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Stability Loss of Rotating Elastoplastic Discs of the Specific Form

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Abstract

A method of calculating a possible stability loss by a rotating circular annular disc of variable thickness is suggested within the theory of perfect plasticity with the help of small parameter method. A characteristic equation for a critical radius of a plastic zone is obtained as a first approximation. The formula for the critical angular velocity, determining the stability loss of the disc according to the self-balanced form, is derived. The method using which we can take into account the disc's geometry and loading parameters is also specified. The efficiency of the proposed method is shown in Section 5 while considering an illustrative example. The values of critical angular velocity of rotating are found numerically for different parameters of the disc.

Keywords: Axisymmetric Elastoplastic Problem, Boundary Shape Perturbation Method, Rotating Circular Annular Disc, Stepped Disc, Stability Loss, Critical Angular Velocity

1. Introduction

The analytical methods of studying the stability loss [1-6] at radial tension are known to be applied to plane discs (with constant thickness) in elastoplastic state. In [7] a method of calculation of possible stability loss was proposed for the case of the simplest non-planar rotating circular disc, namely, the stepped disc, loaded by radial stress on the boundary. This method underlies the present approach to approximate calculation of critical radius of the plastic zone and critical angular velocity of the rotating annular disc of variable thickness. Besides, the real profile is roughly replaced by a step-like one, so that the disc is considered to be composed of partial annular discs of constant thickness. The applicability of the algorithm to the analysis of the small perturbations dynamics in case of the discs with arbitrary profiles is discussed.

2. Problem Statement

Consider a stability loss of the rotating annular disc with an arbitrary smooth profile $y(r)$ (**Figure 1**) as a result of its attaining an equilibrium form, different from a circular one, in the plane of rotation. We will assume the disc to be almost circular, and present the equation of external boundary in its middle plane $y = 0$, being a plane of symmetry of the disc, with the accuracy to the first-order infinitesimals, in the following form

$$r = b + d\cos n\theta,\ n \geq 2,\ d = \text{const},$$

or

$$\rho = 1 + \delta \cos n\theta \qquad (1)$$

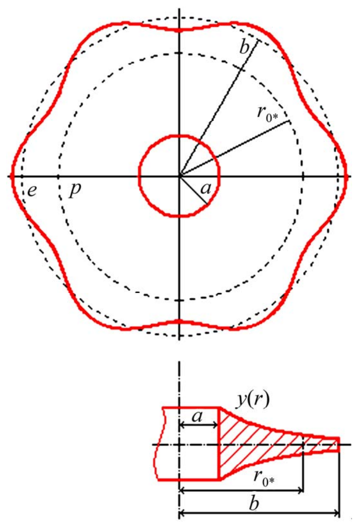

Figure 1. Stability loss of the disc according to self-balancing form.

where b is the external radius of the unperturbed disc (the radius of circumference profile), $\rho = r/b$ is the non-dimensional current radius, δ is a small parameter, $n \in \mathbb{N}$, θ is a polar angle. Let a be the internal radius of the disc, σ_s be the yield strength of the material, E be the modulus of elasticity, γ be the density, v be Poisson's coefficient, ω be the angular velocity of rotation and r_0 be the current radius of the plastic zone for the unperturbed disc.

Let's assume that the maximal thickness of the disc is small as compared to its other dimensions. Based on this assumption, the stresses located on the internal and external boundaries of the disc will be considered as resulted from certain efforts $p_i = p_{i0} + \tilde{p}_i$ and $p_e = p_{e0} + \tilde{p}_e$ [7,8], acting on the disc in its middle plane.

For the boundary form, described by (1), we need to obtain (as a first approximation) the characteristic equation for the critical radius of the plastic zone r_{0*} and to find the corresponding critical angular rotation velocity ω_*.

3. The Unperturbed Elastoplastic State of the Rotating Disc

Consider the equation of quasi-static equilibrium [9]

$$\frac{1}{y}\frac{\mathrm{d}}{\mathrm{d}r}(\sigma_{rr}y) + \frac{\sigma_{rr} - \sigma_{\theta\theta}}{r} = -\frac{\sigma}{b^2}r, \qquad (2)$$

where

$$\sigma = \gamma b^2 \omega^2.$$

Basing on yield condition (of maximum shear theory) and taking into account that the problem statement gives

$$\sigma_{rr}(a) = -p_i,$$

in the plastic region $r \in [a, r_0]$, we present the solution of linear differential Equation (2)

$$\frac{\mathrm{d}\sigma_{rr}}{\mathrm{d}r} + \left(\frac{1}{r} + \frac{1}{y}\frac{\mathrm{d}y}{\mathrm{d}r}\right)\sigma_{rr} = \frac{\sigma_{\theta\theta}}{r} - \frac{\sigma}{b^2}r$$

in general form

$$\sigma_{rr} = x^*(r; a, p_i). \qquad (3)$$

Moreover,

$$\sigma_{\theta\theta} = \sigma_s. \qquad (4)$$

Taking into account the condition on the external boundary

$$\sigma_{rr}(b) = p_e$$

and yield condition (constant stress intensity), suppose that in the disc elastic region $r \in [r_0, b]$ the stress components are

$$\sigma_{rr} = z^*(r, C; b, p_e), \qquad (5)$$

$$\sigma_{\theta\theta} = w^*(r, C; b, p_e). \qquad (6)$$

Here the constant C is to be found.

Having in mind that non-dimensional values will be used in further calculations we refer the values with the dimension of pressure to the yield strength σ_s. The values with the dimension of length will be referred to the characteristic length b. Introducing the notations

$$\beta_0 := r_0/b,\ \beta := a/b,$$

we use the continuity condition for the stress components at transition through the boundary $\rho = \beta_0$. Equating the right-hand sides of (3) and (5), and those of (4) and (6) at $\rho = \beta_0$, we get the system of equations

$$x(\beta_0; \beta, p_i/\sigma_s) = z(\beta_0, C; 1, p_e/\sigma_s),$$

$$1 = w(\beta_0, C; 1, p_e/\sigma_s).$$

Its solution

$$C = \zeta(\beta_0), \sigma/\sigma_s = \chi(\beta_0)$$

fully describes the stress state (3)-(6) and determines the dependence of the angular velocity of disc rotation on the radius of plastic region.

4. Principal Result

Along with relations (3), (5), (6), consider an approximated stress state, obtained at dividing the given disc of an arbitrary profile into partial discs of constant thickness $2h_1, \cdots, 2h_{n_0}$ (**Figure 2**). In [7] it has been shown that the dependences corresponding to (3), (5), (6) for the stepped annular disc are

$$\sigma_{\rho\rho}^{0p} = \begin{cases} 1 - \frac{\sigma}{3\sigma_s}\rho^2 + \frac{C_1}{\rho},\ \rho \in [\beta, \rho_1), \\ 1 - \frac{\sigma}{3\sigma_s}\rho^2 + \frac{C_2}{\rho},\ \rho \in (\rho_1, \rho_2), \\ \cdots \\ 1 - \frac{\sigma}{3\sigma_s}\rho^2 + \frac{C_j}{\rho}, \rho \in (\rho_{j-1}, \beta_0), \end{cases} \qquad (7)$$

$$\sigma_{\rho\rho}^{0e} = \begin{cases} \tilde{C}_{1,j} + \tilde{C}_{2,j}\rho^{-2} - \tilde{\alpha}\rho^2,\ \rho \in (\beta_0, \rho_j), \\ \tilde{C}_{1,j+1} + \tilde{C}_{2,j+1}\rho^{-2} - \tilde{\alpha}\rho^2,\ \rho \in (\rho_j, \rho_{j+1}), \\ \cdots \\ \tilde{C}_{1,n_0} + \tilde{C}_{2,n_0}\rho^{-2} - \tilde{\alpha}\rho^2,\ \rho \in (\rho_{n_0-1}, \rho_{n_0}], \end{cases} \qquad (8)$$

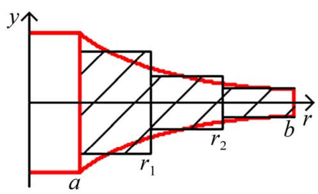

Figure 2. The disc of arbitrary profile divided into partial discs of constant thicknesses.

$$\sigma_{\theta\theta}^{0e} = \begin{cases} \tilde{C}_{1,j} - \tilde{C}_{2,j}\rho^{-2} - \tilde{\beta}\rho^2,\ \rho \in (\beta_0, \rho_j), \\ \tilde{C}_{1,j+1} - \tilde{C}_{2,j+1}\rho^{-2} - \tilde{\beta}\rho^2,\ \rho \in (\rho_j, \rho_{j+1}), \\ \cdots \\ \tilde{C}_{1,n_0} - \tilde{C}_{2,n_0}\rho^{-2} - \tilde{\beta}\rho^2,\ \rho \in (\rho_{n_0-1}, \rho_{n_0}], \end{cases} \quad (9)$$

where $\rho_1 = r_1/b, \cdots, \rho_{n_0-1} = r_{n_0-1}/b$, $\rho_{n_0} = 1$, $\tilde{\alpha} = \sigma(\nu+3)/(8\sigma_s)$, $\tilde{\beta} = \sigma(3\nu+1)/(8\sigma_s)$, and the constants $C_1, \cdots, C_j$ and $\tilde{C}_{1,j}, \tilde{C}_{2,j}, \cdots, \tilde{C}_{1,n_0}, \tilde{C}_{2,n_0}$ are found as solutions of the systems

$$-\frac{p_i}{\sigma_s} = 1 - \frac{\sigma}{3\sigma_s}\beta^2 + \frac{C_1}{\beta},$$

$$h_1\left(1 - \frac{\sigma}{3\sigma_s}\rho_1^2 + \frac{C_1}{\rho_1}\right) = h_2\left(1 - \frac{\sigma}{3\sigma_s}\rho_1^2 + \frac{C_2}{\rho_1}\right),$$

$$h_2\left(1 - \frac{\sigma}{3\sigma_s}\rho_2^2 + \frac{C_2}{\rho_2}\right) = h_3\left(1 - \frac{\sigma}{3\sigma_s}\rho_2^2 + \frac{C_3}{\rho_2}\right),$$

$$\cdots$$

$$h_{j-1}\left(1 - \frac{\sigma}{3\sigma_s}\rho_{j-1}^2 + \frac{C_{j-1}}{\rho_{j-1}}\right) = h_j\left(1 - \frac{\sigma}{3\sigma_s}\rho_{j-1}^2 + \frac{C_j}{\rho_{j-1}}\right)$$

and

$$\tilde{C}_{1,j} + \tilde{C}_{2,j}x_j = s_j^-,$$
$$\tilde{C}_{1,j} - \tilde{C}_{2,j}x_j = t_j^-,$$
$$\tilde{C}_{1,j+1} + \tilde{C}_{2,j+1}x_{j+1} = s_{j+1}^-,$$
$$\tilde{C}_{1,j+1} - \tilde{C}_{2,j+1}x_{j+1} = t_{j+1}^-,$$
$$\cdots$$
$$\tilde{C}_{1,n_0} + \tilde{C}_{2,n_0} = s_{n_0},$$
$$\tilde{C}_{1,n_0} - \tilde{C}_{2,n_0} = t_{n_0}$$

respectively. Here $x_j = 1/\rho_j^2, \cdots, x_{n_0-1} = 1/\rho_{n0-1}^2$,

$$\frac{\sigma}{\sigma_s} = \frac{\omega^2}{q^2} = \frac{8}{\nu+3}\frac{\frac{p_e}{\sigma_s} - \left(RA_{n_0} - R^*B_{n_0}\right)\frac{p_i}{\sigma_s} - (SA_{n_0} - S^*B_{n_0})}{QA_{n_0} - Q^*B_{n_0} + D_{n_0} - 1},$$

$$q = \frac{1}{b}\sqrt{\frac{\sigma_s}{\gamma}},$$

$$Q = d_0\frac{(3\nu+1)\beta_0^2 + 24\delta_2\beta_0^{-1}}{3(\nu+3)} - f_0\frac{(3\nu+1)\beta_0^2}{\nu+3},$$

$$R = d_0\delta_1\beta_0^{-1},$$
$$S = d_0\delta_3\beta_0^{-1} + 1,$$

$$Q^* = -f_0\frac{(3\nu+1)\beta_0^2 + 24\delta_2\beta_0^{-1}}{3(\nu+3)} + d_0\frac{(3\nu+1)\beta_0^2}{\nu+3},$$

$$R^* = -f_0\delta_1\beta_0^{-1},$$
$$S^* = -f_0\delta_3\beta_0^{-1} + 1,$$

$$d_0 = \frac{x_0 + x_j}{2x_0},\ f_0 = \frac{x_j - x_0}{2x_0},\ x_0 = \frac{1}{\beta_0^2},$$

$$\delta_1 = -\frac{h_1}{h_j}\beta,\ \delta_2 = -\frac{1}{3h_j}\sum_{k=1}^{j}(h_{k-1} - h_k)\rho_{k-1}^3,$$

$$\delta_3 = \frac{1}{h_j}\sum_{k=1}^{j}(h_{k-1} - h_k)\rho_{k-1},\ h_0 = 0,\ \rho_0 = \beta,$$

whereas $s_j^-, t_j^-, \cdots, s_{n0-1}^-, t_{n0-1}^-, s_{n_0}, t_{n_0}$ and A_{n_0}, B_{n_0}, D_{n_0}, are found from the recurrence relations

$$s_j^- = Q\tilde{\alpha} + R\frac{p_i}{\sigma_s} + S,\ t_j^- = Q^*\tilde{\alpha} + R^*\frac{p_i}{\sigma_s} + S^*,$$

$$s_{j+1}^- = A_{j+1}s_j^- - B_{j+1}t_j^- + C_{j+1},$$
$$t_{j+1}^- = A_{j+1}^*s_j^- - B_{j+1}^*t_j^- + C_{j+1}^*,$$
$$s_{j+2}^- = A_{j+2}s_j^- - B_{j+2}t_j^- + C_{j+2},$$
$$t_{j+2}^- = A_{j+2}^*s_j^- - B_{j+2}^*t_j^- + C_{j+2}^*,$$
$$\cdots$$
$$s_{n_0} = A_{n_0}s_j^- - B_{n_0}t_j^- + C_{n_0},\ t_{n_0} = A_{n_0}^*s_j^- - B_{n_0}^*t_j^- + C_{n_0}^*,$$

where

$$A_{j+1} = d_ja_j + \nu f_jc_j,\ B_{j+1} = f_j,$$
$$C_{j+1} = d_jb_j - \nu f_jb_j,\ A_{j+1}^* = -f_ja_j - \nu d_jc_j,$$
$$B_{j+1}^* = -d_j,\ C_{j+1}^* = -f_jb_j + \nu d_jb_j,$$
$$A_{j+2} = d_{j+1}a_{j+1}A_{j+1} - f_{j+1}\left(-\nu c_{j+1}A_{j+1} + A_{j+1}^*\right),$$
$$B_{j+2} = d_{j+1}a_{j+1}B_{j+1} - f_{j+1}\left(-\nu c_{j+1}B_{j+1} + B_{j+1}^*\right),$$
$$C_{j+2} = d_{j+1}\left(a_{j+1}C_{j+1} + b_{j+1}\right) - f_{j+1}\left(-\nu c_{j+1}C_{j+1} + C_{j+1}^* + \nu b_{j+1}\right),$$
$$A_{j+2}^* = -f_{j+1}a_{j+1}A_{j+1} + d_{j+1}\left(-\nu c_{j+1}A_{j+1} + A_{j+1}^*\right),$$
$$B_{j+2}^* = -f_{j+1}a_{j+1}B_{j+1} + d_{j+1}\left(-\nu c_{j+1}B_{j+1} + B_{j+1}^*\right),$$
$$C_{j+2}^* = -f_{j+1}\left(a_{j+1}C_{j+1} + b_{j+1}\right) + d_{j+1}\left(-\nu c_{j+1}C_{j+1} + C_{j+1}^* + \nu b_{j+1}\right),$$
$$\cdots$$
$$A_{n_0} = d_{n_0-1}a_{n_0-1}A_{n_0-1} - f_{n_0-1}\left(-\nu c_{n_0-1}A_{n_0-1} + A_{n_0-1}^*\right),$$
$$B_{n_0} = d_{n_0-1}a_{n_0-1}B_{n_0-1} - f_{n_0-1}\left(-\nu c_{n_0-1}B_{n_0-1} + B_{n_0-1}^*\right),$$

$$C_{n_0} = d_{n_0-1}\left(a_{n_0-1}C_{n_0-1} + b_{n_0-1}\right) - f_{n_0-1}\left(-\nu c_{n_0-1}C_{n_0-1} + C^*_{n_0-1} + \nu b_{n_0-1}\right),$$

$$A^*_{n_0} = -f_{n_0-1}a_{n_0-1}A_{n_0-1} + d_{n_0-1}\left(-\nu c_{n_0-1}A_{n_0-1} + A^*_{n_0-1}\right),$$

$$B^*_{n_0} = -f_{n_0-1}a_{n_0-1}B_{n_0-1} + d_{n_0-1}\left(-\nu c_{n_0-1}B_{n_0-1} + B^*_{n_0-1}\right),$$

$$C^*_{n_0} = -f_{n_0-1}\left(a_{n_0-1}C_{n_0-1} + b_{n_0-1}\right) + d_{n_0-1}\left(-\nu c_{n_0-1}C_{n_0-1} + C^*_{n_0-1} + \nu b_{n_0-1}\right),$$

$$C_k = \tilde{\alpha} D_k,\ k = j+1,\cdots,n_0,$$

$$D_{j+1} = g_j\left(d_j - \nu f_j\right),$$

$$D_{j^*} = d_{j^*-1}\left(a_{j^*-1}D_{j^*-1} + g_{j^*-1}\right) - f_{j^*-1}\left(-\nu c_{j^*-1}D_{j^*-1} + D^*_{j^*-1} + \nu g_{j^*-1}\right),$$

$$j^* = j+2,\cdots,n_0,$$

$$a_k = \frac{h_k}{h_{k+1}},\ b_k = \frac{h_{k+1}-h_k}{h_{k+1}}\frac{\tilde{\alpha}}{x_k},$$

$$c_k = \frac{h_{k+1}-h_k}{h_{k+1}},\ d_k = \frac{x_k + x_{k+1}}{2x_k},$$

$$f_k = \frac{x_{k+1}-x_k}{2x_k},\ g_k = \frac{h_{k+1}-h_k}{h_{k+1}}\frac{1}{x_k},$$

$$k = j,\cdots,n_0-1.$$

Dependences (7-9) with account of the relation $\sigma^{0p}_{\theta\theta} = 1$ provide a zeroth approximation to the solution of the problem on plastic equilibrium, determining the position of elastoplastic boundary. In addition,

$$A_1 = \frac{\mathrm{d}\sigma^{0e}_{\rho\rho}(1)}{\mathrm{d}\rho} = \left(A^*_{n_0}Q - B^*_{n_0}Q^* - 3\right)\tilde{\alpha} + \left(A^*_{n_0}R - B^*_{n_0}R^*\right)\frac{p_i}{\sigma_s} - \frac{p_e}{\sigma_s} + A^*_{n_0}S - B^*_{n_0}S^* + C^*_{n_0},$$

$$A_2 = \sigma^{0e}_{\theta\theta}(1) - \sigma^{0e}_{\rho\rho}(1) = A_1 + 3\tilde{\alpha} - \tilde{\beta},$$

where

$$\frac{p_i}{\sigma_s} = \xi(\beta_0) = \frac{\varepsilon_i\tau + \dfrac{8\varkappa_i\left(\beta^{-1}-\beta^2\right)}{\nu+3}\left(\varepsilon_e - \mu\varepsilon_i - \left[SA_{n_0} - S^*B_{n_0}\right]\right)}{\tau - \dfrac{8\varkappa_i\left(\beta^{-1}-\beta^2\right)}{\nu+3}\left(\mu - \left[RA_{n_0} - R^*B_{n_0}\right]\right)},$$

$$0 \le \varepsilon_i < 1,\ 0 \le \varepsilon_e < 1,$$

$$\mu = \frac{\varkappa_e}{\varkappa_i}\frac{\bar{\gamma}}{\gamma}\frac{\varphi^3 - 1}{\beta^{-1}-\beta^2},\ \varkappa_e \ge 0, \bar{\gamma} > 0, \varphi > 1,$$

$$\tau = QA_{n_0} - Q^*B_{n_0} + D_{n_0} - 1,$$

$$p_e/\sigma_s = \varepsilon_e + \mu\left(\xi(\beta_0) - \varepsilon_i\right)$$

for $\varkappa_i \ne 0$, and

$$\frac{p_e}{\sigma_s} = \eta(\beta_0) = \frac{\varepsilon_e\tau - \dfrac{8\varkappa_e\left(\varphi^3-1\right)}{\nu+3}\dfrac{\bar{\gamma}}{\gamma}\left[\left(\varepsilon_i - \mu^{-1}\varepsilon_e\right)\left(RA_{n_0} - R^*B_{n_0}\right) + SA_{n_0} - S^*B_{n_0}\right]}{\tau - \dfrac{8\varkappa_e\left(\varphi^3-1\right)}{\nu+3}\dfrac{\bar{\gamma}}{\gamma}\left[1 - \mu^{-1}\left(RA_{n_0} - R^*B_{n_0}\right)\right]},$$

$$p_i/\sigma_s = \varepsilon_i + \mu^{-1}\left(\eta(\beta_0) - \varepsilon_e\right)$$

for $\varkappa_e \ne 0$.

The number of sections n_0 of the stepped profile, which approximately substitutes a real one, still remains unknown, and constant half-thicknesses of partial annular discs can be introduced by the average theorem:

$$h_j = \frac{n_0}{b-a}\int_{r_{j-1}}^{r_j} y(r)\,\mathrm{d}r,\qquad r_j = a + \frac{b-a}{n_0}j,\ \ j \in \{1,\cdots,n_0\}. \tag{10}$$

First assume that n_0 equals to a certain fixed small natural number. Then, with regard to (10), one has a characteristic equation [2-4,7]

$$\tilde{\Delta}(\beta_0) = 0, \tag{11}$$

where $\tilde{\Delta}(\beta_0)$ is the determinant of the matrix $\left(a_{ij}\right)^4_{i,j=1}$, where

$$a_{11} = 1 + A_1 d_I\left(\rho_{n_0-1},1\right)\sigma_s/E,$$

$$a_{12} = A_1 d_{II}\left(\rho_{n_0-1},1\right)\sigma_s/E,$$

$$a_{13} = A_1 d_{III}\left(\rho_{n_0-1},1\right)\sigma_s/E,$$

$$a_{14} = A_1 d_{IV}\left(\rho_{n_0-1},1\right)\sigma_s/E,$$

$$a_{21} = nA_2 d_I\left(\rho_{n_0-1},1\right)\sigma_s/E,$$

$$a_{22} = nA_2 d_{II}\left(\rho_{n_0-1},1\right)\sigma_s/E,$$

$$a_{23} = 1 + nA_2 d_{III}\left(\rho_{n_0-1},1\right)\sigma_s/E,$$

$$a_{24} = nA_2 d_{IV}\left(\rho_{n_0-1},1\right)\sigma_s/E,$$

$$a_{31} = q_{1,j-1},\ a_{32} = q_{2,j-1},$$

$$a_{33}=q_{3,j-1},\, a_{34}=q_{4,j-1},$$
$$a_{41}=q_{5,j-1},\, a_{42}=q_{6,j-1},$$
$$a_{43}=q_{7,j-1},\, a_{44}=q_{8,j-1},$$

$d_I(\cdot),\cdots,d_{IV}(\cdot)$ are the known functions of two variables, and $A^*_{n_0}$, $B^*_{n_0}$, $C^*_{n_0}$ and $q_{1,j-1},\cdots,q_{8,j-1}$ are found from recurrence relations. Moreover, the critical angular velocity ω_*, corresponding to the critical radius of the plastic zone β_{0*}, $\beta_{0*}\in(\beta,1)$, is obtained from the known formulae [7], depending on the type of contour load p_i, p_e. It remains to be seen whether β_{0*} and ω_* are exact approximations of the corresponding values for the disc with given profile $y(r)$.

Let ε be an arbitrary positive number. Let it be connected with the absolute error of the stress state, appeared due to transition to a stepped disc, by the condition

$$\max\left\{\sup_{\rho\in[\beta,\beta_{0*}]}\left|x-\sigma^{0p}_{\rho\rho}\right|,\ \sup_{\rho\in[\beta_{0*},1]}\left|z-\sigma^{0e}_{\rho\rho}\right|,\ \sup_{\rho\in[\beta_{0*},1]}\left|w-\sigma^{0e}_{\theta\theta}\right|\right\}\le\varepsilon, \tag{12}$$

where the functions ρ (3), (5), (6) and (7)-(9) are taken for $\beta_0=\beta_{0*}$. If for β_{0*} being the solution of characteristic equation (11), inequality (12) fails, one should take $n_0:=n_0+1$, redetermine h_j, and also (7)-(9) according to (10) and solve Equation (11) once again. The fulfillment of condition (12) with new β_{0*} allows to complet solution of the problem on the stability loss of the disc with given profile, with the accuracy of ε. If equality (12) fails, the described procedure must be repeated with $n_0:=n_0+1$ and so on.

5. Example

Let's calculate the stability loss for the disc of a hyperbolic profile

$$y=k/r^s,\ k,s>0. \tag{13}$$

Many real profiles can be approximately expressed by Equation (13). For such discs, as well as for those of constant thickness ($s=0$ in (13)), the stress-strain state can be obtained in a closed form [9].

From Equality (4), Equation (2) in the plastic region is presented as

$$\frac{\mathrm{d}\sigma_{rr}}{\mathrm{d}r}+\frac{1-s}{r}\sigma_{rr}=\frac{\sigma_s}{r}-\frac{\sigma}{b^2}r. \tag{14}$$

Solution (3) of the corresponding initial problem is of the following form

$$\sigma_{rr}=\frac{\sigma_s}{1-s}-\frac{\sigma}{(3-s)b^2}r^2+a^{1-s}\left(-p_i-\frac{\sigma_s}{1-s}+\frac{\sigma a^2}{(3-s)b^2}\right)r^{s-1}$$

or

$$x\left(\rho;\beta,\frac{p_i}{\sigma_s}\right)=\frac{1}{1-s}-\frac{1}{3-s}\frac{\sigma}{\sigma_s}\rho^2+\beta^{1-s}\left(-\frac{1}{1-s}-\frac{p_i}{\sigma_s}+\frac{1}{3-s}\frac{\sigma}{\sigma_s}\beta^2\right)\rho^{s-1}. \tag{15}$$

In the elastic region the stress components of the unperturbed annular disc with a hyperbolic profile can be sought as [9]

$$\begin{aligned}\sigma_{rr}&=\overline{C}_1r^{\rho_1}+\overline{C}_2r^{\rho_2}-\alpha'\frac{\sigma}{b^2}r^2,\\ \sigma_{\theta\theta}&=(\rho_1+1-s)\overline{C}_1r^{\rho_1}+(\rho_2+1-s)\overline{C}_2r^{\rho_2}-\beta'\frac{\sigma}{b^2}r^2,\end{aligned} \tag{16}$$

where $\overline{C}_1$, $\overline{C}_2$, ρ_1, ρ_2, α', β' are yet to be specified. Substitution of the corresponding expressions (16) for σ_{rr} and $\sigma_{\theta\theta}$ in equilibrium Equation (14) gives

$$\alpha'=\frac{\beta'+1}{3-s}. \tag{17}$$

After substitution of expressions (16) into

$$r\frac{\mathrm{d}\sigma_{rr}}{\mathrm{d}r}-mr\frac{\mathrm{d}\sigma_{\theta\theta}}{\mathrm{d}r}+(m+1)(\sigma_{rr}-\sigma_{\theta\theta})=0,\ m=\nu^{-1},$$

obtained by exclusion of radial displacement from the coupling equations for deformations and stresses, based on (17), we get

$$\alpha'=\frac{3m+1}{8m-s(3m+1)},\ \beta'=\frac{m+3}{8m-s(3m+1)}. \tag{18}$$

Besides, using the method of undetermined coefficients, find the indices ρ_1,ρ_2:

$$\rho_{1,2}=\frac{s}{2}-1\mp\sqrt{1+\frac{s}{m}+\frac{s^2}{4}}. \tag{19}$$

The condition on the external boundary leads to the relations

$$\overline{C}_1=b^{-\rho_1}\left(\alpha'\sigma+p_e-\widetilde{C}b^{\rho_2}\right),\ \overline{C}_2=\widetilde{C}.$$

Taking them into account in (16), from the system of stress continuity equations at transition through the elastoplastic boundary, we get

$$C=\frac{1+\beta'\frac{\sigma}{\sigma_s}\beta_0^2-(\rho_1+1-s)\left(\alpha'\frac{\sigma}{\sigma_s}+\frac{p_e}{\sigma_s}\right)\beta_0^{\rho_1}}{(\rho_2+1-s)\beta_0^{\rho_2}-(\rho_1+1-s)\beta_0^{\rho_1}}, \tag{20}$$

$$\frac{\sigma}{\sigma_s} = \left\{ \frac{\rho_1}{1-s}\beta_0^{\rho_1} - \frac{\rho_2}{1-s}\beta_0^{\rho_2} + (\rho_2-\rho_1)\frac{p_e}{\sigma_s}\beta_0^{\rho_1+\rho_2} + \beta^{1-s}\left(-\frac{1}{1-s}-\frac{p_i}{\sigma_s}\right)\beta_0^{s-1}\left(\left(\rho_1+1-s\right)\beta_0^{\rho_1} - \left(\rho_2+1-s\right)\beta_0^{\rho_2}\right)\right\}\left\{\frac{\beta'}{3-s}\left(\left(\rho_2+4\right)\beta_0^{\rho_2+2} - \left(\rho_1+4\right)\beta_0^{\rho_1+2}\right) + \alpha'\left(\rho_1-\rho_2\right)\beta_0^{\rho_1+\rho_2} + \frac{1}{3-s}\beta^{3-s}\beta_0^{s-1}\left(\left(\rho_2+1-s\right)\beta_0^{\rho_2} - \left(\rho_1+1-s\right)\beta_0^{\rho_1}\right)\right\}^{-1}, \quad (21)$$

$$z\left(\rho, C; 1, \frac{p_e}{\sigma_s}\right) = \left(\alpha'\frac{\sigma}{\sigma_s} + \frac{p_e}{\sigma_s} - C\right)\rho^{\rho_1} + C\rho^{\rho_2} - \alpha'\frac{\sigma}{\sigma_s}\rho^2, \quad (22)$$

$$w\left(\rho, C; 1, \frac{p_e}{\sigma_s}\right) = \left(\rho_1+1-s\right)\left(\alpha'\frac{\sigma}{\sigma_s} + \frac{p_e}{\sigma_s} - C\right)\rho^{\rho_1} + \left(\rho_2+1-s\right)C\rho^{\rho_2} - \beta'\frac{\sigma}{\sigma_s}\rho^2, \quad (23)$$

where

$$\frac{p_i}{\sigma_s} = \left\{\tau\varepsilon_i + \varkappa_i\left(\beta^{-1}-\beta^2\right)\left[\frac{\rho_1}{1-s}\beta_0^{\rho_1} - \frac{\rho_2}{1-s}\beta_0^{\rho_2} + \left(\varepsilon_e - \mu\varepsilon_i\right)\left(\rho_2-\rho_1\right)\beta_0^{\rho_1+\rho_2} - \frac{1}{1-s}\beta^{1-s}\beta_0^{s-1} \times\left((\rho_1+1-s)\beta_0^{\rho_1} - (\rho_2+1-s)\beta_0^{\rho_2}\right)\right]\right\} \left\{\tau + \varkappa_i\left(\beta^{-1}-\beta^2\right)\left[\beta^{1-s}\beta_0^{s-1}\left((\rho_1+1-s)\beta_0^{\rho_1} - \left(\rho_2+1-s\right)\beta_0^{\rho_2}\right) - \mu\left(\rho_2-\rho_1\right)\beta_0^{\rho_1+\rho_2}\right]\right\}^{-1},$$

$$\tau = \frac{\beta'}{3-s}\left(\left(\rho_2+4\right)\beta_0^{\rho_2+2} - \left(\rho_1+4\right)\beta_0^{\rho_1+2}\right) + \alpha'\left(\rho_1-\rho_2\right)\beta_0^{\rho_1+\rho_2} + \frac{1}{3-s}\beta^{3-s}\beta_0^{s-1} \times\left(\left(\rho_2+1-s\right)\beta_0^{\rho_2} - \left(\rho_1+1-s\right)\beta_0^{\rho_1}\right),$$

$$\frac{p_e}{\sigma_s} = \varepsilon_e + \mu\left(\frac{p_i}{\sigma_s} - \varepsilon_i\right)$$

for $\varkappa_i > 0$, and

$$\frac{p_e}{\sigma_s} = \left\{\tau\varepsilon_e + \varkappa_e\frac{\overline{\gamma}}{\gamma}\left(\varphi^3-1\right)\left[\frac{\rho_1}{1-s}\beta_0^{\rho_1} - \frac{\rho_2}{1-s}\beta_0^{\rho_2} + \beta^{1-s}\left(-\frac{1}{1-s} - \varepsilon_i + \mu^{-1}\varepsilon_e\right) \times\beta_0^{s-1}\left(\left(\rho_1+1-s\right)\beta_0^{\rho_1} - \left(\rho_2+1-s\right)\beta_0^{\rho_2}\right)\right]\right\} \left\{\tau + \varkappa_e\frac{\overline{\gamma}}{\gamma}\left(\varphi^3-1\right)\left[\beta^{1-s}\mu^{-1}\beta_0^{s-1}\left(\left(\rho_1+1-s\right)\beta_0^{\rho_1} - \left(\rho_2+1-s\right)\beta_0^{\rho_2}\right) - \left(\rho_2-\rho_1\right)\beta_0^{\rho_1+\rho_2}\right]\right\}^{-1},$$

$$\frac{p_i}{\sigma_s} = \varepsilon_i + \mu^{-1}\left(\frac{p_e}{\sigma_s} - \varepsilon_e\right)$$

for $\varkappa_e > 0$.

To analyze the dynamics of small perturbations let's first calculate half-thicknesses h_j. In terms of (10), we have:

$$h_j = \frac{kn_0\left(\left(\beta + j\frac{1-\beta}{n_0}\right)^{1-s} - \left(\beta + (j-1)\frac{1-\beta}{n_0}\right)^{1-s}\right)}{b^s\left(1-\beta\right)\left(1-s\right)}, \quad (24)$$

$j \in \{1, \cdots, n_0\}$.

Then we determine dependences (7)-(9) (for as yet unknown β_0) and characteristic Equation (11) itself. Its solution β_{0*} allows proceeding to the verification of estimation (12) with previously given ε. In some cases the exact upper limits in inequality (12) can be found analytically as the solutions of the global extremum problem for continuously differentiable functions (at discontinuity points $\rho_1, \cdots, \rho_{n_0-1}$ it is necessary to use one-sided limits as the values of function: the right-side limit at ρ_{j-1} and the left-side one at ρ_j of each segment $\left[\rho_{j-1}, \rho_j\right] \subset \left[\beta, 1\right]$). However, the numerical method appears to be easier and more versatile tool to verify condition (12). It is reduced to finding the maximum of the set of limited nonnegative piece-wise continuous functions, given at $\beta_0 = \beta_{0*}$ by using relations (15), (22), (23) and (7)-(9), in the points of quite dense discretization of the corresponding segment.

Table 1 gives the results of problem solution for a hyperbolic disc with $k = 0.005$ and $s = 2$ depending upon n_0. Here $n = 2$, $b = 1$, $\beta = a/b = 0.2$, $\nu = 0.3$, $\sigma_s/E = 0.01$, $\varepsilon_i = 0$, $\varepsilon_e = 0$, $\varkappa_i = 1/3$, $\varkappa_e = 0$.

6. Concluding Remarks

The proposed scheme allows determining the critical

Table 1. Critical radius and squared relative critical velocity.

n_0	3	10	20	25	30
β_{0*}	0.7331	0.8399	0.9199	0.9359	0.9466
ω_*^2/q^2	0.8863	0.5929	0.5195	0.5042	0.4939

radius and critical angular velocity of rotating disc with the given profile for known load parameters. This enables studying the discs, whose unperturbed elastoplastic state can be obtained in a closed form [10]. Except the discs of constant thickness and the hyperbolic discs, conical, exponential and equal resistance discs [9], as well as compound discs of the mentioned profiles are referred to this type. Besides, neglecting the procedure of verification of condition (12) due to sufficient increase of the number of stepped disc sections n_0, we get a method of calculating the possible stability loss (including the eccentric case) for fast rotating elastoplastic annular and solid discs of an arbitrary profile [11].

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