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Estimate of Multiple Attracing Domains for Cohen-Grossberg Neural Network with Distributed Delays

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Abstract

In this paper, we present multiplicity results of exponential stability and attracting domains for Cohen-Grossberg neural network (CGNN) with distributed delays. We establish new criteria for the coexistence of 2^N equilibrium points and estimate their attracting domains. Moreover, we base our criteria on coefficients of the networks and the derivative of activation functions within the attracting domains. It is shown that our results are new and complement corresponding results existing in the previous literature.

Keywords: Cohen-Grossberg Networks, Distributed Delays, Exponential Stability, Attracting Domains

1. Introduction

Cohen-Grossberg neural network (see [1,2]) is usually described by the following differential equations system

$$\frac{\mathrm{d}u_i(t)}{\mathrm{d}t} = -a_i(u_i(t))\left[d_i(u_i(t)) - \sum_{j=1}^{N} b_{ij} g_j(u_j(t))\right],$$

where $i \in \mathbb{N} := \{1, 2, \cdots, N\}$, $N \geq 2$ is the number of neurons in the network; $u_i(t)$ describes the state variable of neuron *i* at time t; $a_i(\cdot)$ represents an amplification function and the function $d_i(\cdot)$ can include a constant term indicating a fixed input to the network; $b_{ij}(t)$ weights the strength of the *j* unit on the *i*th unit at time *t*; the activation function $g_j(\cdot)$ shows how the neurons react to the input. CGNN not only has a wide range of applications in pattern recognition, associative memory and combinatorial optimization but also includes a number of models from neurobiology, population biology and evolution theory. Hence studies on stability of CGNN with or without delays have been vigorously done and many criteria have been obtained so far [3-14].

In the applications of neural network to associative memory storage or pattern recognitions, the coexistence of multiple stable equilibrium points is an important feature [15-19,20-21]. However, few papers focus on the existence of multiple equilibrium points of CGNN and their complex convergence analysis. Hence, we should consider multistability of the following CGNN with distributed delays

$$\frac{\mathrm{d}u_i(t)}{\mathrm{d}t} = -a_i(u_i(t))\Big[d_i(u_i(t)) - \sum_{j=1}^{N} b_{ij} g_j\Big(\int_{-\infty}^{t} k_{ij}(t-s)u_j(s)\,\mathrm{d}s\Big)\Big], \tag{1.1}$$

where the delay kernel function $k_{ij}(t)$ is assumed to be piecewise continuous and satisfying

$$k_{ij}(t) \geq 0, \int_{-\infty}^{t} k_{ij}(s)\,\mathrm{d}s = l_{ij} > 0, \int_{-\infty}^{t} k_{ij}(s)\mathrm{e}^{\sigma s}\mathrm{d}s < \infty,$$

where σ is a positive constant. In this paper, we not only derive new criteria for the existence of 2^N equilibrium points of CGNN (1.1) but also estimate attracting domains for these equilibrium points. When we relax our conditions to be common assumptions, our results improve corresponding results in [12]. Moreover, our results can extend the corresponding results in [3-13] to local exponential stability of multiple equilibrium points of Cohen-Grossberg networks. It is shown that our results are new and complement the existing results in the literature.

The rest of this paper is organized as follows. In Section 2, we should make some preparations by giving some notations, assumptions and a basic lemma. Meanwhile, we discuss the existence of 2^N equilibrium points of CGNN (1.1). In Section 3, we not only discuss local exponential stability of 2^N equilibrium points of CGNN (1.1) but also compare our results with existing ones in the literature. In Section 4, two examples are given to illustrate the new results. Finally, concluding remarks are given in Section 5.

2. Coexistence of Equilibrium Points

In this paper, we denote by $C\left(\left(-\infty,0\right],\mathbb{R}^N\right)$ the set of all continuous and bounded mappings from $\left(-\infty,0\right]$ to $\mathbb{R}^N$ equipped with p-norm $\|\cdot\|_p \left(p\geq 1\right)$ defined by

$$\|\phi\| = \left[\sum_{i=1}^{N}\sup_{-\infty<s\leq 0}\left|\phi_i\left(s\right)\right|^p\right]^p,$$

Where $\phi=\left(\phi_1,\phi_2,\cdots,\phi_N\right)\in C\left(\left(-\infty,0\right],\mathbb{R}^N\right)$, For any given $\phi\in C\left(\left(-\infty,0\right],\mathbb{R}^N\right)$, we denote by $u\left(t;\phi\right)$ the solution of CGNN (1.1) with $u\left(s\right)=\phi\left(s\right)$ for all $s\in\left(-\infty,0\right]$. Given any $\ell\geq 0$, we define $u_\ell\left(s;\phi\right)=u\left(s+\ell;\phi\right)$ for all $s\in\left(-\infty,0\right]$, then

$$u_\ell\left(\cdot;\phi\right)\in C\left(\left(-\infty,0\right],\mathbb{R}^N\right).$$

Throughout this paper we always assume that $\left(S_1\right)$ For each $i\in\mathbb{N}$, $a_i\left(\cdot\right)$ is a continuous function defined on $\mathbb{R}$. Meanwhile, we assume that

$$0<\alpha_i^0\leq a_i\left(v\right)\leq\alpha_i^1<+\infty, v\in\mathbb{R}.$$

$\left(S_2\right)$ For each $i\in\mathbb{N}$, $a_i\left(\cdot\right)$, $g_i\left(\cdot\right)\in C^2\left(\mathbb{R}\right)$ and there exist constants $d_i^\dagger$, $d_i^\perp$, $g_i^\perp$ such that

$$\begin{cases} 0<d_i^\dagger<\dot{d}_i\left(v\right)<d_i^\perp, \lim\limits_{v\to\pm\infty}\left|g_i\left(v\right)\right|\leq g_i^\perp, \\ 0=\lim\limits_{v\to\pm\infty}\dot{g}_i\left(v\right)<\dot{g}_i\left(v\right)\leq\sup\limits_{v\in\mathbb{R}}\dot{g}_i\left(v\right)=\dot{g}_i\left(0\right), v\in\mathbb{R} \\ v\left(\ell_i\ddot{g}_i\left(v\right)-\ddot{d}_i\left(v\right)\right)<0, v\neq 0, v\in\mathbb{R} \end{cases}$$

where $\ell_i=b_{ii}l_{ii}^2$ and $\lim_{v\to\pm\infty}d_i\left(v\right)=\pm\infty$.

Remark 2.1. For Hopfield-type neural networks [5,8-10], we have $a_i\left(v\right)\equiv 1$, $d_i\left(v\right)\equiv d_i v$, $g_i\left(v\right)=\tanh\left(v\right)$, where $d_i>0$ is a constant. Obviously, we can check that $\dot{d}_i\left(v\right)=d_i$, $\ddot{d}_i\left(v\right)=0$, $\lim_{v\to\pm\infty}\left|\tanh\left(v\right)\right|<1$, $1=\sup_{v\in\mathbb{R}}\tanh\left(v\right)'>\tanh\left(v\right)'=1-\tanh^2\left(v\right)>0$, $\lim_{v\to\pm\infty}\tanh\left(v\right)'=0$, $v\tanh\left(v\right)''<0$. Hence, $\left(S_1\right)$ and $\left(S_2\right)$ hold.

We say a constant vector $\boldsymbol{u}=\left(u_1,\cdots,u_N\right)^{\mathrm{T}}\in\mathbb{R}^N$ is an equilibrium point of CGNN (1.1) if for each $i\in\mathbb{N}$,

$$d_i\left(u_i\left(t\right)\right)=\sum_{j=1}^{N}b_{ij}g_j\left(u_j\left(t\right)\right)$$

Consider

$$F_i\left(v\right)=-d_i\left(v\right)+b_{ii}g_i\left(u_i\left(t\right)\right)$$

where $v\in\mathbb{R}$, $i\in\mathbb{N}$. Then it follows

Lemma 2.1. Assume $\left(S_1\right)-\left(S_2\right)$ and the following assumption

$$\left(H_1^A\right)\ d_i^\perp<b_{ii}l_{ii}\sup_{v\in\mathbb{R}}\dot{g}_i\left(v\right),\ i\in\mathbb{N}$$

hold. For each $i\in\mathbb{N}$, there exist only two points v_{i1} and v_{i2} with $v_{i1}<0<v_{i2}$ such that $\dot{F}_i\left(v_{i\varsigma}\right)=0$ and $\dot{F}_i\left(v\right)\cdot\operatorname{sgn}\left\{\left(v-v_{i1}\right)\left(v-v_{i2}\right)\right\}=0$, where $v\in\mathbb{R}$ and $v\neq v_{i\varsigma}$, $\varsigma=1,2$.

Proof. We get from $\left(H_1^A\right)$ and $\left(S_2\right)$ that

$$\dot{F}_i\left(0\right)=-\dot{d}_i\left(0\right)+b_{ii}l_{ii}\dot{g}_i\left(0\right)>0$$

and

$$\dot{F}_i\left(v\right)<-d_i^\dagger+b_{ii}l_{ii}\lim_{v\to\pm\infty}\dot{g}_i\left(v\right)=-d_i^\dagger<0$$

as $v\to\pm\infty$. It follows from $\left(S_2\right)$ that $v\ddot{F}_i\left(v\right)<0$ which implies that $\dot{F}_i\left(v\right)$ is strictly increasing on $\left(-\infty,0\right]$ and is strictly decreasing on $\left[0,+\infty\right)$. Therefore, there exist only two points v_{i1} and v_{i2} with $v_{i1}<0<v_{i2}$ such that $\dot{F}_i\left(v_{i\varsigma}\right)=0$ and

$$\dot{F}_i\left(v\right)\cdot\operatorname{sgn}\left\{\left(v-v_{i1}\right)\left(v-v_{i2}\right)\right\}=0,$$

where $v\in\mathbb{R}$ and $v\neq v_{i\varsigma}$, $\varsigma=1,2$. The proof is complete.

It follows from $\left(H_1^A\right)$ that $b_{ii}>0$ for each $i\in\mathbb{N}$. Now, we consider the following additional assumption:

$$\left(H_2^A\right)\quad\left(-1\right)^\varsigma\cdot F_i\left(v_{i\varsigma}\right)>\sum_{j=1,j\neq i}^{N}\left|b_{ij}\right|g_j^\perp, \varsigma=1,2,\quad i\in\mathbb{N}$$

Take $\varsigma=1$ in $\left(H_2^A\right)$, it is easy for us to get that

$$F_i\left(v_{i1}\right)+\sum_{j=1,j\neq i}^{N}\left|b_{ij}\right|g_j^\perp<0, i\in\mathbb{N} \tag{2.1}$$

Due to $F_i\left(v\right)\to+\infty$ as $v\to-\infty$, Lemma 2.1 and the continuity of $F_i\left(v\right)$, there exists a unique $v_{i1}^\dagger$ with $v_{i1}^\dagger<v_{i1}$ such that

$$\begin{cases} F_i\left(v_{i2}^\dagger\right)-\sum\limits_{j=1,j\neq i}^{N}\left|b_{ij}\right|g_j^\perp=0, \\ F_i\left(v\right)-\sum\limits_{j=1,j\neq i}^{N}\left|b_{ij}\right|g_j^\perp>0, \text{ for all } v\in\left[v_{i2},v_{i2}^\dagger\right) \end{cases} \tag{2.2}$$

Take $\varsigma=2$ in $\left(H_2^A\right)$,by similar argument, we derive that there exists a unique $v_{i2}^\dagger$ with $v_{i2}<v_{i2}^\dagger$ such that

$$\begin{cases} F_i\left(v_{i1}^\dagger\right)+\sum\limits_{j=1,j\neq i}^{N}\left|b_{ij}\right|g_j^\perp=0, \\ F_i\left(v_{i1}\right)+\sum\limits_{j=1,j\neq i}^{N}\left|b_{ij}\right|g_j^\perp<0, \text{ for all } v\in\left(v_{i1}^\dagger,v_{i1}\right] \end{cases} \tag{2.3}$$

Let

$$v_{i1}^\perp=d_i^{-1}\left(-\sum_{j=1}^{N}\left|b_{ij}\right|g_j^\perp\right), v_{i2}^\perp=d_i^{-1}\left(\sum_{j=1}^{N}\left|b_{ij}\right|g_j^\perp\right)$$

Due to the monotonicity of $d_i\left(\cdot\right)$, it is easy for us to check that $v_{i1}^{\perp} < v_{i1}^{\dagger} < 0 < v_{i2}^{\dagger} < v_{i2}^{\perp}$. For each $i \in \mathbb{N}$, define $\Delta_{i1} = \left[v_{i1}^{\perp}, v_{i1}^{\dagger}\right], \Delta_{i2} = \left[v_{i2}^{\dagger}, v_{i2}^{\perp}\right]$. Hence, we can construct 2^N subsets $\Delta^{\Sigma} = \Delta_{1\varsigma_1} \times \Delta_{2\varsigma_2} \times \cdots \times \Delta_{N\varsigma_N} \subset \mathbb{R}^N$, where

$$\Sigma = (\varsigma_1, \varsigma_2, \cdots, \varsigma_N) \text{ with } \varsigma_i = 1, 2, \ i \in \mathbb{N}.$$

With these notations, we have the following theorem:

Theorem 2.1. Under the assumptions $(S_1)-(S_2)$ and $\left(H_1^A\right)-\left(H_2^A\right)$, there exist at least 2^N equilibrium points of CGNN (1.1).

Proof. For each $\Sigma = (\varsigma_1, \varsigma_2, \cdots, \varsigma_N)$ with $\varsigma_i = 1, 2$, $i \in \mathbb{N}$, we define $F^{\Sigma} = \left(F_1^{\Sigma}, F_2^{\Sigma}, \cdots, F_N^{\Sigma}\right)$

With $F^{\Sigma} : \Delta^{\Sigma} \to \mathbb{R}^N$ by

$$F_i^{\Sigma}(\boldsymbol{u}) = d_i^{-1}\left[\sum_{j=1}^{N} b_{ij} g_j\left(l_{ij} u_j\right)\right],$$

where $\boldsymbol{u} = (u_1, u_2, \cdots, u_N)^T \in \Delta^{\Sigma}$. It is obvious that for any $u \in \Delta^{\Sigma}$, we have $F_i^{\Sigma}(\boldsymbol{u}) \in \left[v_{i1}^{\perp}, v_{i2}^{\perp}\right]$ for each $i \in \mathbb{N}$. For each ς_i, $i \in \mathbb{N}$, there are two cases for us to further discussion.

Case I: If $\varsigma_i = 1$, *i.e.*, $u_i \le v_{i1}^{\dagger}$, then from $\left(H_1^A\right)$ and (2.2) we get

$$\begin{aligned} F_i^{\Sigma}(u) &= d_i^{-1}\left[\sum_{j=1}^{N} b_{ij} g_j\left(l_{ij} u_j\right)\right] \\ &\le d_i^{-1}\left[b_{ii} g_i\left(l_{ii} v_{i1}^{\dagger}\right) + \sum_{j=1, j\ne i}^{N} \left|b_{ij}\right| g_j^{\perp}\right] = v_{i1}^{\dagger} \end{aligned} \tag{2.4}$$

Case II: If $\varsigma_i = 2$, *i.e.*, $u_i \ge v_{i2}^{\dagger}$, then from $\left(H_1^A\right)$ and (2.3) we get

$$\begin{aligned} F_i^{\Sigma}(u) &= d_i^{-1}\left[\sum_{j=1}^{N} b_{ij} g_j\left(l_{ij} u_j\right)\right] \\ &\ge d_i^{-1}\left[b_{ii} g_i\left(l_{ii} v_{i2}^{\dagger}\right) - \sum_{j=1, j\ne i}^{N} \left|b_{ij}\right| g_j^{\perp}\right] = v_{i2}^{\dagger} \end{aligned} \tag{2.5}$$

Hence $F_i^{\Sigma}(\boldsymbol{u}) \in \Delta_{i\varsigma_i}$, that is $F^{\Sigma}(\boldsymbol{u}) \in \Delta^{\Sigma}$ for all $\boldsymbol{u} \in \Delta^{\Sigma}$. By Brouwer's fixed point theory, for each Σ, there exist at least one $u^{\Sigma} \in \Delta^{\Sigma}$ such that $F^{\Sigma}(\boldsymbol{u}) = u^{\Sigma}$. Therefore, there exist at least 2^N equilibrium points of CGNN (1.1). The proof is complete.

Next we should make some preparations for the coming section. For each $i \in \mathbb{N}$, we define the following subsets of $C((-\infty, 0], \mathbb{R})$ as

$$H_{i1} = \left\{\phi \in C\left((-\infty, 0], \mathbb{R}^N\right) \middle| \phi(s) \le v_{i1}^{\dagger}\right\}$$

$$H_{i2} = \left\{\phi \in C\left((-\infty, 0], \mathbb{R}^N\right) \middle| \phi(s) \ge v_{i2}^{\dagger}\right\}$$

Hence, we have construct 2^N subsets $H^{\Sigma} = H_{1\varsigma_1} \times H_{2\varsigma_2} \times \cdots \times H_{N\varsigma_N} \subset C\left((-\infty, 0], \mathbb{R}^N\right)$. Given any $\Sigma = (\varsigma_1, \varsigma_2, \cdots, \varsigma_N)$ with $\varsigma_i = 1, 2$, we define semi-close subsets for each $i \in \mathbb{N}$,

$$O_i^{\Sigma} = \left\{v \in \mathbb{R} \middle| (-1)^{\varsigma_1} v \ge (-1)^{\varsigma_1} v_{i\varsigma_i}^{\dagger}\right\}.$$

For any $\phi \in H^{\Sigma}$, we get $O^{\Sigma} = O_{1\varsigma_1} \times O_{2\varsigma_2} \times \cdots \times O_{N\varsigma_N}$ for all $s \in (-\infty, 0]$.

3. Stability and Estimation of Attracting Domains

Theorem 3.1. Assume that assumptions $(S_1)-(S_2)$ and $\left(H_1^A\right)-\left(H_2^A\right)$ hold. For each Σ, if $\phi \in H^{\Sigma}$, then $u_t(\cdot;\phi) \in H^{\Sigma}$ for all $t \in \mathbb{R}$.

Proof. Fix $\Sigma = (\varsigma_1, \varsigma_2, \cdots, \varsigma_N)$. For any $\phi \in H^{\Sigma}$, we should prove that $u_t(\cdot;\phi) \in H^{\Sigma}$ for all $t \in \mathbb{R}$. For each $i \in \mathbb{N}$, we only consider the case $\varsigma_i = 2$, *i.e.*, $\phi_i(s) \ge v_{i2}^{\dagger}$ for all $s \in (-\infty, 0]$. We assert that, for any sufficiently small $\varepsilon > 0\left(\varepsilon \le v_{i2}^{\dagger} - v_{i2}\right)$, the solution $u_i(t;\phi) \ge v_{i2}^{\dagger} - \varepsilon$ holds for all $t \ge 0$. If this is not true, there exists a $t^* > 0$ such that $u_i(t^*) \ge v_{i2}^{\dagger} - \varepsilon$, $\dot{u}_i(t^*) \le 0$ and $u_i(t) \ge v_{i2}^{\dagger} - \varepsilon$ for $t \in (-\infty, t^*]$. Due to $\left(H_1^A\right)$, (2.3) and the monotonicity of $g_i(\cdot)$, we derive from CGNN (1.1) that

$$\begin{aligned} &\frac{\mathrm{d}u_i(t^*)}{\mathrm{d}t} \\ &\ge -a_i\left(v_{i2}^{\dagger} - \varepsilon\right) \\ &\quad\times\left[d_i\left(v_{i2}^{\dagger} - \varepsilon\right) - b_{ii} g_i \int_{-\infty}^{t} k_{ii}(t-s) u_j(s)\,\mathrm{d}s - \sum_{j=1}^{N} \left|b_{ij}\right| g_j^{\perp}\right] \\ &\ge -a_i\left(v_{i2}^{\dagger} - \varepsilon\right) \\ &\quad\times\left[d_i\left(v_{i2}^{\dagger} - \varepsilon\right) - b_{ii} g_i\left(l_{ii}\left(v_{i2}^{\dagger} - \varepsilon\right)\right) - \sum_{j=1}^{N} \left|b_{ij}\right| g_j^{\perp}\right] \\ &\ge -a_i\left(v_{i2}^{\dagger} - \varepsilon\right)\left[F_i\left(v_{i2}^{\dagger} - \varepsilon\right) - \sum_{j=1}^{N} \left|b_{ij}\right| g_j^{\perp}\right] > 0 \end{aligned} \tag{3.1}$$

which leads to a contradiction. Since the choice of ε is arbitrary, for each $i \in \mathbb{N}$, if $\phi_i(s) \ge v_{i2}^{\dagger}$ for all $s \in (-\infty, 0]$, then $u_i(t;\phi) \ge v_{i2}^{\dagger}$ holds for all $t \ge 0$. When $\varsigma_i = 1$, similar argument can be performed to show that if $\phi_i(s) \le v_{i1}^{\dagger}$ for all $s \in (-\infty, 0]$, then $u_i(t;\phi) \le v_{i1}^{\dagger}$ holds for all $t \ge 0$. Hence, for any $\phi \in H^{\Sigma}$, we have that $u_t(\cdot;\phi) \in H^{\Sigma}$ for all $t \ge 0$. The proof is complete.

Definition 3.1. Let $u^{\Sigma} = \left(u_1^{\Sigma}, u_2^{\Sigma}, \cdots, u_N^{\Sigma}\right)$ be an equilibrium point of CGNN (1.1) and $Y\left(u^{\Sigma}\right) \subset C\left((-\infty, 0], \mathbb{R}^N\right)$ be a neighborhood of u^{Σ}. If there exist $\kappa > 0$ and

$M>1$ such that for any $\boldsymbol{\psi}=(\psi_1,\psi_2,\cdots,\psi_N)^{\mathrm{T}}\in Y(u^{\Sigma})$ and all $t\ge 0$,

$$\left[\sum_{i=1}^{N}\left|u_i(t;\psi)-u_i^{\Sigma}\right|^p\right]^{\frac{1}{p}}\le M\mathrm{e}^{-\kappa t}\left[\sum_{i=1}^{N}\sup_{s\in(-\infty,0]}\left|\psi_i(s)-u_i^{\Sigma}\right|^p\right]^{\frac{1}{p}}$$

Where $\boldsymbol{u}(t;\psi)=(u_1(t;\psi),u_2(t;\psi),\cdots,u_N(t;\psi))^{\mathrm{T}}$ is the solution of CGNN (1.1), then u^{Σ} is said to be locally exponentially stable and $Y(u^{\Sigma})$ is contained in the attracting domain of u^{Σ}.

The existence of 2^N equilibrium points of CGNN (1.1) follows from Theorem 2.1. For any given Σ, let $x_i(t)=u_i(t;\phi)-u_i^{\Sigma}$ with $\phi\in H^{\Sigma}$ and $\boldsymbol{u}\in\Delta^{\Sigma}$, where $i\in\mathbb{N}$. Then CGNN (1.1) can be written as

$$\dot{x}_i(t)=-\hat{a}_i(x_i(t))\left[\hat{d}_i(x_i(t))-\sum_{j=1}^{N}b_{ij}\hat{g}_j\left(\int_{-\infty}^{t}k_{ij}(t-s)x_j(s)\mathrm{d}s\right)\right],\tag{3.2}$$

where

$$\hat{a}_i(x_i(t))=\hat{a}_i(x_i(t)+u_i^{\Sigma}),$$
$$\hat{d}_i(x_i(t))=d_i(x_i(t)+u_i^{\Sigma})-d_i(u_i^{\Sigma}),$$
$$\hat{g}_j\left(\int_{-\infty}^{t}k_{ij}(t-s)x_j(s)\,\mathrm{d}s\right)=g_j\left(\int_{-\infty}^{t}k_{ij}(t-s)x_j(s)\,\mathrm{d}s\right)-g_j(l_{ij}u_j^{\Sigma})$$

Theorem 3.2. Assume that assumptions $(S_1)-(S_2)$ and $(H_1^A)-(H_2^A)$ hold. For each Σ, if there exist positive constants $\lambda_i\ (i\in\mathbb{N})$ and ρ such that for each $i\in\mathbb{N}$

$$\begin{aligned}&\alpha_i^0\inf_{v\in O_i^{\Sigma}}\dot{d}_i(v)-\rho\\&>\frac{p-1}{p}\alpha_i^1\sum_{j=1}^{N}\left|b_{ij}\right|\sup_{v\in O_j^{\Sigma}}\dot{g}_j(v)\int_0^{\infty}k_{ij}(s)\mathrm{e}^{\rho s}\,\mathrm{d}s\\&\quad+\frac{1}{p}\sum_{j=1}^{N}\frac{\lambda_j}{\lambda_i}\alpha_j^1\left|b_{ij}\right|\sup_{v\in O_i^{\Sigma}}\dot{g}_i(v)\int_0^{\infty}k_{ji}(s)\mathrm{e}^{\rho s}\,\mathrm{d}s\end{aligned}\tag{3.3}$$

Then u^{Σ} is locally exponentially stable and H^{Σ} is contained in the attracting domain of u^{Σ}.

Proof. Fix $\Sigma=(\varsigma_1,\cdots,\varsigma_N)$. Let $\boldsymbol{u}(t)=(u_1(t;\psi),\cdots,u_N(t;\psi))^{\mathrm{T}}$ be a solution of CGNN (1.1) with $\phi\in H^{\Sigma}$ and u^{Σ} is an equilibrium point of CGNN (1.1) in Δ^{Σ}. By Theorem 3.1, we know that $u_t(\cdot;\phi)\in H^{\Sigma}$ for all $t\ge 0$. It is obvious that $u^{\Sigma}\in O^{\Sigma}$ and $u(t)\in O^{\Sigma}$ for all $t\ge 0$. For each $i\in\mathbb{N}$, let $x_i(t)=u_i(t;\phi)-u_i^{\Sigma}$ and let $X_i(t)=\mathrm{e}^{\rho t}\left|x_i(t)\right|$. In view of (3.2), we obtain

$$\begin{aligned}D^+X_i(t)&\le\rho X_i(t)+\mathrm{e}^{\rho t}\Big[-\dot{d}_i(\eta_i)\alpha_i^0\left|x_i(t)\right|\\&\quad+\alpha_i^1\sum_{j=1}^{N}\left|b_{ij}\right|\dot{g}_j(\zeta_j)\int_{-\infty}^{t}k_{ij}(t-s)x_j(s)\,\mathrm{d}s\Big],\end{aligned}$$

i.e.,

$$\begin{aligned}D^+X_i(t)&\le\left(\rho-\dot{d}_i(\eta_i)\alpha_i^0\right)X_i(t)\\&\quad+\alpha_i^1\sum_{j=1}^{N}\left|b_{ij}\right|\dot{g}_j(\zeta_j)\int_{-\infty}^{t}k_{ij}(t-s)X_j(s)\mathrm{d}s.\end{aligned}\tag{3.4}$$

where $\dot{d}_i(\eta_i)=\inf_{v\in O_i^{\Sigma}}\dot{d}_i(v)$ and $\dot{g}_j(\zeta_j)=\sup_{v\in O_j^{\Sigma}}\dot{g}_j(v)$. Now we define a Lyapunov-Kravsovskii function $V(\cdot)$ as follows

$$\begin{cases}V(t)=V_1(t)+V_2(t),\quad V_1(t)=\sum_{i=1}^{N}\lambda_iX_i^p(t)\\V_1(t)=\sum_{i=1}^{N}\lambda_i\alpha_i^1\sum_{j=1}^{N}\left|b_{ij}\right|\dot{g}_j(\zeta)(t)\\\qquad\left[\int_0^{\infty}k_{ij}(s)\mathrm{e}^{\rho s}\left(\int_{t-s}^{t}X_j^p(w)\mathrm{d}w\right)\mathrm{d}s\right]\end{cases}\tag{3.5}$$

From (3.5), it is easy for us to estimate

$$\begin{aligned}V(0)\le\sum_{i=1}^{N}\lambda_i\Bigg[&\left(1+\frac{\lambda_j}{\lambda_i}\left|b_{ij}\right|\dot{g}_j(\zeta)\alpha_j^1\int_0^{\infty}k_{ji}(s)s\mathrm{e}^{\rho s}\,\mathrm{d}s\right)\\&\sup_{s\in(-\infty,0]}X_i^p(s)\Bigg].\end{aligned}\tag{3.6}$$

From (3.4), (3.5) and by simple calculations, we obtain

$$\begin{aligned}D^+V(t)\le\sum_{i=1}^{N}\lambda_i\Big[&pX_i^p(t)\left(\rho-\dot{d}_i(\eta)\alpha_i^0\right)\\&+pX_i^{p-1}(t)\alpha_i^1V_1(t)\alpha_i^1\sum_{j=1}^{N}\left|b_{ij}\right|\dot{g}_j(\zeta)\\&\times\left[\int_0^{\infty}k_{ij}(s)\mathrm{e}^{\rho s}X_j(t-s)\mathrm{d}s\right]\\&+\alpha_i^1\sum_{j=1}^{N}\left|b_{ij}\right|\dot{g}_j(\zeta)\int_0^{\infty}k_{ij}(s)\mathrm{e}^{\rho s}\\&\times\left(X_j^p(t)-X_j^p(t-s)\right)\mathrm{d}s\end{aligned}\tag{3.7}$$

By using the basic inequality $pa^{p-1}b\le(p-1)a^p+b^p$ and (3.7), we obtain that

$$D^+V(t) \le \sum_{i=1}^{N} \lambda_i \Big[p\big(\rho - \dot{d}_i(\eta)\alpha_i^0\big) + (p-1)\alpha_i^1 \sum_{j=1}^{N} |b_{ij}| \dot{g}_j(\zeta) \int_0^\infty k_{ij}(s) e^{\rho s} ds + \sum_{j=1}^{N} |b_{ij}| \frac{\lambda_j}{\lambda_i} \alpha_j^1 \dot{g}_j(\zeta) \int_0^\infty k_{ji}(s) e^{\rho s} ds \Big] X_i^p(t) \quad (3.8)$$

It follows from (3.3) that $D^+V(t) < 0$. Hence $V(t) \le V(0)$ for any $t \ge 0$ which leads to

$$\sum_{i=1}^{N} \lambda_i e^{p\rho t} |x_i(t)|^p \le \sum_{i=1}^{N} \lambda_i \left[\left(1 + \frac{\lambda_j}{\lambda_i} |b_{ij}| \dot{g}_j(\zeta) \alpha_j^1 \int_0^\infty k_{ji}(s) s e^{\rho s} ds \right) \sup_{s\in(-\infty,0]} X_i^p(s) \right].$$

That is,

$$\sum_{i=1}^{N} |x_i(t)|^p \le \Lambda e^{-p\rho t} \sum_{i=1}^{N} \sup_{s\in(-\infty,0]} |x_i(s)|^p .$$

where

$$\Lambda = \frac{\max_{i\in\mathbb{N}}\{\lambda_j\}}{\min_{i\in\mathbb{N}}\{\lambda_i\}} \max_{i\in\mathbb{N}} \left\{ 1 + \sum_{j=1}^{N} \frac{\lambda_j}{\lambda_i} |b_{ij}| \dot{g}_i(\zeta) \alpha_j^1 \int_0^\infty k_{ji}(s) s e^{\rho s} ds \right\}.$$

Therefore, we have

$$\left[\sum_{i=1}^{N} \left| u_i(t;\psi) - u_i^\Sigma \right|^p \right]^{\frac{1}{p}} \le M e^{-\rho t} \left\| \psi - u^\Sigma \right\|_p$$

where $M = \sqrt[p]{\Lambda}$. That is u^Σ is locally exponentially stable and H^Σ is contained in the attracting domain of u^Σ. The proof is complete.

Take $\lambda_i = 1$ and $p = 1$, it is easy for us to have the following corollary.

Corollary 3.1. Assume that $(S_1)-(S_2)$ and $(H_1^A)-(H_2^A)$ hold. For any given Σ, if there exists a constant ρ such that for each $i \in \mathbb{N}$,

$$\alpha_i^0 \inf_{v\in O_i^\Sigma} \dot{d}_i(v) - \rho > \sum_{j=1}^{N} \alpha_j^1 |b_{ji}| \sup_{v\in O_i^\Sigma} \dot{g}_i(v) \int_0^\infty k_{ji}(s) e^{\rho s} ds \quad (3.9)$$

Then u^Σ is locally exponentially stable and H^Σ is contained in the attracting domain of u^Σ.

Remark 3.1 For each $j \in \mathbb{N}$, if $g_j(\cdot)$ is globally Lipschitz continuous with a Lipschitz constant L_j and there exists a constant $\gamma_j > 0$ such that $\big(d_j(u) - d_j(u)\big)(u-v)^{-1} \ge \gamma_j$ for all $u, v \in \mathbb{R}$. It is obvious that we have $\sup_{v\in O_j^\Sigma} \dot{g}_j(v) \le L_j$ and $\inf_{v\in O_j^\Sigma} \dot{d}_j(v) \ge \gamma_j$. If we replace $\inf_{v\in O_j^\Sigma} \dot{d}_j(v)$ by γ_j, $\sup_{v\in O_j^\Sigma} \dot{g}_j(v)$ by L_j in (3.3), then we get

$$\alpha_i^0 \gamma_i - \rho > \frac{p-1}{p} \alpha_i^1 \sum_{j=1}^{N} |b_{ij}| L_j \int_0^\infty k_{ij}(s) e^{\rho s} ds + \frac{1}{p} \sum_{j=1}^{N} \frac{\lambda_j}{\lambda_i} \alpha_j^1 |b_{ij}| L_i \int_0^\infty k_{ji}(s) e^{\rho s} ds \quad (3.10)$$

Take $\rho = 0$, $p = 1$ and $l_{ij} = 1$ in (3.10), [9] proved that there exists a unique equilibrium point of CGNN (1.1) which is globally asymptotically stable. It is obvious that our criteria only base on parameters of the network and the derivative of activation functions within the confines of attracting domains, they can be easily checked. However, when we relax condition (3.10) to be (3.3), results in [7,9,11] are not applicable for CGNN (1.1). It is obvious that our results are new and complement the corresponding results in [9].

From above remark, it is easy for us to have the following corollary.

Corollary 3.2. Under the following basic assumptions (S_1^*). For each $i \in \mathbb{N}$, $a_i(\cdot)$ is a continuous function on $\mathbb{R}$. Meanwhile, we assume that $0 < \alpha_i^0 \le a_i(v) \le \alpha_i^1$ and $d_i(\cdot)$ is continuous increasing with

$$\big(d_i(u) - d_i(u)\big)(u-v)^{-1} \ge \gamma_i > 0 .$$

(S_2^*) For each $i \in \mathbb{N}$, $g_i(\cdot)$ is globally Lipschitz continuous with a Lipschitz constant L_i and there exists a constant $g_i^\perp > 0$ such that $|g_i(v)| \le g_i^\perp$ for all $v \in \mathbb{R}$.

If there exist positive constants $\lambda_i (i \in \mathbb{N})$ and ρ such that for each $i \in \mathbb{N}$,

$$\alpha_i^0 \gamma_i - \rho > \frac{p-1}{p} \alpha_i^1 \sum_{j=1}^{N} |b_{ij}| L_j \int_0^\infty k_{ij}(s) e^{\rho s} ds + \frac{1}{p} \sum_{j=1}^{N} \frac{\lambda_j}{\lambda_i} \alpha_j^1 |b_{ij}| L_i \int_0^\infty k_{ji}(s) e^{\rho s} ds \quad (3.11)$$

Then there exists at least a unique equilibrium point of CGNN (1.1). which is globally exponentially stable.

Proof. With assumptions $(H_1^A)-(H_2^A)$, we only consider mapping $F = (F_1, F_2, \cdots, F_N)$ from Δ to Δ defined by

$$F_i(\boldsymbol{u}) = d_i^{-1} \left[\sum_{j=1}^{N} b_{ij} g_j(l_{ij} u_j) \right],$$

where $u \in \Delta^\Sigma = \Delta_1 \times \cdots \times \Delta_N$ and $\Delta_i = [v_{i1}^\perp, v_{i2}^\perp], i \in \mathbb{N}$. By Brouwer's fixed point theory, there exists at least one equilibrium point $\hat{u} \in \Delta$ such that $F(\hat{u}) = \hat{u}$. Similarly as Theorem 3.2, by (3.11) we can show that $\hat{u} \in \Delta$ is the unique equilibrium point of CGNN (1.1) which is globally exponentially stable. The proof is complete.

Remark 3.2. Take $\rho = 0$ and $l_{ij} = 1$. When $p = 1, 2$, Corollary 3.2 leads to Theorem 3.2 and Theorem 3.3 in [17]. It is obvious that Theorem 3.2-3.5 in [9] are only our special cases of Corollary 3.2. Furthermore, our results can extend the existence of multiple equilibrium points and their attracting behavior of Cohen-Grossberg networks to [3-12].

4. Examples

Consider the following Cohen-Grossberg networks with distributed delays

$$\begin{cases} \frac{\mathrm{d}u_1(t)}{\mathrm{d}t} = -a_1(u_1(t))d_1(u_1(t)) \\ \qquad - \sum_{j=1}^{N} b_{11} g_1\left(\int_{-\infty}^{t} k_{11}(t-s)u_1(s)\,\mathrm{d}s\right) \\ \qquad + \sum_{j=1}^{N} b_{12} g_2\left(\int_{-\infty}^{t} k_{12}(t-s)u_2(s)\,\mathrm{d}s\right) \\ \frac{\mathrm{d}u_2(t)}{\mathrm{d}t} = -a_2(u_2(t))d_2(u_2(t)) \\ \qquad - \sum_{j=1}^{N} b_{21} g_1\left(\int_{-\infty}^{t} k_{21}(t-s)u_1(s)\,\mathrm{d}s\right) \\ \qquad + \sum_{j=1}^{N} b_{22} g_2\left(\int_{-\infty}^{t} k_{22}(s)u_2(s)\,\mathrm{d}s\right) \end{cases} \quad (4.1)$$

Example 4.1 Consider

$$a_1(u_1(t)) = a_2(u_2(t)) = 1,\ d_1(u_1(t)) = 1.4u_1(t),$$
$$d_2(u_2(t)) = 3.1u_2(t),\ b_{11} = 6,$$
$$k_{12}(s) = 0.125\mathrm{e}^{-s},\ k_{21}(s) = 0.25\mathrm{e}^{-s},$$
$$k_{11}(s) = k_{22}(s) = \mathrm{e}^{-s},\ g_1(v) = g_2(v) = \tanh(v),$$
$$b_{21} = 3.6742,\ b_{12} = 3.1458,\ b_{22} = 10.$$

Since $|b_{i1}|L_1 + |b_{i2}|L_2 > \gamma_i$ (L_i and γ_i are defined in Corollary 3.2), most results reported in [7,9-11] could not applicable for CGNN (4.1) even though we take delay kernels into consideration. It is obvious that $(S_1)-(S_2)$ hold and we have

$$l_{11} = l_{22} = \int_0^\infty k_{11}(s)\,\mathrm{d}s = 1,\ l_{21} = 2l_{12} = \int_0^\infty k_{12}(s)\,\mathrm{d}s = 1$$
$$F_1(v) = -1.4v + 6g_1(v),\ F_2(v) = -3.1v + 10g_2(v),$$
$$g_1^\perp = g_2^\perp = 1$$

From some computations, we get $v_{11} = -1.3565$, $v_{12} = 1.3565$, $v_{21} = -1.19$, $v_{22} = 1.19$ such that $\dot{F}_i(v_{i\varsigma}) = 0$, hence $F_1(v_{1\varsigma}) = (-1)^\varsigma 3.3544$ and $F_2(v_{2\varsigma}) = (-1)^\varsigma 4.6168$, where $\varsigma = 1, 2$. From (2.2)-(2.3), we can estimate

$$v_{11}^\dagger = -1.8190\ v_{12}^\dagger = 1.8190\ v_{21}^\dagger = -1.9\ v_{22}^\dagger = 1.9$$

It is easy for us to get $\dot{g}_1(v) = 0.0999,\ \dot{g}_2(v) = 0.0086$.

Therefore, CGNN (4.1) satisfied our assumptions in Theorem 3.2 and assumptions $(H_1^A)-(H_2^A)$ hold. Let $p = 1, \rho = 1/16$ and $\lambda_1 = \lambda_2 = 1$. From some calculations, we can check that (3.3) holds. By Theorem 3.2, there exist only four locally exponentially stable equilibrium points of CGNN (4.1) located in Δ^Σ. Moreover, their attracting domains can be estimated by

$$H^{(1,1)} = \left\{(\psi_1,\psi_2) \in C\left((-\infty,0],\mathbb{R}^2\right) \middle| \psi_1(s) \le -1.1819,\ \psi_2(s) \le -1.9\right\}$$
$$H^{(1,2)} = \left\{(\psi_1,\psi_2) \in C\left((-\infty,0],\mathbb{R}^2\right) \middle| \psi_1(s) \le -1.1819,\ \psi_2(s) \ge 1.9\right\}$$
$$H^{(2,1)} = \left\{(\psi_1,\psi_2) \in C\left((-\infty,0],\mathbb{R}^2\right) \middle| \psi_1(s) \ge 1.1819,\ \psi_2(s) \le -1.9\right\}$$
$$H^{(2,2)} = \left\{(\psi_1,\psi_2) \in C\left((-\infty,0],\mathbb{R}^2\right) \middle| \psi_1(s) \ge 1.1819,\ \psi_2(s) \ge 1.9\right\}$$

Example 4.2 Consider

$$a_1(u_1(t)) = a_2(u_2(t)) = 1,$$
$$d_1(u_1(t)) = 1.4u_1(t) - 0.02\tanh(u_1(t)),$$
$$d_2(u_2(t)) = 3.1u_2(t) - 0.02\tanh(u_2(t)),$$
$$b_{11} = 6,\ k_{11}(s) = k_{12}(s) = k_{21}(s) = k_{22}(s) = \mathrm{e}^{-s},$$
$$g_1(v) = g_2(v) = \tanh(v),\ k_{12}(s) = \frac{\mathrm{e}^{-s}}{8},$$
$$b_{21} = 3.6724,\ b_{12} = 3.6724,\ b_{22} = 10,\ b_{12} = 3.1458$$

We can estimate that

$$d_1^\dagger = 1.38 < \dot{d}_1(v) = 1.4 - 0.02\tanh(v)' < d_1^\perp = 1.4,$$
$$d_2^\dagger = 3.08 < \dot{d}_2(v) = 3.1 - 0.02\tanh(v)' < d_2^\perp = 3.1,$$
$$v\ddot{d}_1(v) = -0.02\tanh(v)'' > 0 > vb_{11}\tanh(v)'',$$
$$v\ddot{d}_2(v) = -0.02\tanh(v)'' > 0 > vb_{12}\tanh(v)''.$$

So, $(S_1)-(S_2)$ hold. Similarly as Example 4.1, we get $v_{11} = -1.37$, $v_{12} = 1.37$, $v_{21} = -1.21$, $v_{22} = 1.21$ such that $\dot{F}_i(v_{i\varsigma}) = 0$, hence $F_1(v_{1\varsigma}) = (-1)^\varsigma 3.372$ and

$F_2\left(v_{2\varsigma}\right)=\left(-1\right)^{\varsigma}4.633$, where $\varsigma=1,2$. From (2.2)-(2.3), we can estimate

$$v_{11}^{\dagger}=-1.85\ v_{12}^{\dagger}=1.85\ v_{21}^{\dagger}=-1.91\ v_{22}^{\dagger}=1.91$$

We can check that assumptions $\left(H_1^A\right)-\left(H_2^A\right)$ and (3.3) hold. By Theorem 3.2, there exist only four stable equilibrium points of CGNN (4.1) located in Δ^{Σ}. Moreover, their attracting domains H^{Σ} can be estimated as Example 4.1.

5. Concluding Remarks

In this paper, some new criteria are derived for the coexistence of 2N equilibrium points and attracting domain is also given for each equilibrium point. It is shown that our results are new and complement the previous results in [7,9-11]. Furthermore, our results can extend corresponding ones reported in [22-23] to local exponential stability of multiple equilibrium points of neural network.

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