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Two Theorems about Nilpotent Subgroup

Lijiang Zeng
Department of Mathematics, Zunyi Normal College, Zunyi, China
E-mail: *zlj4383@sina.com*
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Abstract

In the paper, we introduce some concepts and notations of Hall π-subgroup etc, and prove some properties about finite *p*-group, nilpotent group and Sylow *p*-subgroup. Finally, we have proved two interesting theorems about nilpotent subgroup.

Keywords: Hall π-Subgroup, Sylow *p*-Subgroup, Normalizer, Nilpotent Group

In this paper, we introduced some concepts and notations such as Hall π-subgroup and so on. Using concepts, terms and notations in group theory, we have proved some properties about finite group, nilpotent group and Sylow *p*-subgroup, and proved two interesting theorems about nilpotent subgroup in these properties.

Let π be a set of some primes and the supplementary set of π in the set of all primes be notated π′, When π contains only one prime *p* we notate π and π′ as *p* and *p*′, When all prime factor of integer *n* be in π we called *n* as a π-number, If the order $|H|$ of *G*'s subgroup be a π-number we called *H* as a π-subgroup.

Definition 1. If *H* be a π-subgroup of *G* and $|G:H|$ be a π′- number, we called *H* as a Hall π-subgroup of *G*.

Lemma 1. A nontrivial finite *p*-group has a nontrivial center.

Proof. Let $p^m = n_1 + \cdots + n_k$ be the class equation [1] of the group; n_i divides p^m and hence is a power of *p*. If the center were trivial, only n_i would equal 1 and $p^m \equiv 1(\bmod p)$, which is impossible since $p^m > 1$. □

Definition 2. A group *G* is called nilpotent [2] if it has a central series [2], that is, a normal series

$$1 = G_0 \le G_1 \le \cdots \le G_n = G$$

such that G/G_i is contained in the center of G/G_i for all *i*. The length of a shortest central series of *G* is the nilpotent class of *G*.

A nilpotent group of class 0 has order 1 of course, while nilpotent groups of class at most 1 are abelian. Whereas nilpotent groups are obviously soluble, an example of a non nilpotent soluble group is S_3 (its centre is trivial). The great source of finite nilpotent groups is the class [3] of groups whose orders [4] are prime powers.

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Lemma 2. A finite *p*-group is nilpotent.

Proof. Let *G* be a finite *p*-group of order > 1. Then Lemma 1 shows that $\zeta G \ne 1$. Hence $G/\zeta G$ is nilpotent by induction on $|G|$. By forming the preimages of the terms of a central series of $G/\zeta G$ under the natural homomorphism [5] $G \to G/\zeta G$ and adjoining [6] 1, we arrive at a central series of *G*. □

Lemma 3. The class of nilpotent groups is closed under the formation of subgroups, images, and finite direct products.

The proof can be found in Reference [1].

Lemma 4. Let *P* be a Sylow *p*-subgroup [7] of a finite group *G*.

i) If $N_G(P) \le H \le G$, then $H = N_G(H)$.

ii) If $N \lhd G$, then $P \cap N$ is a Sylow *p*-subgroup of *N* and PN/N is a Sylow *p*-subgroup of G/N.

Proof. i) Let $x \in N_G(H)$. Since $P \le H \lhd N_G(H)$, we have $P_x \le H$. Obviously *P* and P^x are Sylow *p*-subgroup of *H*, so $P^x = P^h$ for some $h \in H$. Hence $xh^{-1} \in N_G(P) \le H$ and $x \in H$. It follows that $H = N_G(H)$.

ii) In the first place $|N : P \cap N| = |PN : P|$, which is prime to *p*. Since $P \cap N$ is a *p*-subgroup, it must be a Sylow *p*-subgroup of *N*. For PN/N the argument is similar. □

Lemma 5. Let *G* be a finite group. Then the following properties are equivalent:

i) *G* is nilpotent;

ii) every subgroup of *G* is subnormal [8];

iii) *G* satisfies the normalizer [9] condition;

iv) every maximal subgroup [8] of G is normal;

v) G is the direct product of its Sylow subgroups.

Proof: i)→ii) Let G be nilpotent with class c. If $H \le G$, then $H\zeta_i G \lhd H\zeta_{i+1}G$ since

$$\zeta_{i+1}G/\zeta_i G = \zeta\left(G/\zeta_i G\right).$$

Hence

$$H = H\zeta_0 G \lhd H\zeta_1 G \lhd \cdots \lhd H\zeta_c G = G$$

and H is subnormal in G in c steps.

ii)→iii) Let $H \le G$. Then H is subnormal in G and there is a series $H = H_0 \lhd H_1 \lhd \cdots \lhd H_n = G$. If i is the least positive integer such that $H \ne H_i$. Then $H = H_{i-1} \lhd H_i$ and $H_i \le N_G\left(H\right)$.

iii)→iv) If M is a maximal subgroup of G, then $M < N_G\left(M\right)$, so by maximality $N_G\left(M\right) = G$ and $M \lhd G$.

iv)→v) Let P be a Sylow subgroup of G. If P is not normal in G, then $N_G\left(P\right)$ is a proper subgroup of G and hence is contained in a maximal subgroup of G, say M. Then $M \lhd G$; however this contradicts Lemma 4. Therefore each Sylow subgroup of G is normal and there is exactly one Sylow p-subgroup for each prime p since all such are conjugate. The product of all the Sylow subgroups is clearly direct and it must equal G.

v)→i) by Lemma 2 and Lemma 3.

Theorem 1. Assume that every maximal subgroup of a finite group G itself is not nilpotent. Then:

i) G is soluble;

ii) $\left|G\right| = p^m q^n$ where p and q are unequal primes;

iii) there is a unique Sylow p-subgroup P and a Sylow q-subgroup Q is cyclic. Hence $G = QP$ and $P \lhd G$.

Proof. i) Let G be a counterexample of least order. If N is a proper nontrivial normal subgroup, both N and G/N are soluble, whence G is soluble. It follow that G is a simple group.

Suppose that every pair of distinct maximal subgroups of G intersects in 1. Let M be any maximal subgroup: then certainly $M = N_G\left(M\right)$ If $\left|G\right| = n$ and $\left|M\right| = m$, then M has n/m conjugates [10] every pair of which intersect trivially. Hence the conjugates of M account for exactly

$$\frac{\left(m-1\right)n}{m} = n - \frac{n}{m}$$

nontrivial elements. Since $m \ge 2$, we have

$$n - \frac{n}{m} \ge \frac{n}{2} \ge \frac{n-1}{2}$$

in addition it is clear that

$$n - \frac{n}{m} \le n - 2 < n - 1.$$

Since each nonidentity element of G belongs to exactly one maximal subgroup, $n - 1$ is the sum of integers lying strictly between $\frac{n-1}{2}$ and $n - 1$. This is plainly impossible.

It follows that there exist distinct maximal subgroups M_1 and M_2 whose intersection I is nontrivial. Let M_1 and M_2 be chosen so that I has maximum order [8]. Write $N = N_G\left(I\right)$. Since M is nilpotent, $I \ne N_{M_1}\left(I\right)$ by Lemma 5, so that $I < N \cap M_1$. Now I cannot be normal in G; thus N is proper and is contained in a maximal subgroup M. Then $I < N \cap M_1 \le M \cap M_1$, which contradicts the maximality of $\left|I\right|$.

ii) Let $\left|G\right| = p_1^{e_1} \cdots p_k^{e_k}$, where $e_i > 0$ and the p_i are distinct primes. Assume that $k \ge 3$. If M is a maximal normal subgroup, its index is prime since G is soluble; let us say $\left|G : M\right| = p_1$. Let P_i be a Sylow p_i-subgroup of G. If $i > 1$, then $P_i \le M$ and, since M is nilpotent, it follows that $P_i \lhd G$; also the since $k \ge 3$. Hence $P_1 P_i$ is nilpotent and thus $\left[P_1, P_i\right] = 1$ (by Lemma 5). It follows that $N_G\left(P_1\right) = G$ and $P_1 \lhd G$. This means that all Sylow subgroup of G are normal, so G is nilpotent. By this contradiction $k = 2$ and $\left|G\right| = p_1^{e_1} p_2^{e_2}$. We shall write $p = p_2$ and $q = p_1$.

iii) Let there be a maximal normal subgroup M with index [6] q. Then the Sylow p-subgroup P of M is normal in G and is evidently also a Sylow p-subgroup of G. Let Q be a Sylow q-subgroup of G. Then $G = QP$. Suppose that Q is not cyclic. If $g \in Q$, then $\left\langle g, P\right\rangle \ne G$ since otherwise $Q \simeq G/P$, which is cyclic [6]. Hence $\left\langle g, P\right\rangle$ is nilpotent and $\left\langle g, P\right\rangle = 1$. But this means that $\left[P, Q\right] = 1$ and $G = P \times Q$, a nilpotent group. Hence Q is cyclic. □

In an insoluble group [3] Hall π-subgroups, even if they exist, may not be conjugate: for example, the simple group **PSL (2, 11)** of order **660** has subgroups isomorphic with D_{12} and A_4: these are nonisomorphic [10] Hall $\left\{2,3\right\}$-subgroups and they are certainly not conjugate. However the situation is quite different when a nilpotent Hall π-subgroup is present.

Theorem 2. Let the finite group G possess a nilpotent Hall π-subgroup H. Then every π-subgroup of G is contained in a conjugate of H. In particular all Hall π-subgroups of G are conjugate.

Proof. Let K be a π-subgroup of G. We shall argue by induction on $\left|K\right|$, which can be assumed greater than 1. By the induction hypothesis a maximal subgroup of K is contained in a conjugate of H and is therefore nilpotent. If K itself is not nilpotent, Theorem 1 may be applied to produce a prime q in π dividing $\left|K\right|$ and a Sylow q-subgroup Q which has a normal complement L in K. Of course, if K is nilpotent, this is still true by Lemma 5.

Now write $H = H_1 \times H_2$ where H_1 is the unique Sylow q-subgroup of H. Since $L \ne K$, the induction hypothesis shows that

$$L \le H^g = H_1^g \times H_2^g$$

for some $g \in G$. Thus $L < H_2^g$ because L is a q'-group. Consequently $N = N_G(L)$ contains $\langle H_1^g, K \rangle$. Observe that $|G : H_1|$ is not divisible by q; hence H_1^g is a Sylow q-subgroup of N and by Sylow's Theorem $Q \le (H_1^g)^x$ for some $x \in N$. But $L = L^x$ and, using $L \le H_2^g$, we obtain

$$K = QL = QL^x \le H_1^{gx} H_2^{gx} = H^{gx}$$

as required. □

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