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Pressure/Saturation System for Immiscible Two-Phase Flow: Uniqueness Revisited

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Abstract

We give a sufficient condition for uniqueness for the pressure/saturation system. We establish this condition through analytic arguments, and then construct "mobilities" (or mobility-like functions) that satisfy the new condition (when the parameter μ is 2). For the constructed "mobilities", we do graphical experiments that show, empirically, that this condition could be satisfied for other values of $1 < \mu < 2$. These empirical experiments indicate that the usual smoothness condition on the fractional flow function (and on the total mobility), for uniqueness and convergence, might not be necessary. This condition is also sufficient for the convergence of a family of perturbed problems to the original pressure/saturation problem.

Keywords: Porous Medium, Uniqueness of a Solution, Degenerate Equation, Immiscible Two-Phase Flow, Regularization, Phase Mobility.

1. Introduction

Consider the coupled nonlinear problem (1), with $0 \le S^0(x) \le 1$, which arises from modeling incompressible two-phase immiscible (water/oil, for example) flow through a porous medium (see [1,2], for instance). The problem considered, here, is in one of its simplified problem.

The conductivity of the medium is denoted by $\boldsymbol{k}$ while $\boldsymbol{u}$ is the total Darcy's velocity for the two-phase flow, $\boldsymbol{f}$ is the fractional flow function, $\boldsymbol{S}$ the saturation of the invading fluid (or wetting phase), $\boldsymbol{P}$ is the global pressure, and ϕ the porosity of the medium. For the present analysis and for simplicity, we let $\phi \equiv 1$.

The set Ω is a sufficiently smooth bounded domain of R^n, $n = 1$, 2 or 3, although this analysis focuses more on the case $n = 2$.

Obviously, Problem 1 cannot, in general, be solved analytically: One needs to proceed through numerical approximations. Before attempting any solution method, one needs to investigate whether the problem has a solution and, if it does, whether the solution is unique. The main purpose of this paper is to revisit the uniqueness question of Problem 1, exhibit sufficient conditions for which the problem has a unique solution, and construct examples for which these conditions are satisfied. Those

$$\begin{cases} u = -a(S)\nabla p & \text{in } \Omega\times(0,T) \\ \operatorname{div}(u) = Q_1 & \text{in } \Omega\times(0,T) \\ u\cdot\eta = 0 & \text{on } \partial\Omega\times[0,T] \\ \int_\Omega p\,\mathrm{d}x = 0 & \text{for all } t\in[0,T] \\ \varphi\dfrac{\partial S}{\partial t} + \nabla\cdot\left(f(S)u - k(S)\nabla S\right) = Q(S) \equiv 0 & \text{in } \Omega\times(0,T) \\ k(S)\dfrac{\partial S}{\partial \eta} = 0 & \text{on } \partial\Omega\times[0,T] \\ S(x,0) = S^0(x) & \text{in } \Omega \end{cases} \tag{1}$$

conditions generalize the ones considered in [3] and in [4] for the uniqueness of the problem and for the convergence of a family of perturbed problems. This work constitutes, in some way, a complement to [3]. In addition, and on the applied side, the mobility-like functions that we construct can be used in testing codes for two-phase flow through porous media.

The following conditions are usually imposed on the data (see [3], for instance).

$$\boldsymbol{k}(\boldsymbol{0}) = \boldsymbol{k}(\boldsymbol{1}) = \boldsymbol{0} \tag{2}$$

$$\boldsymbol{k}(\xi) \geq \begin{cases} c_1 \xi^{\mu}, & \text{if } 0 \leq \xi \leq \alpha_1 \\ c_2, & \text{if } \alpha_1 < \xi < \alpha_2 \\ c_3(1-\xi)^{\mu}, & \text{if } \alpha_2 \leq \xi \leq 1 \end{cases} \tag{3}$$

where $0 < \alpha_1 < \frac{1}{2} < \alpha_2 < 1$, and $0 < \mu \leq 2$, for some α_1 and α_2.

The function $\boldsymbol{f}$ satisfies the following.

$$f \in \boldsymbol{C}^2([0,1]) \text{ in the variable } s \tag{4}$$

and

$$f'(0) = f'(1) = 0. \tag{5}$$

Conditions (2) through (5) imply the following (see [5, 6]).

$$\left| f(b) - f(a) \right|^2 \leq C\left(K(b) - K(a)\right)(b-a), \tag{6}$$

for all $0 \leq a \leq b \leq 1$

where

$$K(s) = \int_0^s k(\tau)\,\mathrm{d}\tau \tag{7}$$

Condition (6) has been used (as a sufficient condition, among other conditions) for the proof of the wellposedness for the saturation equation, the convergence for a regularization of that equation, and the convergence of numerical approximations of the same equation [7-9].

The rest of the paper is articulated as follows. In Section 2, we establish a new sufficient condition for (6) to hold, therefore for uniqueness of a solution of Problem 1. We also show that conditions (2) through (5) imply this new condition. In Section 3, we revisit the pressure saturation problem, to show, indeed, that, under this new condition (defined in Section 2), there is uniqueness for Problem 1. In Section 4, we construct examples of relative "mobilities" (mobility-like functions) and show that we have uniqueness under the special case $\mu = 2$, with μ defined as in (3). We also explore experimentally, through graphs, the uniqueness problem for the pressure/saturation problem, for these examples, for other values of μ, $1 < \mu < 2$, though condition (4) is not satisfied for the corresponding total mobility a, or fractional flow function f.

In this work we use standard notations. In particular, we use $\|\boldsymbol{v}\|_{L^2}$ for $\|\boldsymbol{v}\|_{L^2(\Omega)}$, $\|\boldsymbol{v}\|_{L^2(L^2)}$ for $\|\boldsymbol{v}\|_{L^2(0,T;L^2(\Omega))}$, the norm of the function $v(x,t)$ as an L^2 function in the variable t on $[0,T]$ with values in $L^2(\Omega)$. If $\boldsymbol{v} = (v_1, v_2, \cdots, v_n)$ is a vector, we denote by $\|\boldsymbol{v}\|_{L^p}$ the norm $\|\boldsymbol{v}\|_{(L^p(\Omega))^N}$.

2. Sufficient Condition for Uniqueness for the Pressure/Saturation System

Lemma 2.1 *Let $g \geq 0$ and $h \geq 0$ be two functions defined on the interval $[0,1]$. Assume that g is continuously differentiable on $[0,1]$, and that h is continuous on $[0,1]$. Define*

$$H(s) = \int_0^s h(\tau)\,\mathrm{d}\tau \tag{8}$$

and suppose that

$$g'(x)\left(g(x) - g(a)\right) \leq \frac{C}{2}\left(h(x)(x-a) + H(x) - H(a)\right), \tag{9}$$

for all $x \in [a,1]$ and for all $a \in [0,1]$,

for some constant $C > 0$, then

$$\left(g(b) - g(a)\right)^2 \leq C\left(H(b) - H(a)\right)(b-a), \tag{10}$$

for all $0 \leq a \leq b \leq 1$.

Proof.

We use a calculus argument. If $a = 1$, then the only value that x can assume is 1, and (10) is obvious. For $0 \leq a < 1$, define the function

$$G_a(x) = \left(g(x) - g(a)\right)^2 - C\left(H(x) - H(a)\right)(x-a). \tag{11}$$

for $x \geq a$.

Then, $G_a(a) = 0$, and

$$G_a{}'(x) = 2g'(x)\left(g(x) - g(a)\right) \tag{12}$$

Clearly, if (9) holds, then $G_a{}'(x) \leq 0$ for all $x \geq a$, *i.e.*, G_a is decreasing on the interval $[a,1]$. Since $G_a(a) = 0$, we must have $G_a(x) \leq 0$ for all $a \leq x \leq 1$. This is true for any $0 \leq a \leq 1$. Hence the lemma is proved.

In [9], for one space variable and the unilateral case ($k(0) = 0$), and in [6] for several variables and the bilateral case ($k(0) = k(1) = 0$), it is proved that if f and k satisfy conditions (2) through (5), then f and k satisfy (6), for $f \in C^2([0,1])$. In the next lemma, we show that if (2) through (5) hold, then the couple (f,k) satisfy (9), and therefore (6).

Note: The above lemma is more general than what is known so far, since we do not require any of the conditions (2) through (5) to hold, nor do we require that f be in $C^2(0,1)$. However, if those conditions are satisfied, we have the following lemma.

Lemma 2.2 *Under conditions (2) through (5), and under the assumption that the function* f *is twice continuously differentiable on* $[0,1]$, *we have*

$$f'(x)\left(f(x)-f(a)\right)\le C\left(k(x)(x-a)+K(x)-K(a)\right), \tag{13}$$

for all $x\in[a,1]$ and for all $a\in[0,1]$,

for some constant $C>0$.

Thus, the combination of Lemma 2.1 and Lemma 2.2 gives an alternative way of proving that (6) holds, which in turns leads to uniqueness for Problem 1.

Proof. We follow the lines of the proof of Proposition 3.2 of [6], with some modification. For the proof, it suffices to bound the quantity

$$\frac{f'(x)\left(f(x)-f(a)\right)}{k(x)(x-a)+K(x)-K(a)},$$

independently of a and x.

Thanks to the symmetry implied by (3), we prove this for $0\le a\le x\le\alpha_1$ only, without lost of generality; the rest of the prove can be obtained by the change of variable $x\to 1-x$, for $\alpha_2\le a\le x\le 1$, and by using the fact that $k(x)\ge c_2$ for $\alpha_1\le x\le\alpha_2$. Using (7) and (3), we obtain

$$\begin{aligned}K(x)-K(a)&=\int_a^x k(s)\,\mathrm{d}s\ge c_1\int_a^x s^{\mu}\,\mathrm{d}s\\&\ge\frac{c_1}{\mu+1}\left(x^{\mu+1}-a^{\mu+1}\right).\end{aligned}\tag{14}$$

Therefore, since $x\ge a$, $k(x)\ge 0$, and K is increasing,

$$\begin{aligned}\left|\frac{f'(x)\left(f(x)-f(a)\right)}{k(x)(x-a)+K(x)-K(a)}\right|&\le\frac{\left|f'(x)\right|\left|f(x)-f(a)\right|}{K(x)-K(a)}\\&\le\frac{\mu+1}{c_1}\frac{\left|f'(x)\right|\left|f(x)-f(a)\right|}{x^{\mu+1}-a^{\mu+1}}\\&\le\frac{\mu+1}{c_1}\frac{\left|f'(x)\right|\left|\dfrac{f(x)-f(a)}{x-a}\right|}{\dfrac{x^{\mu+1}-a^{\mu+1}}{x-a}}\end{aligned}\tag{15}$$

By the Mean-Value Theorem, there exist $a<c,d<x$ such that

$$\frac{f(x)-f(a)}{x-a}=f'(c)$$

and

$$\frac{x^{\mu+1}-a^{\mu+1}}{x-a}=(\mu+1)d^{\mu}.$$

Because the function $s\to s^{\mu}$ is increasing, its mean value occurs on the interval $[a,x]$ after the midpoint $\frac{a+x}{2}$, so $d>\frac{a+x}{2}$. Hence

$$d>\frac{c}{2}\tag{16}$$

and

$$d>\frac{x}{2}.\tag{17}$$

Going back to (15), we get

$$\begin{aligned}\left|\frac{f'(x)\left(f(x)-f(a)\right)}{k(x)(x-a)+K(x)-K(a)}\right|&\le\frac{1}{c_1}\frac{\left|f'(x)\right|\left|f'(c)\right|}{d^{\mu}}\\&\le\frac{1}{c_1}\frac{\left|f'(x)\right|\left|f'(c)\right|}{d^{\frac{\mu}{2}}d^{\frac{\mu}{2}}}\\&\le\frac{2^{\mu}}{c_1}x^{1-\frac{\mu}{2}}c^{1-\frac{\mu}{2}}\left|\frac{f'(x)}{x}\right|\left|\frac{f'(c)}{c}\right|\\&\le C(\mu)\left\|f''\right\|_{L^\infty}^2,\end{aligned}\tag{18}$$

where we have used (16), (17), and the fact that $f'(0)=0$. Therefore the lemma is proved.

3. Uniqueness of a Solution and Convergence of the Regularized Problem

3.1. Uniqueness

We give an existence and uniqueness result for the case when a and k satisfy (9), *i.e.*

$$a'(s)\left(a(s)-a(c)\right)\le C\left(k(s)(s-c)+K(s)-K(c)\right)\tag{19}$$

for all $c\ge 0$, and for all $x\ge c$. We also give a convergence result for a perturbation of Problem 1 to a nondegenerate case in the next subsection.

Under condition (19) and the analogue for the fractional flow function f, its is easy to see, through the proof of Theorem 6.1 of [3], that the following holds.

Theorem 3.1 *Suppose the data* a, f, *and* k *are Lipschitz continuous in their argument* s. *Then Problem 1 has a solution* (p,S), *with*

$$\begin{cases}\dfrac{\partial S}{\partial t}\in L^2\left(0,T,\left(H^1(\Omega)^*\right)\right), \text{ and}\\ 0\le S(x,t)\le 1 \qquad a.e.\ \Omega\times[0,T].\end{cases}\tag{20}$$

Furthermore, if the pairs (f,k) and (a,k) satisfy (9), respectively, and if we assume that $a\left(S(\cdot,\cdot)\right)\nabla p(\cdot,\cdot)\in L^\infty\left(L^\infty\right)$, then the solution is unique.

3.2. Convergence of the Regularized Problem

To get around the difficulties from the degeneracies of the problem, we perturb the diffusion coefficient, k, to k_β in such that a way that $k_\beta \to k$ strongly as $\beta \to 0$. Define

$$K_\beta(s) = \int_0^s k_\beta(\tau)\,\mathrm{d}\tau. \tag{21}$$

Then under the condition (19), the family of solutions (p_β, S_β) converges to the unique solution (p, S) of (1). More precisely.

Theorem 3.2 *Under the conditions of Theorem 3.1, let* (p, S) *be the solution to* (1). *For* $\beta > 0$ *small, say* $0 < \beta < \frac{1}{2}$, *let* (p_β, S_β) *be the solution of* (1) *when* k *is replaced by* k_β, *with* k_β *as described above. Then*

$$\left\|\sqrt{a(S)}\nabla(p_\beta - p)\right\|_{L^2\left(0,T,L^2(\Omega)\right)} \le C\left\|a(S_\beta) - a(S)\right\|_{L^2\left(0,T,L^2(\Omega)\right)}, \tag{22}$$

and

$$\begin{aligned} &\left\|S_\beta - S\right\|^2_{L^\infty\left(0,T,(H^1(\Omega))^*\right)} \\ &+ \eta\int_0^T \left(K_\beta(S_\beta) - K_\beta(S), S_\beta - S\right)(\tau)\,\mathrm{d}\tau \\ &\le C\left\{\left\|K_\beta(\cdot) - K(\cdot)\right\|^\gamma_{L^\infty(0,1)}\right\} \end{aligned} \tag{23}$$

where $\gamma = \dfrac{2+\mu}{\mu}$, with K and K_β defined by (7) and (21), respectively, and for some $\eta > 0$.

4. Examples of Uniqueness

In this Section, we describe the physical meanings of the parameters in Problem 1 and give an example that satisfies conditions (2) through (3). These are purely mathematical examples that might not correspond exactly to models derived through physical experiments. Nevertheless, the shapes of the graphs of the mobilities, the fractional flow function, and the conductivity, as functions of the saturation S, resemble the ones obtained through experiments. See **Figures 1-3**, for $\mu = 3/2$.

For more details on the physical meanings of these parameters, see [1,2,10-12], for instance. We retain the simplicity of the examples below for the mathematical analysis in this paper. For these examples, the diffusion coefficient (also called the total mobility) a of the pressure equation of (1), as well as the fractional flow function, f, satisfy (5).

Physically

$$a(s) = k_1(s) + k_2(s) \tag{24}$$

where k_1 is the mobility of the wetting phase, and the k_2 the mobility of the nonwetting phase. The conductivity of the porous medium is defined by

$$k(s) = \frac{k_1(s)k_2(s)}{k_1(s)+k_2(s)}\frac{\mathrm{d}p_c}{\mathrm{d}s}, \tag{25}$$

where p_c is the capillary pressure. Assuming $\mathrm{d}p_c/\mathrm{d}s$ is bounded and bounded away from 0, we will define, for this analysis,

$$k(s) = \frac{k_1(s)k_2(s)}{k_1(s)+k_2(s)}, \tag{26}$$

dropping, in this manner, the factor dp/ds. The fractional flow function is defined by

$$f(s) = \frac{k_1(s)}{k_1(s)+k_2(s)} \tag{27}$$

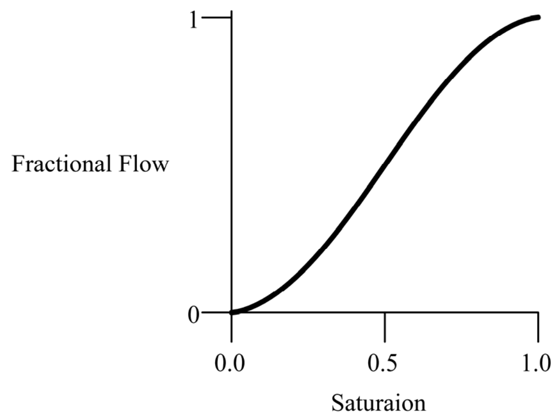

Figure 1. Fractional Flow.

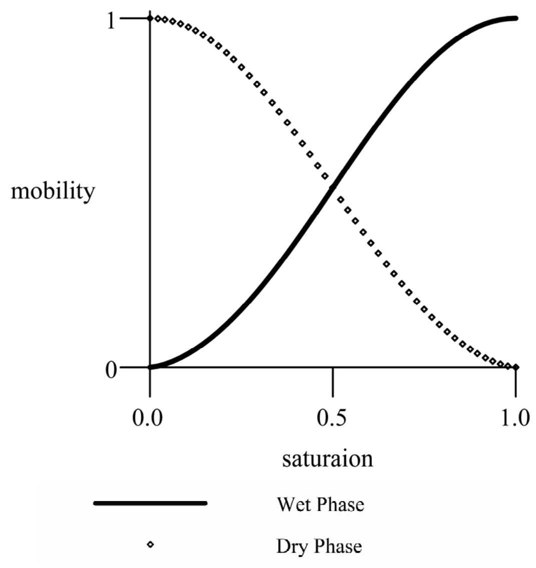

Figure 2. Mobilities.

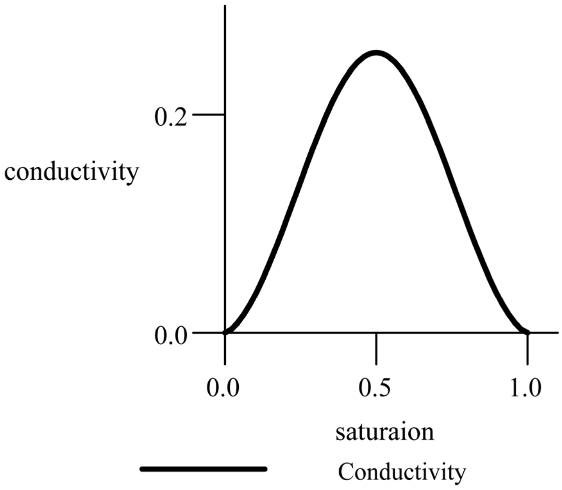

Figure 3. Conductivity of the Medium.

and a, the total mobility, is given by (24).

For numerical modeling of immiscible two-phase flow through porous media, it has been used the following mobilities (see [13], for example).

$$k_1(s) = s^\mu \tag{28}$$

for the wetting, and

$$k_2(s) = (1-s)^\mu \tag{29}$$

for the nonwetting phase, up to multiplicative constants (or bounded functions). For a mathematical analysis purpose, and in order to get an example of uniqueness of a solution of Problem 1, we multiply both (28) and (29) by a bounded function of s on the interval $[0,1]$.

4.1. A case of Uniqueness

We define our new mobilities (up to the same multiplicative constant) by the following. For $1 < \mu \le 2$, let

$$k_1(s) = s^\mu \mathrm{e}^{\mu(s-s^2)}, \tag{30}$$

for the wetting phase, and

$$k_2(s) = (1-s)^\mu \mathrm{e}^{\mu(s-s^2)}, \tag{31}$$

for the non wetting phase. Then, the total mobility (up to a multiplicative constant K, the absolute permeability, which we take here to be 1) is given by

$$a(s) = \left(s^\mu + (1-s)^\mu\right)\mathrm{e}^{\mu(s-s^2)}, \tag{32}$$

while the conductivity of the medium (up to the same multiplicative constant K) is given by

$$k(s) = \frac{s^\mu (1-s)^\mu \mathrm{e}^{\mu(s-s^2)}}{s^\mu + (1-s)^\mu}, \tag{33}$$

and the fractional flow function is given by

$$f(s) = \frac{s^\mu}{s^\mu + (1-s)^\mu}. \tag{34}$$

It is clearly seen that k, defined by (26), satisfies (2) and (3), and that f and a satisfy (5) for $1 < \mu \le 2$. One also checks that if $\mu = 2$, then

$$\left\|a''(\cdot)\right\|^2_{L^\infty([0,1])} + \left\|f''(\cdot)\right\|^2_{L^\infty([0,1])} \le C. \tag{35}$$

Therefore, for k defined by (33), f defined by (34), and a defined by (32), we have that the couple (a,k) and (f,k) satisfy (6), by Lemma 2.1 and Lemma 2.2. Hence the following.

Corollary 4.1 *Under the conditions (26) through (34), if* $\mu = 2$, *Problem 1 has a unique solution* (p,S). *Furthermore, the family of regularized solutions* (p_β, S_β), *as defined by Theorem 3.2, converges to the unique solution* (p,s) *of (1).*

Conditions (32) through (35) and Corollary 4.1 show that condition (9) is not empty, neither is condition (6), which is often used in the proof of the well-posedness of problem 1 or the like and for the convergence of the regularization of the same type of problems ([3,4,6,7,14].

4.2. Graphical Experiments for Uniqueness

One can check, through computations, that a and f, as defined by (32) and (34), respectively, are not twice continuously differentiable, for $1 < \mu < 2$. They fail to be twice differentiable at $s = 0$ and $s = 1$. For some values of μ, $1 < \mu < 2$, namely for the values $\mu = 3/2$ and $\mu = 4/3$, we show graphically, experimenting with several values of c, that condition (19) seems to hold for a and f defined by (32) and (34). So this is an indication that Corollary 4.1 could hold for these values of μ (and, maybe, for $1 < \mu < 2$). We emphasize that this does **not** constitute a rigorous mathematical proof that Corollary 4.1 holds for these values of μ, but it does point to the conjecture that this could be the case.

For our graphical illustrations, we define the functions $G(s)$ and $F(s)$ on the interval $[0,1]$, for a given $1 < \mu < 2$ by,

$$G(s) := a'(s)\left(a(s) - a(c)\right) - c_1^*\left(k(s)(s-c) + K(s) - K(c)\right) \tag{36}$$

and, in the same way,

$$F(s) := f'(s)\left(f(s) - f(c)\right) - c_2^*\left(k(s)(s-c) + K(s) - K(c)\right), \tag{37}$$

for a parameter $0 \le c \le 1$. Here c_1^* and c_2^* are fixed positive constants that are independent of c, but could depend on μ.

For a given $1 < \mu < 2$, for (19) to hold for the pairs (a,k) and (f,k) respectively, we need that $G(s) \le 0$ and $F(s) \le 0$ for all $s \ge c$ and for all, fixed, $0 \le c < 1$. Through the graphs in the **Figures 4** and **5**, we show that this seems to be the case, at least for the chosen values of μ. Here we do this just for two values μ ($\mu = 3/2$ and $\mu = 4/3$), and for each such μ, four values of c and only for the function $G(s)$. However, one can check our claim, by plotting $F(s)$ and $G(s)$ for other values of μ and different choices c for each chosen value of μ.

Here we have used the value $c_1^* = 10$ for both values $\mu = 3/2$ and $\mu = 4/3$. One can see, through **Figures 4-7**, for $\mu = 3/2$, and **Figures 8-11**, for $\mu = 4/3$, that, for these graphs, $G(s) \le 0$ for $s \ge c$ for the chosen values of *c*.

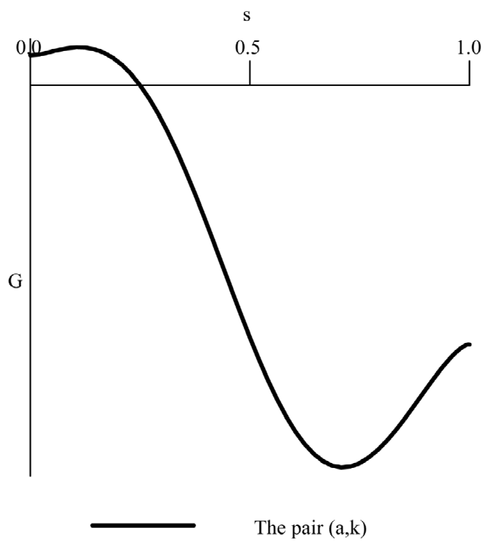

Figure 4. Case. $\mu = 3/2$ **and** $c = 1/4$.

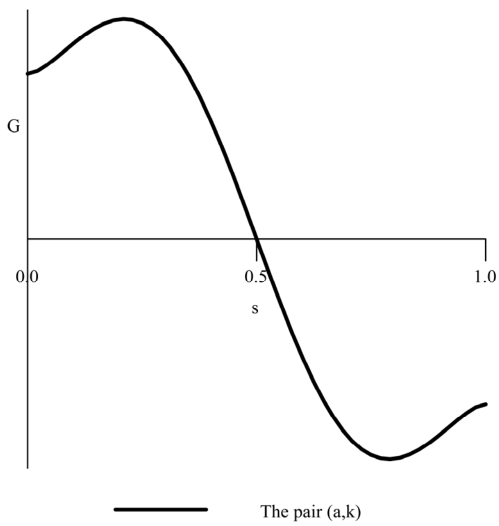

Figure 5. Case. $\mu = 3/2$ **and** $c = 1/2$.

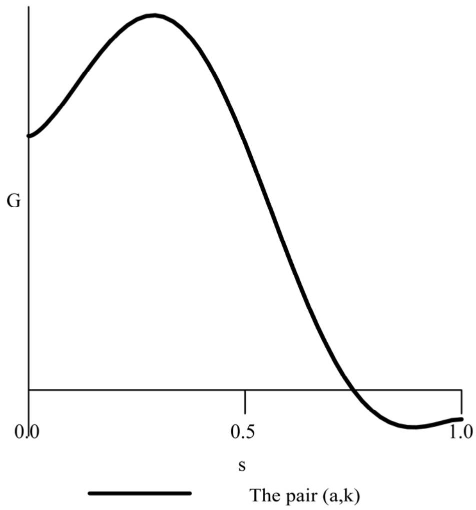

Figure 6. Case. $\mu = 3/2$ **and** $c = 3/4$.

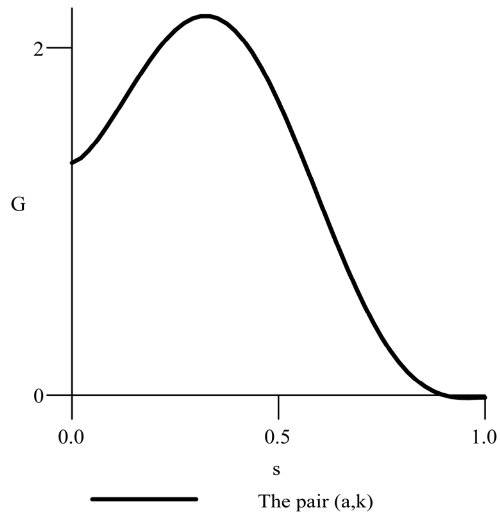

Figure 7. Case: $\mu = 3/2$ **and** $c = 9/10$.

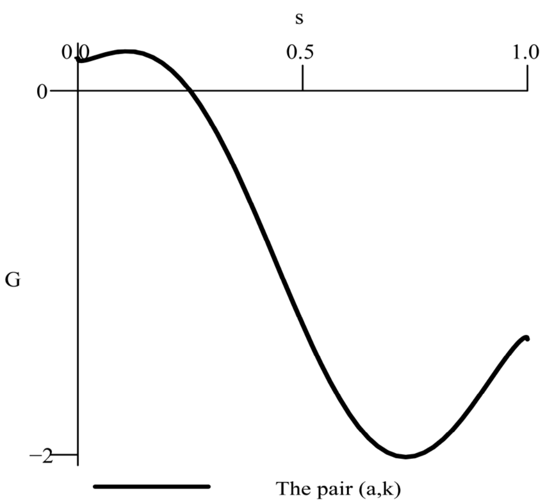

Figure 8. Case: $\mu = 4/3$ **and** $c = 1/4$.

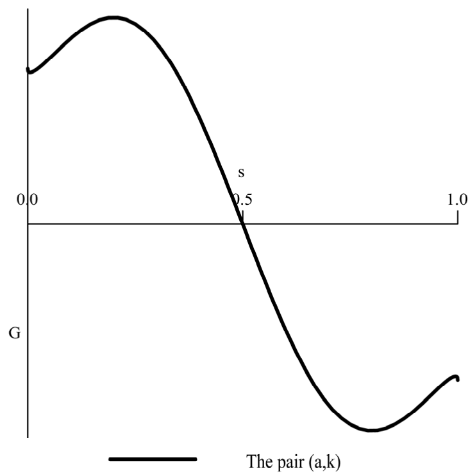

Figure 9. Case: $\mu = 4/3$ and $c = 1/2$.

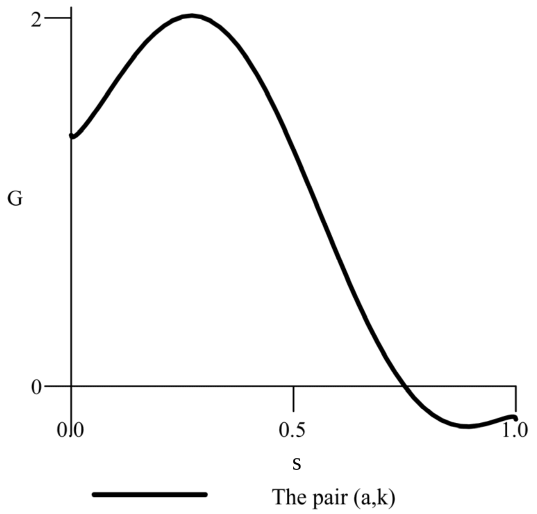

Figure 10. Case: $\mu = 4/3$ and $c = 3/4$.

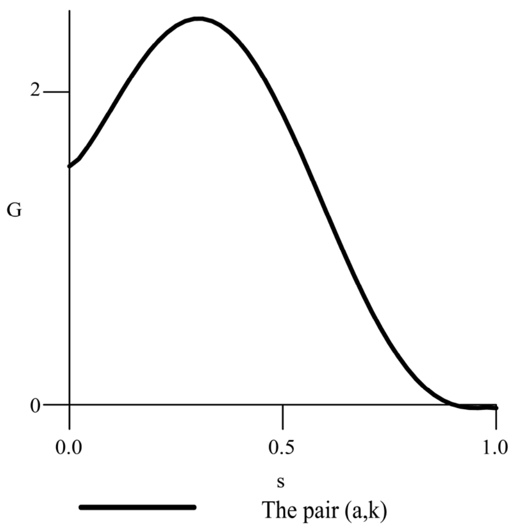

Figure 11. Case: $\mu = 4/3$ and $c = 9/10$.

4.3. Other Graphical Experiments for Uniqueness for $1 < \mu < 2$

In this subsection, we use other arguments to show that the hypotheses of Lemma 2.1 seem to hold for the functions a and f, respectively, for $1 < \mu < 2$. We consider the following functions.

$$F(x,y) := \frac{f'(x)\big(f(x) - f(y)\big)}{k(x)(x-y) + K(x) - K(y)}, \tag{38}$$

and

$$G(x,y) := \frac{a'(x)\big(a(x) - a(y)\big)}{k(x)(x-y) + K(x) - K(y)}. \tag{39}$$

Clearly, condition (9) holds for f and a, if the functions F and G, defined respectively by (38) and (39) are bounded above independently of x and y on the region enclosed by the triangle with vertices (0,0), (1,0), and (1,1) *i.e.* the region $R := \left\{(x,y) \in [0,1]^2, x > y\right\}$. Notice that the common denominator of both functions is positive in the interior of the region R. See **Figure 12** below.

Functions F and G are very complex by their definition, especially for non integer values of μ. They involve the integral-defined function K. They are difficult to handle algebraically. For the present work, we sketch the surfaces representing the two functions, above the region R, for some values of μ, using Maple Software, in order to analyze their boundedness. This is illustrated through the **Figures 13 through 18.**

We notice the smoothness of the surfaces corresponding to the case $\mu = 2$. This suggests that the two functions are definitely bounded in this case. For $\mu = 2$, we show directly that this is indeed the case, *i.e.* that Corollary 4.1 holds. We prove this through the following lemma.

Lemma 4.2 *For* $\mu = 2$, *functions* F *and* G, *defined by* (38) *and* (39), *respectively, are bounded independently of* (x,y) *over the region* **R**.

Proof of Lemma 4.2. From (32) and (34), it is easily seen that

$$f'(x) = \frac{\mu x^{\mu-1}(1-x)^{\mu-1}}{\left(x^{\mu} + (1-x)^{\mu}\right)^2} \tag{40}$$

and

$$a'(x) = \mu x(1-x)\left(x^{\mu-2}(1+2x) + (1-x)^{\mu-2}(2x-3)\right)e^{\mu\left(x-x^2\right)}. \tag{41}$$

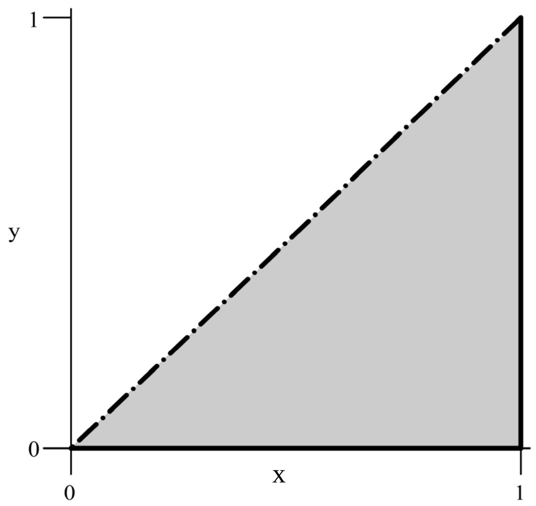

Figure 12. Region R.

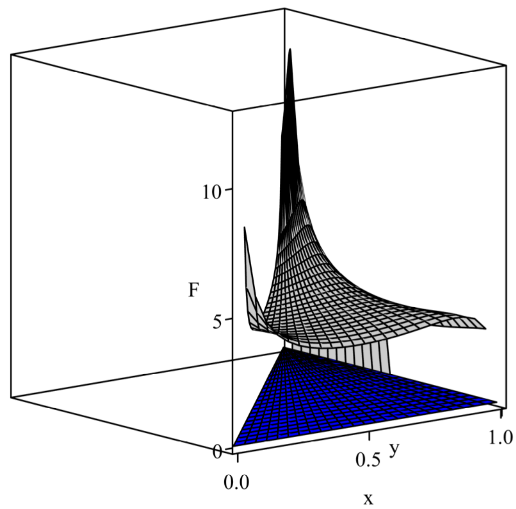

Figure 15. Surface $z = F(x, y)$, over region R, for $\mu = 4/3$.

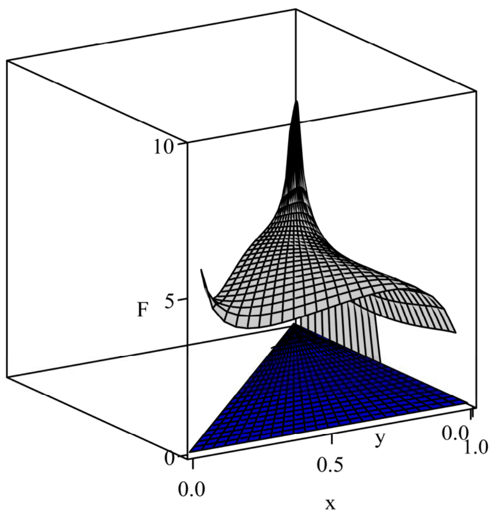

Figure 13. Surface $z = F(x, y)$, over region R, for $\mu = 3/2$.

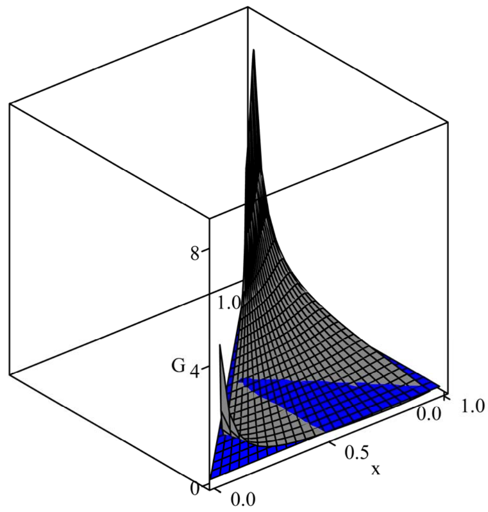

Figure 16. Surface $z = G(x, y)$, over region R, for $\mu = 4/3$.

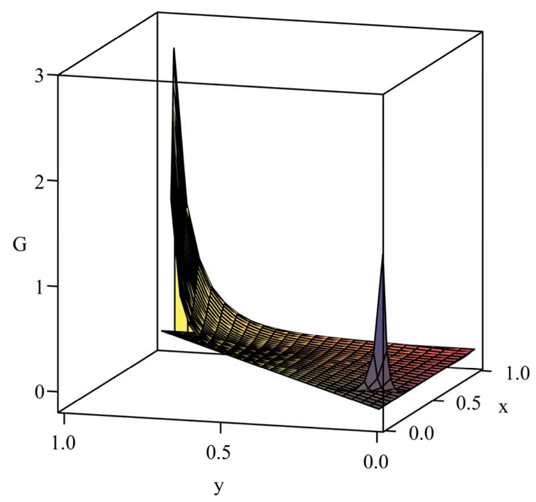

Figure 14. Surface $z = G(x, y)$ over region R, for $\mu = 3/2$.

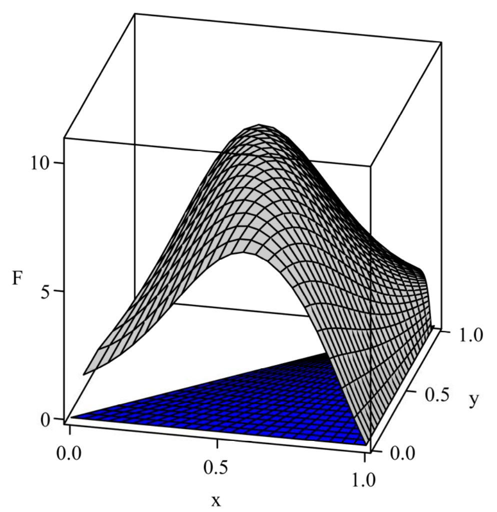

Figure 17. Surface $z = F(x, y)$, over region R, for $\mu = 2$.

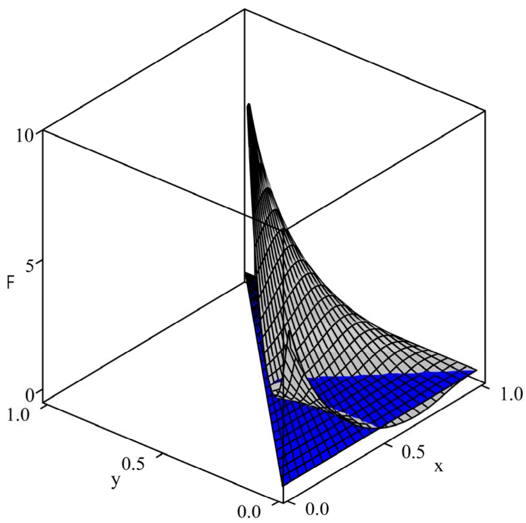

Figure 18. Surface $z = G(x, y)$, over region R, for $\mu = 2$.

On the other hand, by the Mean-Value Theorem, we have

$$F(x,y) = \frac{f'(x) f'(\xi_1)(x-y)}{k(x)(x-y)+k(\xi_2)(x-y)} = \frac{f'(x) f'(\xi_1)}{k(x)+k(\xi_2)} \tag{42}$$

and

$$G(x,y) = \frac{a'(x) a'(\xi_3)(x-y)}{k(x)(x-y)+k(\xi_4)(x-y)} = \frac{a'(x) a'(\xi_3)}{k(x)+k(\xi_4)} \tag{43}$$

where ξ_i, $1 \le i \le 4$, are between x and y, and where we have used (7). We obtain from (42) and (43) that

$$|F(x,y)| \le \frac{|f'(x)||f'(\xi_1)|}{2\sqrt{k(x)k(\xi_2)}} \tag{44}$$

and

$$|G(x,y)| \le \frac{|a'(x)||a'(\xi_3)|}{2\sqrt{k(x)k(\xi_4)}} \tag{45}$$

Combining (33),(40), (41), (44), and (45), we obtain

$$|F(x,y)| = O\left(x^{\frac{\mu}{2}-1}(1-x)^{\frac{\mu}{2}-1}\right) \tag{46}$$

and

$$|G(x,y)| = O\left(x^{1-\frac{\mu}{2}}(1-x)^{1-\frac{\mu}{2}}\right), \tag{47}$$

as $y \to x$, $y < x$. Hence, if $\mu = 2$, $F(x,y)$ and $G(x,y)$ are bounded on R. This proves the lemma.

Conjecture. *Corollary 4.1 also holds for* $\frac{3}{2} \le \mu < 2$.

We hope, in a future work, to be able to prove this claim or give a counterexample that disproves it. If this claim happens to be true, that would give examples of functions a and f that are not very smooth but for which there is uniqueness for the problem (1).

5. Conclusions

In this paper, we have revisited the problem of uniqueness for the pressure/saturation system. A new sufficient condition for uniqueness has been established and we have showed that the old conditions for uniqueness (conditions (3), (5), and (35)) imply the new condition. An example of a family of data (depending on a parameter μ) has been constructed that gives uniqueness for $\mu = 2$. For the general case $1 < \mu < 2$, we have illustrated graphically (without a rigorous proof) that there could be uniqueness for these cases. A sequel of this paper should concern itself with a rigorous proof (or disproof) of this claim. It should also concern itself with the general case, especially the case of convection dominated flow.

6. References

[1] G. Chavent and J. Jaffre, "Mathematical Models and Finite Element for Reservoir Simulation: Single Phase, Multiphase and Multicomponent Flows through Porous Media," North-Holland, New York, 1986.

[2] R. E. Ewing, "Problems Arising in the Modeling of Processes for Hydrocarbon Recovery," In R. E. Ewing, Ed., *The Mathematics of Reservoir Simulation*, S.I.A.M., Philadelphia, 1983, pp. 3-34.

[3] K. B. Fadimba, "On Existence and Uniqueness for a Coupled System Modeling Immiscible flow through a Porous Medium," *Journal of Mathematical Analysis and Applications*, Vol. 328, No. 2, 2007, pp. 1034-1056. doi:10.1016/j.jmaa.2006.06.012

[4] Z. J. Chen and N. L. Khlopina, "Degenerate Two-Phase Flow Problems: Error Estimates," *Communication in Applied Analysis*, Vol. 5, 2001, pp. 503-521.

[5] K. B. Fadimba, "Regularization and Numerical Methods for a Class of Porous Medium Equations," PhD Thesis, University of South Carolina, Columbia, 1993.

[6] K. B. Fadimba and R. C. Sharpley, "A Priori Estimates and Regularization for a Class of Porous Medium Equations," *Nonlinear World*, Vol. 2, 1995, pp. 13-41.

[7] Z. Chen and R. E. Ewing, "Degenerate Two-Phase Incompressible Flow-III: Sharp Error Estimates," *Numerische Mathematik*, Vol. 90, No. 2, 2001, pp. 215-240. doi:10.1007/s002110100291

[8] K. B. Fadimba and R. C. Sharpley, "Galerkin Finite Element Method for a Class of Porous Medium Equations," *Nonlinear Analysis*: *Real World Applications*, Vol. 5, No. 2, 2004, pp. 355-387. doi:10.1016/j.nonrwa.2003.07.001

[9] M. E. Rose, "Numerical Methods for Flows through Porous Media-I," *Mathematics of Computations*, Vol. 40, No. 162, 1983, pp. 437-467.

[10] J. Bear and A. Verruijt, "Modeling Groundwater Flow and Pollution," D. Reidel Publication Company, Dodreich, 1987.

[11] R. E. Ewing and H. Wang, "A Summary of Numerical Methods for Time-Dependent Advection Dominated Partial Differential Equations," *Journal of Computational and Applied Mathematics*, Vol. 128, No. 1-2, 2001, pp. 423-445. doi:10.1016/S0377-0427(00)00522-7

[12] Z. Chen, G. Huan, and Y Ma, "Computational Methods for Multiphase Flows in Porous Media," SIAM Computational Science and Engineering, Philadelphia, 2006.

[13] D. L. Smylie, "A near Optimal Order Approximation to a Class of Two-Sided Nonlinear Parabolic Partial differential Equations," PhD Thesis, University of Wyoming, Laramie, 1989.

[14] Z. Chen, "Degenerate Two-Phase Incompressible Flow-I: Existence, Uniqueness, and Regularity of a Weak Solution," *Journal of Differential Equations*, Vol. 171, 2001, pp. 203-232. doi:10.1006/jdeq.2000.3848