinear $h(.,.)$ to a corresponding function

That is linear with respect to a new control variable. In Section 3, the new problem is converted to a discrete-time problem via discretization. In Section 4, numerical examples are presented to illustrate the effectiveness of this proposed method. Finally conclusions are given in Section 5.

2. Linearization

In this section, problems (1)-(2) is transformed to an equivalent linear problem. First, we state and prove the following two theorems:

Theorem 2.1: Let $h_i:\left[t_0,t_f\right]\times U\to\mathbb{R}$ for $i=1,2,\cdots,n$ be a continuous function where U is a compact and connected subset of $\mathbb{R}^m$, then for any arbitrary (but fixed) $t\in\left[t_0,t_f\right]$ the set $\left\{h_i(t,u):u\in U\right\}$ is a closed interval in $\mathbb{R}$.

Proof: Assume that $t\in\left[t_0,t_f\right]$ be given. Let $\phi_i(u)=h_i(t,u)$ for $i=1,2,\cdots,n$. Obviously $\phi_i(.)$ is a continuous function on U. Since continuous functions preserve compactness and connectedness properties, $\left\{\phi_i(u):u\in U\right\}$ is compact and connected in $\mathbb{R}$. Therefore $\left\{h_i(t,u):u\in U\right\}$ is a closed interval in $\mathbb{R}$.

Now, for any $t\in\left[t_0,t_f\right]$, suppose that the lower and upper bounds of the closed interval $\left\{h_i(t,u):u\in U\right\}$ are $g_i(t)$ and $w_i(t)$ respectively. Thus for $i=1,2,\cdots,n$:

$$g_i(t)\le h_i(t,u)\le w_i(t),\ t\in\left(t_0,t_f\right) \tag{3}$$

In other words

$$g_i(t)=\min_u\left\{h_i(t,u):u\in U\right\},\ t\in\left[t_0,t_f\right] \tag{4}$$

$$w_i(t)=\max_u\left\{h_i(t,u):u\in U\right\},t\in\left[t_0,t_f\right] \tag{5}$$

Theorem 2.2: Let functions $g_i(.)$ and $w_i(.)$ for $i=1,2,\cdots,n$ be defined by relations (4) and (5). Then they are uniformly continuous on $\left[t_0,t_f\right]$.

Proof: We will show that $g_i(.)$ for $i=1,2,\cdots,n$ is uniformly continuous. It is sufficient to show that for any given $\varepsilon>0$, there exists $\delta>0$ such that if $s_1\in N_\delta(s_2)$ then $\left|g_i(s_1)-g_i(s_2)\right|<\varepsilon$ where $N_\delta(z)$ is a δ – neighborhood of z. Since any continuous function on a compact set is uniformly continuous, the function $h_i(.,.)$ on the compact set $\left[t_0,t_f\right]\times U$ is uniformly continuous, *i.e.* for any $\varepsilon>0$ there exists $\delta>0$, such that if $(s_1,u)\in N_\delta(s_2,u)$ then $\left|h_i(s_1,u)-h_i(s_2,u)\right|<\varepsilon$. Thus $h_i(s_1,u)<h_i(s_2,u)+\varepsilon$. In addition, by (4), $g_i(s_1)\le h_i(s_1,u)$ and so $g_i(s_1)\le h_i(s_2,u)+\varepsilon$. Now, by taking infimum on the right hand side of the latter inequality $g_i(s_1)\le g_i(s_2)+\varepsilon$. By a similar argument we have also $g_i(s_2)-g_i(s_1)\le\varepsilon$. Thus $\left|g_i(s_1)-g_i(s_2)\right|\le\varepsilon$. The proof of uniformly continuity of $w_i(.)$ for $i=1,2,\cdots,n$ is similar.

By linear combination property of intervals and relation (4), for any $t\in\left[t_0,t_f\right]$:

$$h_i(t,u(t))=\beta_i(t)\lambda_i(t)+g_i(t),\ \lambda_i(t)\in[0,1] \tag{6}$$

where $\beta_i(t)=w_i(t)-g_i(t)$ for $i=1,2,\cdots,n$. Thus, we transform problems (1)-(2) by relations (4), (5) and (6) to the following problem:

$$\min\int_{t_0}^{t_f}c(t)x(t)dt$$

$$\text{subject to}\quad \dot{x}_k(t)=\sum_{r=1}^{n}a_{kr}(t)x_r(t)+\beta_k(t)\lambda_k(t)+g_k(t),$$

$$0\le\lambda_k(t)\le1,\quad t\in\left[t_0,t_f\right],\ k=1,2,\cdots,n$$

$$x(t_0)=\alpha,\ x\left(t_f\right)=\eta \tag{7}$$

where $a_{kr}(.)$ is the k^{th} row and r^{th} column component of matrix $A(.)$. Note that on the problem (7), which is a linear optimal control problem, $\lambda(.)=\left(\lambda_1(.),\lambda_2(.),\cdots,\lambda_n(.)\right)$ is the new control variable.

Next section, converts the latter problem to the corresponding discrete-time problem.

Corollary 2.3: Let the pair of $\left(x^*(.),\lambda^*(.)\right)$ be the optimal solution of problem (7). Then, there exists $u^*(.)$ such that the pair of $(x^*(.),u^*(.))$ is the optimal solution of problems (1)-(2).

Proof: Let $u^*(.)$ satisfies system of (6), where $\lambda^*(.)$ is replaced by $\lambda(.)$. Thus, the pair of $(x^*(.),u^*(.))$ satisfies constraints of problems (1)-(2). Since the objective function of problems (1)-(2) is the same of problem (7), the pair of $(x^*(.),u^*(.))$ is the optimal solution of (1)-(2) evidently.

3. Discrete-Time Problem

Now, discretization method enables us transforming continuous problem (7) to the corresponding discrete form. Consider equidistance points $t_0 = s_0 < s_1 < s_2 < \cdots < s_N = t_f$ on $[t_0, t_f]$ which defined as $s_j = t_0 + \delta j$ for all $j = 0,1,\cdots,N$ with length step $\delta = \frac{t_f - t_0}{N}$ where N is a given large number. We use the trapezoidal approximation in numerical integration and the following approximations to change problem (7) to the corresponding discrete form:

$$\dot{x}_k(s_j) \approx \frac{x_k(s_{j+1}) - x_k(s_j)}{\delta},\ \dot{x}_k(s_N) \approx \frac{x_k(s_N) - x_k(s_{N-1})}{\delta},$$
$$k = 1,2,\cdots,n \quad j = 1,2,\cdots,N-1.$$

Thus we have the following discrete-time problem with unknown variables x_{kj} and λ_{kj} for $k = 1,2,\cdots,n$ and $j = 0,1,2,\cdots,N$:

$$\min \frac{\delta}{2}\sum_{k=1}^{n}\left(c_{k0}\, x_{k0} + c_{kN}\, x_{kN}\right) + \delta\sum_{k=1}^{n}\sum_{j=1}^{N-1} c_{kj}\, x_{kj}$$

subject to

$$\begin{aligned}
& x_{k,j+1} - \left(1+\delta a_{kkj}\right)x_{kj} - \sum_{\substack{r=1\\ r\neq k}}^{n}\delta a_{krj}x_{rj} - \delta\beta_{kj}\lambda_{kj} = \delta g_{kj},\\
& j = 0,1,\cdots,N-1,\ k = 1,2,\cdots,n\\
& \left(1-\delta a_{kkN}\right)x_{kN} - x_{k,N-1} - \sum_{\substack{r=1\\ r\neq k}}^{n}\delta a_{krN}x_{rN} - \delta\beta_{kN}\lambda_{kN} = \delta g_{kN},\\
& k = 1,2,\cdots,n \quad 0 \le \lambda_{kj} \le 1,\quad j = 0,1,\cdots,N,\\
& k = 1,2,\cdots,n \quad x_{k0} = \alpha_k,\quad x_{kN} = \eta_k,\quad k = 1,2,\cdots,n
\end{aligned} \tag{8}$$

where

$$x_{kj} = x_k(s_j),\ c_{kj} = c_k(s_j),\ a_{krj} = a_{kr}(s_j),$$
$$\lambda_{kj} = \lambda_k(s_j),\ g_{kj} = g_k(s_j),\ \beta_{kj} = \beta_k(s_j),$$

for all $k = 1,2,\cdots,n$ and $j = 0,1,\cdots,N$. By solving problem (8), which is a linear programming problem, we are able to obtain optimal solutions λ^*_{kj} and x^*_{kj} for all $j = 1,2,\cdots,N$ and $k = 1,2,\cdots,n$. Note that, for evaluating the control function $u^*(.)$, we must use the following system:

$$h\left(t,u^*(t)\right) = \beta(t)\lambda^*(t) + g(t) \tag{9}$$

Remark 3.1: The most important reason of LCPI (linear combination property of intervals) consideration is that problem (8) is an (finite-dimensional) LP problem and has at least a global optimal solution (by the assumptions of the problems (1)-(2)). However, if problems (1)-(2) be discretized directly then, we reach to an NLP problem which its optimal solution may be a local solution.

Remark 3.2: In Equation (8) if $h(.,.)$ is a well-define function with respect to control $u(.)$ we can obtain optimal control $u^*(.)$ directly. Otherwise, one has to apply numerical technique such as Newton and fixed-point methods for approximating $u^*(.)$ after obtaining $\lambda^*(.)$.

4. Numerical Examples

Here, we use our approach to obtain approximate optimal solutions of the following three nonlinear optimal control problems by solving linear programming (LP) problem (8), via simplex method [20]. All the problems are programmed in MATLAB and run on a PC with 1.8 GHz and 1GB RAM. Moreover, comparisons of our solutions with the method that argued in [11] are included in **Tables 1**, **2** and **3** respectively for each example.

Example 4.1: Consider the following nonlinear optimal control problem:

$$\begin{aligned}
& \min\int_0^1 \sin(3\pi t)\,x(t)\,\mathrm{d}t\\
\text{subject to}\quad & \dot{x}(t) = \cos(2\pi t)\,x(t) - \tan\left(\frac{\pi}{8}u^3(t) + t\right),\\
& 0 \le u(t) \le 1,\quad t\in[0,1]\\
& x(0) = 1,\ x(1) = 0.
\end{aligned} \tag{10}$$

Here, $h(t,u) = -\tan\left(\frac{\pi}{8}u^3 + t\right)$, $c(t) = \sin(3\pi t)$ and $A(t) = \cos(2\pi t)$ for $(t,u)\in[0,1]\times[0,1]$. Thus by (4) and (5) for all $t\in[0,1]$

$$g(t) = \min_{u\in[0,1]}\left\{-\tan\left(\frac{\pi}{8}u^3 + t\right)\right\} = -\tan\left(\frac{\pi}{8} + t\right),$$
$$w(t) = \max_{u\in[0,1]}\left\{-\tan\left(\frac{\pi}{8}u^3 + t\right)\right\} = -\tan(t).$$

Hence

$$\beta(t) = w(t) - g(t) = -\tan(t) + \tan\left(t + \frac{\pi}{8}\right).$$

Let $N=100$. Then $\delta=0.01$ and $s_j=\frac{j}{100}$ for $j=0,1,2,\cdots,100$. The optimal solutions x_j^* and λ_j^*, $j=0,1,2,\cdots,100$. Of problem (10) is obtained by solving problem (8) which is illustrated in **Figures 1** and **2** respectively. Here, the value of optimal solution of objective function is 0.0977. In addition, the corresponding Equation (9) of this example is

$$-\tan\left(\frac{\pi}{8}u_j^{*3}+s_j\right)=\beta(s_j)\lambda_j^*+g(s_j)\quad j=0,1,2,\cdots,100$$

Therefore for $j=0,1,2,\cdots,100$

$$u_j^*=\left(\frac{8}{\pi}\left(\tan^{-1}\left(-\beta(s_j)\lambda_j^*-g(s_j)\right)-s_j\right)\right)^{1/3},$$

The optimal control u_j^*, $j=0,1,2,\cdots,100$ of problem (10) is showed in **Figure 3**.

Example 4.2: Consider the following nonlinear optimal control problem:

$$\min\int_0^1\frac{1}{2}\left(e^{-t}-2t\right)x(t)\,\mathrm{d}t$$

$$\begin{aligned}\text{subject to}\quad &\dot{x}(t)=-tx(t)+\ln\left(u(t)+t+3\right),\\ &u(t)\in[-1,1],\quad t\in[0,1]\\ &x(0)=0,\ \ x(1)=0.8\end{aligned}\tag{11}$$

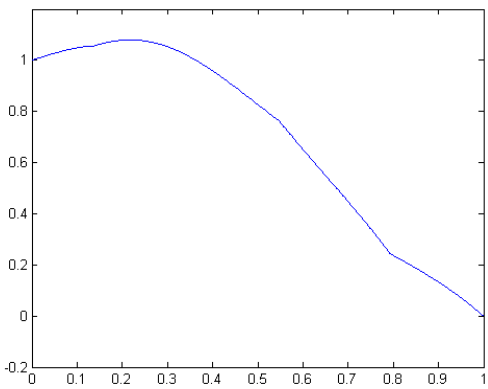

Figure 1. Optimal state $x^*(.)$ of Ex. 4.1.

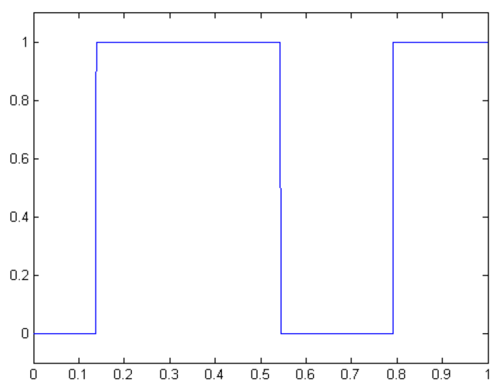

Figure 2. Corresponding optimal control $\lambda^*(.)$ of Ex. 4.1.

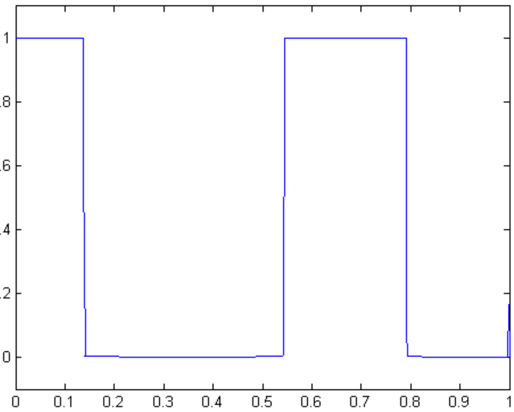

Figure 3. Optimal control $u^*(.)$ of Ex. 4.1.

By relations (4) and (5) for $t\in[0,1]$

$$g(t)=\min_{u\in[-1,1]}\left\{\ln(u+t+3)\right\}=\ln(t+2),$$

$$w(t)=\max_{u\in[-1,1]}\left\{\ln(u+t+3)\right\}=\ln(t+4).$$

Hence

$$\beta(t)=w(t)-g(t)=\ln(t+4)-\ln(t+2)$$

Let $N=100$. Then $\delta=0.01$ and $s_j=\frac{j}{100}$ for all $j=0,1,2,\cdots,100$. We obtain the optimal solutions x_j^* and λ_j^*, $j=0,1,2,\cdots,100$ of this problem by solving corresponding problem (8) which is illustrated in **Figures 4** and **5** respectively. In addition, by relation (9) the corresponding $u^*(.)$ of this example is

$$u_j^*=e^{\beta(s_j)\lambda_j^*+g(s_j)}-s_j-3,\quad j=0,1,2,\cdots,100$$

The optimal controls u_j^*, $j=0,1,2,\cdots,100$ of problem (11) is shown in **Figure 6**. Here, The value of optimal solution of objective function is –0.1829.

Example 4.3: Consider the following nonlinear optimal control problem:

$$\min\int_0^1\left(\left|\sin(2\pi t)\right|-e^{-t}\right)x(t)\,\mathrm{d}t$$

$$\begin{aligned}\text{subject to}\quad &\dot{x}(t)=\left(t^5-t^2+t\right)x(t)-|u(t)|^3\,\mathrm{e}^{\sin(2\pi t)},\\ &u(t)\in[-1,1],\quad t\in[0,1]\\ &x(0)=0.9,\ \ x(1)=0.4\end{aligned}\tag{12}$$

Since $h(t,u)=-|u(t)|^3\,\mathrm{e}^{\sin(2\pi t)}$ is a non-smooth function, the methods that discussed in [2,6] cannot solve the problem (14) correctly. However, by relations (4) and (5), we have for all $t\in[0,1]$:

$$g(t)=\min_{u\in[-1,1]}\left\{-|u(t)|^3\,\mathrm{e}^{\sin(2\pi t)}\right\}=-\mathrm{e}^{\sin(2\pi t)},$$

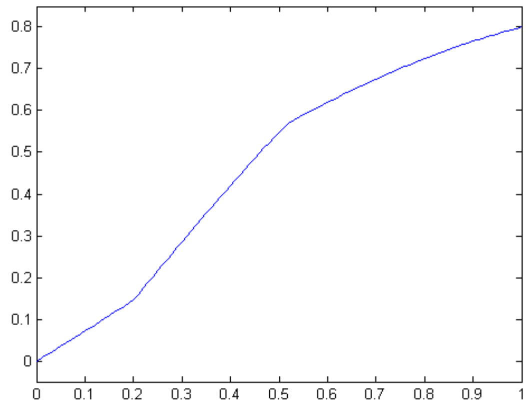

Figure 4. Optimal state $x^*(.)$ of Ex. 4.2.

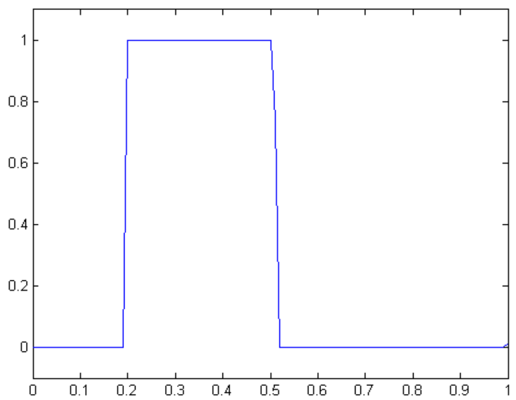

Figure 5. Corresponding optimal control $\lambda^*(.)$ of Ex. 4.2.

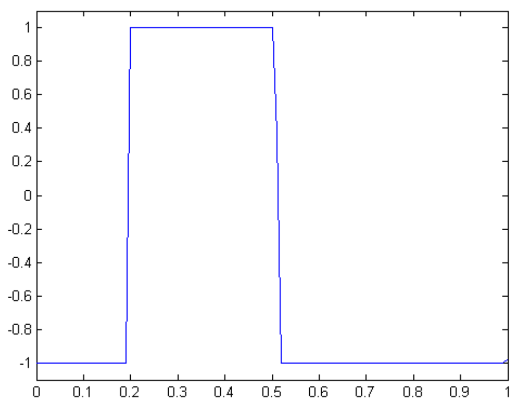

Figure 6. Optimal control $u^*(.)$ of Ex. 4.2.

$$w(t) = \max_{u\in[-1,1]} \left\{-|u(t)|^3 \mathrm{e}^{\sin(2\pi t)}\right\} = 0,$$

thus

$$\beta(t) = w(t) - g(t) = \mathrm{e}^{\sin(2\pi t)}.$$

Let $N = 100$. Then $\delta = 0.01$ and $s_j = \dfrac{j}{100}$ for all $j = 0,1,2,\cdots,100$. We obtain the optimal solutions x_j^* and λ_j^*, $j = 0,1,2,\cdots,100$ of this problem by solving corresponding problem (8), which is illustrated in **Figures 7** and **8** respectively. In addition, by relation (9) the corresponding $u^*(.)$ of this example is

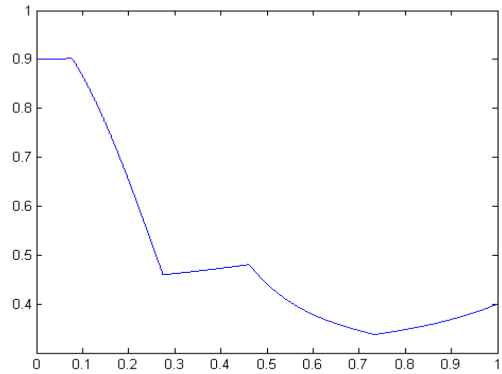

Figure 7. Optimal state $x^*(.)$ of Ex. 4.3.

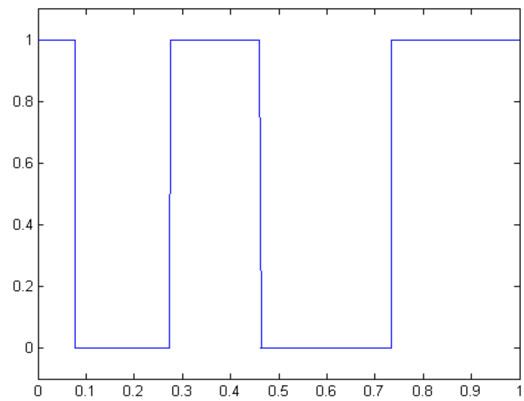

Figure 8. Corresponding optimal control $\lambda^*(.)$ of Ex. 4.3.

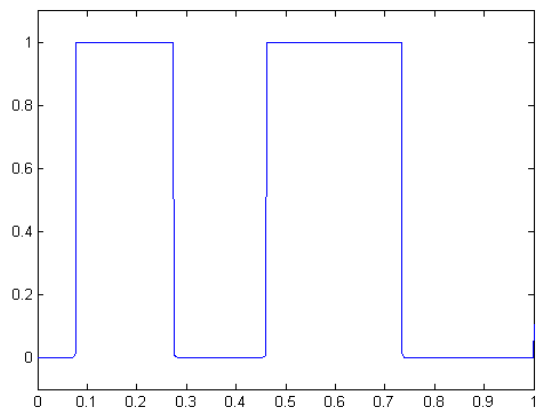

Figure 9. Optimal control $u^*(.)$ of Ex. 4.3.

$$u_j^* = \left(-\left(\beta(s_j)\lambda_j^* + g(s_j)\right)\mathrm{e}^{-\sin(2\pi s_j)}\right)^{\frac{1}{3}},\ j = 0,1,2,\cdots,100$$

Table 1. Solutions comparison of the Ex. 10.

N =100	Discretization method [11]	Presented approach
Objective value	0.1180	0.0980
CPU Times (Sec)	5.281	0.047

Table 2. Solutions comparison of the Ex. 11.

N =100	Discretization method [11]	Presented approach
Objective value	–0.1808	–0.1830
CPU Times (Sec)	95.734	0.125

Table 3. Solutions comparison of the Ex. 12.

N =100	Discretization method [11]	Presented approach
Objective value	–0.0261	–0.0434
CPU Times (Sec)	6.680	0.078

The optimal controls $u_j^*, \quad j = 0,1,2,\cdots,100$ of problem (12) is shown in **Figure 9**. Here, the value of optimal solution of objective function is –0.0435.

5. Conclusions

In this paper, we proposed a different approach for solving a class of nonlinear optimal control problems which have a linear functional and nonlinear dynamical control system. In our approach, the linear combination property of intervals is used to obtain the new corresponding problem which is a linear optimal control problem. The new problem can be converted to an LP problem by discretezation method. Finally, we obtain an approximate solution for the main problem. By the approach of this paper we may solve a wide class of nonlinear optimal control problems.

6. References

[1] M. Diehl, H. G. Bock and J. P. Schloder, "A Real-Time Iteration Scheme for Nonlinear Optimization in Optimal Feedback Control," *Siam Journal on Control and Optimization*, Vol. 43, No. 5, 2005, pp.1714-1736. doi:10.1137/S0363012902400713

[2] M. Diehl, H. G. Bock, J. P. Schloder, R. Findeisen, Z. Nagyc and F. Allgower, "Real-Time Optimization and Nonlinear Model Predictive Control of Processes Governed by Differential-Algebraic Equations," *Journal of Process Control*, Vol. 12, No. 4, 2002, pp. 577-585.

[3] M. Gerdts and H. J. Pesch, "Direct Shooting Method for the Numerical Solution of Higher-Index DAE Optimal Control Problems," *Journal of Optimization Theory and Applications*, Vol. 117, No. 2, 2003, pp. 267-294. doi:10.1023/A:1023679622905

[4] H. J. Pesch, "A Practical Guide to the Solution of Real-Life Optimal Control Problems," 1994. http://citeseerx.ist.psu.edu/viewdoc/download?doi=10.1.1.53.5766&rep=rep1&type=pdf

[5] J. A. Pietz, "Pseudospectral Collocation Methods for the Direct Transcription of Optimal Control Problems," Master Thesis, Rice University, Houston, 2003.

[6] O. V. Stryk, "Numerical Solution of Optimal Control Problems by Direct Collocation," *International Series of Numerical Mathematics*, Vol. 111, No. 1, 1993, pp. 129-143.

[7] A. H. Borzabadi, A. V. Kamyad, M. H. Farahi and H. H. Mehne, "Solving Some Optimal Path Planning Problems Using an Approach Based on Measure Theory," *Applied Mathematics and Computation*, Vol. 170, No. 2, 2005, pp. 1418-1435.

[8] M. Gachpazan, A. H. Borzabadi and A. V. Kamyad, "A Measure-Theoretical Approach for Solving Discrete Optimal Control Problems," *Applied Mathematics and Computation*, Vol. 173, No. 2, 2006, pp. 736-752.

[9] A.V. Kamyad, M. Keyanpour and M. H. Farahi, "A New Approach for Solving of Optimal Nonlinear Control Problems," *Applied Mathematics and Computation*, Vol. 187, No. 2, 2007, pp. 1461-1471.

[10] A. V. Kamyad, H. H. Mehne and A. H. Borzabadi, "The Best Linear Approximation for Nonlinear Systems," *Applied Mathematics and Computation*, Vol. 167, No. 2, 2005, pp. 1041-1061.

[11] K. P. Badakhshan and A. V. Kamyad, "Numerical Solution of Nonlinear Optimal Control Problems Using Nonlinear Programming," *Applied Mathematics and Computation*, Vol. 187, No. 2, 2007, pp. 1511-1519.

[12] K. P. Badakhshan, A. V. Kamyad and A. Azemi, "Using AVK Method to Solve Nonlinear Problems with Uncertain Parameters," *Applied Mathematics and Computation*, Vol. 189, No. 1, 2007, pp. 27-34.

[13] W. Alt, "Approximation of Optimal Control Problems with Bound Constraints by Control Parameterization," *Control and Cybernetics*, Vol. 32, No. 3, 2003, pp. 451-472.

[14] T. M. Gindy, H. M. El-Hawary, M. S. Salim and M. El-Kady, "A Chebyshev Approximation for Solving Optimal Control Problems," *Computers & Mathematics with Applications*, Vol 29, No. 6, 1995, pp 35-45. doi:10.1016/0898-1221(95)00005-J

[15] H. Jaddu, "Direct Solution of Nonlinear Optimal Control Using Quasilinearization and Chebyshev Polynomials Problems," *Journal of the Franklin Institute*, Vol. 339, No. 4-5, 2002, pp. 479-498.

[16] G. N. Saridis, C. S. G. Lee, "An Approximation Theory of Optimal Control for Trainable Manipulators," *IEEE Transations on Systems*, Vol. 9, No. 3, 1979, pp. 152-159.

[17] P. Balasubramaniam, J. A. Samath and N. Kumaresan, "Optimal Control for Nonlinear Singular Systems with Quadratic Performance Using Neural Networks," *Applied Mathematics and Computation*, Vol. 187, No. 2, 2007, pp. 1535-1543.

[18] T. Cheng, F. L. Lewis, M. Abu-Khalaf, "A Neural Network Solution for Fixed-Final Time Optimal Control of

Nonlinear Systems," *Automatica*, Vol. 43, No. 3, 2007, pp. 482-490.

[19] P. V. Medagam and F. Pourboghrat, "Optimal Control of Nonlinear Systems Using RBF Neural Network and Adaptive Extended Kalman Filter," *Proceedings of American Control Conference Hyatt Regency Riverfront*, St. Louis, 10-12 June 2009, pp. 355-360.

[20] D. Luenberger, "Linear and Nonlinear Programming," Kluwer Academic Publishers, Norwell, 1984.