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Efficiency and Duality in Nondifferentiable Multiobjective Programming Involving Directional Derivative

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Abstract

In this paper, we introduce a new class of generalized d_I-univexity in which each component of the objective and constraint functions is directionally differentiable in its own direction d_i for a nondifferentiable multiobjective programming problem. Based upon these generalized functions, sufficient optimality conditions are established for a feasible point to be efficient and properly efficient under the generalised d_I-univexity requirements. Moreover, weak, strong and strict converse duality theorems are also derived for Mond-Weir type dual programs.

Keywords: Multiobjective Programming, Nondifferentiable Programming, Generalized d_I-Univexity, Sufficiency, Duality

1. Introduction

The field of multiobjective programming, also known as vector programming, has grown remarkably in different directions in the setting of optimality conditions and duality theory. It has been enriched by the applications of various types of generalizations of convexity theory, with and without differentiability assumptions, and in the framework of continuous time programming, fractional programming, inverse vector optimization, saddle point theory, symmetric duality and vector variational inequalities etc.

Hanson [1] introduced a class of functions by generalizing the difference vector $x - \bar{x}$ in the definition of a convex function to any vector function $\eta(x, \bar{x})$. These functions were named invex by Craven [2] and η-convex by Kaul and Kaur [3]. Hanson and Mond [4] defined two new classes of functions called Type I and Type II functions, which were further generalized to pseudo Type I and quasi Type I functions by Rueda and Hanson [5]. Zhao [6] established optimality conditions and duality in nonsmooth scalar programming problems assuming Clarke [7] generalized subgradients under Type I functions.

Kaul *et al.* [8] extended the concept of type I and its generalizations for a multiobjective programming problem. They investigated optimality conditions and derived Wolfe type and Mond-Weir type duality results. Suneja and Srivastava [9] introduced generalized d-type I functions in terms of directional derivative for a multiobjective programming problem and discussed Wolfe type and Mond-Weir type duality results. In [10], Kuk and Tanino derived optimality conditions and duality theorems for non-smooth multiobjective programming problems involving generalized Type I vector valued functions. Gulati and Agarwal [11] discussed sufficiency and duality results for nonsmooth multiobjective problems under $(F, \alpha, \rho, d$ -type I functions. Agarwal *et al.* [12] established sufficient conditions and duality theorems for nonsmooth multiobjective problems under V-type I functions. Recently, Jayswal *et al.* [13] obtained some optimality conditions and duality results for nonsmooth multiobjective problems involving generalized $(F, \alpha, \rho, \theta)$ $-d-V$ -univexity.

Antczak [14] studied d-invexity is one of the generalization of invex function, which is introduced by [15]. In [14], Antczak established, under weaker assumptions than Ye, the Fritz John type and Karush-Kuhn-Tucker type necessary optimality conditions for weak Pareto optimality and duality results which have been stated in terms of the right differentials of functions involved in the considered multiobjective programming problem. Some authors [16-18] proved that the Karush-Kuhn-Tucker type necessary conditions [14] are sufficient under various generalized d-invex functions. Antczak [19] corrrected the Karush-Kuhn-Tucker necessary conditions in [14] and discussed the sufficiency and duality under $d-r-$type I functions. Recently, Silmani and Radjef [20] introduced generalzed d_I-invexity in which each component of the objective and constraint functions is directionally differentiable in its own direction and established the necessary and sufficient conditions for efficient and properly efficient solutions. The duality results for a Mond-Weir type dual are also derived in [20]. They also observed that the Karush-Kuhn-Tucker sufficient conditions discussed in [16-18] are not applicable. More recently, Agarwal *et al.* [21] introduced a new class of generalized $d-\rho-(\eta,\theta)-$ type I for a non-smooth multiobjective programming problem and discussed optimality conditions and duality results.

In this paper, we introduce d_I-V -univexity and generalized d_I-V -univexity in which each component of the objective and constraint functions of a multiobjective programming problem is semidirectionally differentiable in its own direction d_i. Various Karush-Kuhn-Tucker sufficient optimality conditions for efficient and properly efficient solutions to the problem are established involving new classes of semidirectionally differentiable generalized type I functions. Moreover, usual duality theorems are discussed for a Mond-Weir type dual involving aforesaid assumptions. The results in this paper extend many earlier work appeared in the literature [9,10,12,14-16, 19].

2. Preliminaries and Definitions

The following conventions for equalities and inequalities will be used. If $\boldsymbol{x}=(x_1,\cdots,x_n)$, $\boldsymbol{y}=(y_1,\cdots,y_n)\in\mathbb{R}^n$, then $\boldsymbol{x}=\boldsymbol{y}\Leftrightarrow x_i=y_i, i=1,\cdots,n$; $\boldsymbol{x}<\boldsymbol{y}\Leftrightarrow x_i<y_i, i=1,\cdots,n$; $\boldsymbol{x}<\boldsymbol{y}\Leftrightarrow x_i<y_i, i=1,\cdots,n$; $\boldsymbol{x}\le\boldsymbol{y}\Leftrightarrow\boldsymbol{x}\le\boldsymbol{y}$ and $\boldsymbol{x}\ne\boldsymbol{y}$, We also note $\mathbb{R}^q_{\leqq}$ (resp. $\mathbb{R}^q_{\geq}$ or $\mathbb{R}^q_{\gneq}$) the set of vectors $\boldsymbol{y}\in\mathbb{R}^q$ with $\boldsymbol{y}\geqq 0$ (resp. $\boldsymbol{y}\ge 0$ or $\boldsymbol{y}>0$).

Definition 1 [22]. Let D be a nonempty subset of $\mathbb{R}^n$, $\eta:D\times D\to\mathbb{R}^n$ and let x_0 be an arbitrary point of D.The set D is said to be invex at x_0 with respect to η, if for each $x\in D$,

$$x_0+\lambda\eta(x,x_0)\in D,\forall\lambda\in[0,1].$$

D is said to be an invex set with respect to η, if D is invex at each $x_0\in D$ with respect to the same η.

Definition 2 [23]. Let $D\subseteq\mathbb{R}^n$ be an invex set with respect to $\eta:D\times D\to\mathbb{R}^n$ .A function $f:D\to\mathbb{R}$ is called pre-invex on D with respect to η, if for all $x,x_0\in D$,

$$\lambda f(x)+(1-\lambda)f(x_0)\geqq f(x_0+\lambda\eta(x,x_0)),\forall\lambda\in[0,1].$$

Definition 3 [14]. Let $D\subseteq\mathbb{R}^n$ be an invex set with respect to $\eta:D\times D\to\mathbb{R}^n$. A m-dimensional vector valued function $\boldsymbol{\Psi}:D\to\mathbb{R}^m$ is pre-invex with respect to η, if each of its components is pre-invex on D with respect to the same function η.

Definition 4 [7]. Let D be a nonempty open set in $\mathbb{R}^n$. A function $f:D\to\mathbb{R}$ is said to be locally Lipschitz at $x_0\in D$, if there exist a neighborhood $\upsilon(x_0)$ of x_0 and a constant $K>0$ such that

$$|f(y)-f(x)|\geqq K\|y-x\|,\quad\forall x,y\in\upsilon(x_0),$$

where $\|.\|$ denotes the Euclidean norm. We say that f is locally Lipschitz on D if its locally Lipschitz at any point of D.

Definition 5 [7]. If $f:D\subseteq\mathbb{R}^n\to R$ is locally Lipschitz at $x_0\in D$, the Clarke generalized directional derivative of f at x_0 in the direction $d\in\mathbb{R}^n$, denoted by

$$f^0(x_0;d)=\lim_{\substack{y\to x_0\\ t\downarrow 0}}\sup\left[\frac{f(y+td)-f(y)}{t}\right].$$

And the usual one-sided directional derivative of f at x_0 in the direction d is defined by

$$f'(x_0;d)=\lim_{\lambda\to 0^+}\frac{f(x_0+\lambda d)-f(x_0)}{\lambda},$$

whenever this limit exists. Obviously,

$$f^0(x_0;d)\geqq f'(x_0;d).$$

We say that f is directionally differentiable at x_0 if its directional derivative $f'(x_0;d)$ exists finite for all $d\in\mathbb{R}^n$.

Definition 6 [15]. Let $f:D\to\mathbb{R}^N$ be a function defined on a nonempty open set $D\subset\mathbb{R}^n$ and directionally differentiable at $x_0\in D.$ f is called d-invex at x_0 on D with respect to η, if there exists a vector function $\eta:D\times D\to\mathbb{R}^n$, such that for any $x\in D$,

$$|f(y)-f(x)|\geqq K\|y-x\|,$$
$$\forall x,y\in\upsilon(x_0)\ f_i(x)-f_i(x_0)\geqq f_i'(x_0;\eta(x,x_0)),$$
for all $i=1,\cdots,N$,

where $f_i'(x_0;\eta(x,x_0))$ denotes the directional derivative of f_i at x_0 in the direction

$$\eta(x,x_0): f_i'(x_0;\eta(x,x_0))$$
$$=\lim_{\lambda\to 0^+}\frac{f_i(x_0+\lambda\eta(x,x_0))-f_i(x_0)}{\lambda}.$$

If Inequalities (1) are satisfied at any point $x_0\in D$, then f is said to be d-invex on D with respect to η.

Definition 7 [20]. Let D be a nonempty set in $\mathbb{R}^n$ and $\phi:D\times D\to\mathbb{R}^n$ a function.

- We say that $f:D\to\mathbb{R}$ is a semi-directionally differentiable at $x_0 \in D$,if there exist a nonempty subset $S \subset \mathbb{R}^n$ such that $f'(x_0;d)$ exists finite for all $d \in S$

- We say that f is a semi-directionally differentiable at $x_0 \in D$ in the direction $\phi(x,x_0)$, if its directional derivative $f'(x_0;\phi(x,x_0))$ exists finite for all $x \in D$.

Definition 8 [20]. Let $f: D \to \mathbb{R}^N$ be a function defined on a nonempty open set $D \subset \mathbb{R}^n$ and for all $i = 1,\cdots,N$, f_i is semi-directionally differentiable at $x_0 \in D$ in the direction $\eta_i : D \times D \to \mathbb{R}^n$. f is called d_I-invex at x_0 on D with respect to $(\eta_i)_{i=\overline{1,N},}$ if for any $x \in D$,

$$f_i(x) - f_i(x_0) \geqq f_i'(x_0;\eta_i(x,x_0)), \text{for all } i, 2, \cdots, N,$$

where $f_i'(x_0;\eta_i(x,x_0))$ denotes the directional derivative of f_i at x_0 in the direction

$$\eta_i(x,x_0): f_i'(x_0;\eta_i(x,x_0)) = \lim_{\lambda \to 0^+} \frac{f_i(x_0 + \lambda\eta_i(x,x_0)) - f_i(x_0)}{\lambda}.$$

If Inequalities (2) are satisfied at any point $x_0 \in D$, then f is said to be d_I-invex on D with respect to $(\eta_i)_{i=\overline{1,N},}$

Consider the following multiobjective programming problem

$$(MP) \text{ Minimize } f(x) = (f_1(x), f_2(x), \cdots, f_N(x))$$

$$\text{Subject to } g(x) \leqq 0,$$

$$x \in D,$$

where $f: D \to \mathbb{R}^N, g: D \to \mathbb{R}^k$, D is a nonempty open subset of $\mathbb{R}^n$. Let $X = \{x \in D : g(x) \leqq 0\}$ be the set of feasible solutions of (MP). For $x_0 \in D$, we denote by $J(x_0)$ the set $\{j \in \{1,2,\cdots,k\} : g_j(x_0) = 0\}, J = |J(x_0)|$ and by $\tilde{J}(x_0)(\text{resp.}\overline{J}(x_0))$ the set $\{j \in \{1,2,\cdots,k\} : g_j(x_0) < 0 \ \ (\text{resp. } \ g_j(x_0) > 0)\}$. we have $J(x_0) \cup \tilde{J}(x_0) \cup \overline{J}(x_0) = \{1,2,\cdots,k\}$ and if $x_0 \in X$, $\overline{J}(x_0) = \varnothing$.

We recall some optimality concepts, the most often studied in the literature, for the problem (MP).

Definition 9. A point $x_0 \in X$ is said to be a local weakly efficient solution of the problem (MP), if there exists a neighborhood $N(x_0)$ around x_0 such that

$$f(x) \not< f(x_0) \text{ for all } x \in N(x_0) \cap X$$

Definition 10. A Point $x_0 \in X$ is said to be a weakly efficient (an efficient) solution of the problem (MP), if there exists no $x \in X$ such that

$$f(x) < f(x_0)(f(x) \leq f(x_0)).$$

Definition 11. An efficient solution $x_0 \in X$ of (MP) is said to be properly efficient, if there exists a positive real number M such that inequality

$$f_i(x_0) - f_i(x) \leqq M[f_j(x) - f_j(x_0)]$$

is verified for all $i \in \{1,\cdots,N\}$ and $x \in X$ such that $f_i(x) < f_i(x_0)$, and for a certain $j \in \{1,\cdots,N\}$ such that $f_j(x) > f_j(x_0)$.

Following Jeyakumar and Mond [24], Kaul *et al*. [8] and Slimani and Radjef [20], we give the following definitions.

Definition 12. (f,g) is $d_I - V$ -univex type I at $x_0 \in D$ if there exist positive real valued functions α_i and β_j defined on $X \times D$, nonnegative functions b_0 and b_1, also defined on $X \times D, \phi_0 : R \to R, \phi_1 : R \to R; \eta_i : X \times D \to R^n$, and $\theta_j : X \times D \to R^n$ such that

$$b_0(x,x_0)\phi_0[f_i(x) - f_i(x_0)] \geqq \alpha_i(x,x_0) f_i'(x_0;\eta_i(x,x_0)) \tag{3}$$

and

$$-b_1(x_1,x_0)\phi_1[g_j(x_0)] \geqq \beta_j(x,x_0) g_j'(x_0;\theta_j(x,x_0)) \tag{4}$$

for every $x \in X$ and for all $i = 1,2,\cdots,N$ and $j = 1,2,\cdots,k$.

If the inequality in (3) is strict (whenever $x \neq x_0$), we say that (MP) is of semistrictly $d_I - V$ -univex type I at x_0 with respect to $(\eta_i)_{i=\overline{1,N}}$ and $(\theta_j)_{j=\overline{1,k}}$.

Definition 13. (f,g) is quasi-$d_I - V$ -univex type I at $x_0 \in D$ if there exist positive real valued functions α and β_j, defined on $X \times D$, nonnegative functions b_0 and b_1, also defined on $X \times D$, $\varphi_0 : R \to R$, $\varphi_1 : R \to R$ and $(N+k)$ dimensional vector functions $\eta_i : X \times D \to R^n, i = \overline{1,N}$ and $\theta_j : X \times D \to R^n, j = \overline{1,k}$ such that for some vectors $\lambda \in R^N_{\geqq}$ and $\mu \in R^k_{\geqq}$:

$$b_0(x,x_0)\phi_0\left[\sum_{i=1}^{N}\lambda_i\alpha_i(x,x_0)(f_i(x) - f_i(x_0))\right] \leqq 0 \Rightarrow \sum_{i=1}^{N}\lambda_i f_i'(x_0;\eta_i(x,x_0)) \leqq 0 \quad \forall x \in X \tag{5}$$

and

$$b_1(x,x_0)\phi_1\left[\sum_{j=1}^{k}\mu_j\beta_j(x,x_0)g_j(x_0)\right] \geqq 0 \Rightarrow \sum_{j=1}^{k}\mu_j g_j'(x_0;\theta_j(x,x_0)) \geqq 0 \quad \forall x \in X. \tag{6}$$

If the second inequality in (5) is strict $(x \neq x_0)$, we say that (MP) is of semi-strictly quasi d_I-V-univex type I at X with respect to $(\eta_i)_{i=\overline{1,N}}$ and $(\theta_j)_{j=\overline{1,k}}$.

Definition 14. (f,g) is pseudo-d_I-V-univex type I at $x_0\in D$ if there exist positive real valued functions α_i and β_j, defined on $X\times D$, nonnegative functions b_0 and b_1, also defined on $X\times D$, $\phi_0:R\to R$, $\phi_1:R\to R$ and $(N+k)$ dimensions vector functions $\eta_i:X\times D\to R^n, i=\overline{1,N}$ and $\theta_j:X\times D\to R^n$, $j=\overline{1,k}$ such that for some vectors $\lambda\in R^N_{\geq}$ and $\mu\in R^k_{\geq}$:

$$\sum_{i=1}^{N}\lambda_i f_i'\left(x_0;\eta_i\left(x,x_0\right)\right)\geqq 0\Rightarrow$$
$$b_0\left(x,x_0\right)\phi_0\left[\sum_{i=1}^{N}\lambda_i\alpha_i\left(x,x_0\right)\left(f_i\left(x\right)-f_i\left(x_0\right)\right)\right]\leqq 0\quad\forall x\in X \tag{7}$$

and

$$\sum_{j=1}^{k}\mu_j g_j'\left(x_0;\theta_j\left(x,x_0\right)\right)\geqq 0\Rightarrow$$
$$b_1\left(x,x_0\right)\phi_1\left[\sum_{j=1}^{k}\mu_j\beta_j\left(x,x_0\right)g_j\left(x_0\right)\right]\leqq 0\quad\forall x\in X. \tag{8}$$

Definition 15. (f,g) is quasi pseudo-d_I-V-univex type I at $x_0\in D$ if there exist positive real valued functions α_i and β_j, defined on $X\times D$, nonnegative functions b_0 and b_1, also defined on $X\times D$, $\phi_0:R\to R$, $\phi_1:R\to R$ and $(N+k)$ dimensions vector functions $\eta_i:X\times D\to R^n$, $i=\overline{1,N}$ and $\theta_j:X\times D\to R^n$, $j=\overline{1,k}$ such that the relation (5) and (8) are satisfied. If the second inequality in (8) is strict ($x\neq x_0$, we say that (*VP*) is of quasi strictly-pseudo d_I- V –*type I* at x_0 with respect to $\left(\eta_i\right)_{i=\overline{1,N}}$ and $\left(\theta_j\right)_{j=\overline{1,k}}$.

Definition 16. (f,g) is pseudoquasi -d_I-V-univex type I at $x_0\in D$ if there exist positive real valued functions α_i and β_j, defined on $X\times D$, nonnegative functions b_0 and b_1, also defined on $X\times D$, $\phi_0:R\to R$, $\phi_1:R\to R$ and $(N+k)$ dimensions vector functions $\eta_i:X\times D\to R^n$, $i=\overline{1,N}$ and $\theta_j:X\times D\to R^n$, $j=\overline{1,k}$, such that $\mu\in R^k_{\geqq}$ the relations (7) and (6) are satisfied. If the second inequality in (7) is strict $(x\neq x_0)$, we say that (MP) is of strictly-pseudo quasi d_I – V – type I at x_0 with respect to $\left(\eta_i\right)_{i=\overline{1,N}}$ and $\left(\theta_j\right)_{j=\overline{1,k}}$.

3. Optimality Conditions

In this section, we discuss some sufficient conditions for a point to be an efficient or properly efficient for (MP) under generalized d_I-V – univex type I assumptions.

Theorem 3.1. Let x_0 be a feasible solution for (MP) and suppose that there exist $(\boldsymbol{N}+\boldsymbol{J})$ vector functions $\eta_i:X\times X\to R^n, i=\overline{1,N}$, $\theta_j:X\times X\to R^n, j\in J\left(x_0\right)$ and scalars $\bar\lambda_i\geqq 0$, $i=\overline{1,N}$, $\sum_{i=1}^{N}\bar\lambda_i=1$; $\bar\mu_j\geqq 0$, $j\in J\left(x_0\right)$ such that

$$\sum_{i=1}^{N}\bar\lambda_i f_i'\left(x_0;\eta_i\left(x,x_0\right)\right)+\sum_{j\in J\left(x_0\right)}\bar\mu_j g_j'\left(x_0;\theta_j\left(x,x_0\right)\right)\geqq 0,$$
$$\forall x\in X, \tag{9}$$

Further, assume that one of the following conditions is satisfied:

a) i) (f,g) is quasi strictly-pseudo d_I-V-univex type I at x_0 with respect to $\left(\eta_i\right)_{i=\overline{1,N}},\left(\theta_j\right)_{j\in J\left(x_0\right)},\bar\lambda,\bar\mu$ and for some positive functions $\alpha_i, i=\overline{1,N}$, β_j, $j\in J\left(x_0\right)$,

ii) for any $u\in R$, $u\leqq 0\Rightarrow\varphi_0\left(u\right)\leqq 0$; $\varphi_1\left(u\right)<0\Rightarrow u<0$; $b_0\left(x,x_0\right)>0$, $b_1\left(x,x_0\right)>0$;

b) i) (f,g) is strictly-pseudo d_I-V-univex type I at x_0 with respect to $\left(\eta_i\right)_{i=\overline{1,N}}$, $\left(\theta_j\right)_{j\in J\left(x_0\right)},\bar\lambda,\bar\mu$ and for some positive functions $\alpha_i, i=\overline{1,N},\beta_j$, $j\in J\left(x_0\right)$,

ii) for any $u\in R$, $\varphi_0(u)>0\Rightarrow u>0$; $u\geqq 0\Rightarrow\varphi_1(u)\geqq 0$, $b_0(x,x_0)>0$, $b_1(x,x_0)\geqq 0$.

Then x_0 is an efficient solution for (MP).

Proof: Condition a). Suppose that x_0 is not an efficient solution of (MP). Then there exists an $x\in X$ such that

$$f\left(x\right)\leq f\left(x_0\right),$$

which implies that

$$\sum_{i=1}^{N}\bar\lambda_i\alpha_i\left(x,x_0\right)\left[f_i\left(x\right)-f_i\left(x_0\right)\right]\leqq 0. \tag{10}$$

Since $b_0\left(x,x_0\right)>0$; $u\geqq 0\Rightarrow\varphi_0\left(u\right)\geqq 0$, the above inequality gives

$$b_0\left(x,x_0\right)\phi_0\left[\sum_{i=1}^{N}\bar\lambda_i\alpha_i\left(x,x_0\right)\left[f_i\left(x\right)-f_i\left(x_0\right)\right]\right]\leqq 0.$$

From the above inequality and Hypothesis i) of a), we have

$$\sum_{i=1}^{N}\bar\lambda_i f_i'\left(x_0;\eta_i\left(x,x_0\right)\right)\leqq 0.$$

By using the Inequality (9) we deduce that

$$\sum_{j\in J\left(x_0\right)}\bar\mu_j g_j'\left(x_0;\theta_j\left(x,x_0\right)\right)\geqq 0,$$

which implies from the condition part ii) of a) that

$$b_1\left(x,x_0\right)\phi_1\left[\sum_{j\in J\left(x_0\right)}\bar\mu_j\beta_j\left(x,x_0\right)g_j\left(x_0\right)\right]<0.$$

Since $b_1\left(x,x_0\right)>0$; $\phi_1\left(u\right)<0\Rightarrow u<0$, we get

$$\sum_{j\in J(x_0)} \bar{\mu}_j \beta_j (x, x_0) g_j (x_0) < 0. \quad (11)$$

As $\bar{\lambda} \geqq 0$ and $g_j(x_0) = 0;\ \forall j \in J(x_0)$, it follows that $\bar{\lambda}_j g_j(x_0) = 0,\ \forall j \in J(x_0)$, which implies that

$$\sum_{j\in J(x_0)} \bar{\mu}_j \beta_j (x, x_0) g_j (x_0) = 0.$$

The above equation contradicts Inequality (11) and hence the conclusion of the theorem follows:

Condition b): Since $g_j(x_0) = 0,\ \bar{\mu}_j \geqq 0,\ \forall j \in J(x_0)$, and $\beta_j(x, x_0) > 0,\ j \in J(x_0)$, we obtain

$$\sum_{j\in J(x_0)} \bar{\mu}_j \beta_j (x, x_0) g_j (x_0) = 0, \quad \forall x \in X.$$

By Hypothesis ii) of b), we get

$$b_1(x, x_0)\phi_1\left[\sum_{j\in J(x_0)} \bar{\mu}_j \beta_j (x, x_0)\right] \geqq 0.$$

From the above inequality and the Hypothesis i) of b)(in view of reverse implication in (8), if follows that

$$\sum_{j\in J(x_0)} \bar{\mu}_j g_j'\left(x_0; \theta_j(x, x_0)\right) < 0, \quad \forall x \in X \setminus \{x_0\}.$$

By using Inequality (9), we deduce that

$$\sum_{i=1}^{N} \bar{\lambda}_i f_i'\left(x_0; \eta_i(x, x_0)\right) > 0, \quad \forall x \in X \setminus \{x_0\}, \quad (12)$$

which by virtue of relation (7) implies that

$$b_0(x, x_0)\phi_0\left[\sum_{i=1}^{N} \bar{\lambda}_i \alpha_i(x, x_0)\left(f_i(x) - f_i(x_0)\right)\right] > 0,$$
$$\forall x \in X \setminus \{x_0\}.$$

The above inequality along with Hypothesis ii) of b) gives

$$\sum_{i=1}^{N} \bar{\lambda}_i \alpha_i(x, x_0)\left(f_i(x) - f_i(x_0)\right) > 0, \forall x \in X \setminus \{x_0\}. \quad (13)$$

Since (10) and (13) contradicts each other, and hence the conclusion follows:

Theorem 3.2. Let x_0 be a feasible solution for (MP) and suppose that there exist $(N+J)$ vector functions $\eta_i : X \times X \to R^n, i = \overline{1,N}, \theta_j : X \times X \to R^n,\ j \in J(x_0)$ and scalars $\bar{\lambda}_i \geqq 0, i = \overline{1,N}, \sum_{i=1}^{N} \bar{\lambda}_i = 1, \bar{\mu}_j \geqq 0,\ j \in J(x_0)$ such that Inequality (9) of Theorem 3.1 is satisfied.

Moreover, assume that one of the following conditions is satisfied.

a) i) (f, g) is pseudo quasi $d_I - V -$ univex type I at x_0 with respect to $(\eta_i)_{i=\overline{1,N}}, (\theta_j)_{j\in J(x_0)}, \bar{\lambda}, \bar{\mu}$ and for some positive functions $\alpha_i, i = \overline{1,N}$ and $\beta_j,\ j \in J(x_0)$,

ii) for any $u \in R$,

$$u \geqq 0 \Rightarrow \phi_1(u) \geqq 0, \phi_0(u) \geqq 0 \Rightarrow u \geqq 0,$$
$$b_0(x, x_0) > 0, b_1(x, x_0) \geqq 0;$$

b) i) (f, g) is strictly pseudo $d_I - V -$ univex type I at x_0 with respect to $(\eta_i)_{i=\overline{1,N}}, (\theta_j)_{j\in J(x_0)}, \bar{\lambda}, \bar{\mu}$ and for positive functions $\alpha_i = \overline{1,N}$ and $\beta_j,\ j \in J(x_0)$,

ii) for any $u \in R$

$$u \leqq 0 \Rightarrow \phi_0(u) \leqq 0;\ u \geqq 0 \Rightarrow \phi_1(u) \geqq 0;$$
$$b_0(x, x_0) > 0,\ b_1(x, x_0) \geqq 0.$$

Then x_0 is an efficient solution for (MP). Further Suppose that these exist positive real numbers n_i, m_i such that $n_i < \alpha_i(x, x_0) < m_i,\ i = \overline{1,N}$ for all feasible x .Then x_0 is a properly efficient solution for (MP)

Proof: Condition a). Suppose that x_0 is not an efficient solution of (MP). Then there exists an $x \in X$ such that $f(x) \leq f(x_0)$ which implies that

$$\sum_{i=1}^{N} \bar{\lambda}_i \alpha_i(x, x_0)\left(f_i(x) - f_i(x_0)\right) < 0. \quad (14)$$

Since $g_j(x_0) = 0,\ \bar{\mu}_j \geqq 0$ and

$$\beta_j(x, x_0) > 0, \forall j \in J(x_0)$$

we obtain

$$\sum_{j\in J(x_0)} \mu_j \beta_j (x, x_0) g_j (x_0) = 0.$$

From the above inequality and Hypothesis ii) of a), we have

$$b_1(x, x_0)\phi_1\left[\sum_{j\in J(x_0)} \bar{\mu}_j \beta_j (x, x_0) g_j(x_0)\right] \geqq 0.$$

Using Hypothesis i) of a), we deduce that

$$\sum_{j\in J(x_0)} \bar{\mu}_j \beta_j (x, x_0) g_j'\left(x_0; \theta_j(x, x_0)\right) \leqq 0. \quad (15)$$

The Inequalities (9) and (14) yield that

$$\sum_{i=1}^{N} \bar{\lambda}_i f_i'\left(x_0; \eta_i(x, x_0)\right) \geqq 0,$$

which by Hypothesis i) of a), we obtain

$$b_0(x, x_0)\phi_0\left[\sum_{i=1}^{N} \bar{\lambda}_i \alpha_i(x, x_0)\left(f_i(x) - f_i(x_0)\right)\right] \geqq 0, \quad (16)$$

The Inequality (16) and Hypothesis ii) of a) give

$$\sum_{i=1}^{N} \bar{\lambda}_i \alpha_i(x, x_0)\left(f_i(x) - f_i(x_0)\right) \geqq 0. \quad (17)$$

Since (14) and (17) contradict each other, we conclude that x_0 is not an efficient solution of (MP). The properly efficient solution follows as in Hanson *et al.* [25]. For the proof of part b), we proceed as in part b) of Theorem 3.1, we get Inequality (17). Thus complete the proof.

4. Mond-Weir Type Duality

Consider the following multiobjective dual to problem (MP)

(MD) Maximize $f(y)=(f_1(y), f_2(y),\cdots, f_N(y))$

subject to

$$\sum_{i=1}^{N}\lambda_i f_i'(y;\eta_i(x,y))+\sum_{j=1}^{k}\mu_j g_j'(y;\theta_j(x,y))\geqq 0, \forall x\in X$$

$$\mu_j g_j(y)\geqq 0, j=1,2,\cdots,k,\ y\in D,\ \lambda\in R^N_{\geqq}, \mu\in R^k_{\geqq}$$

$$\eta_i: X\times D\to R^n,\ \ \forall i=1,2,\cdots,N,$$

$$\theta_j: X\times D\to R^n,\ j=1,2,\cdots,k.$$

Let Y be the set of feasible solutions of problem (MD); that is,

$$Y=\Big\{\Big(y,\lambda,\mu,(\eta_i)_i,(\theta_j)_j\Big):$$

$$\sum_{i=1}^{N}\lambda_i f_i'(y;\eta_2(x,y))+\sum_{j=1}^{k}\mu_j g_j'(y;\theta_j(x,y))\geqq 0,$$

$$\mu_j g_j(y)\geqq 0,\ \ \forall x\in X;\ y\in D,\lambda\in R^N_{\geqq},\ \mu\in R^k_{\geqq};$$

$$\eta_i: X\times D\to R^n\ \ \ \forall i=1,2,\cdots,N;$$

$$\theta_j: X\times D\to R^n,\forall j=1,2,\cdots,k\Big\}.$$

We denote by $P_{rD}Y$, the projection of set Y on D.

Theorem 4.1. (Weak Duality). Let x and $\left(y,\lambda,\mu,(\eta_i)_{i=\overline{1,N}},(\theta_j)_{j=\overline{1,k}}\right)$ be feasible solution for (MP) and (MD) respectively. Moreover, assume that one of the following conditions is satisfied:

a) i) (f,g) is pseudo quasi d_I-V-univex type I at y with respect to $\lambda>0,\ \mu,\ (\eta_i)_{i=\overline{1,N}},\ (\theta_j)_{j=\overline{1,k}}$ and for some positive functions α_i,β_j for $i=1,2,\cdots,N$ and $j=1,2,\cdots,k$,

ii) for any $u\in R$

$$\phi_0(u)\geqq 0\Rightarrow u\geqq 0;\ u\geqq 0\Rightarrow \phi_1(u)\geqq 0;$$
$$b_0(x,y)>0, b_1(x,y)\geqq 0$$

b) i) (f,g) is strictly-pseudo quasi d_I-V-univex type *I* at *y* with respect to $\lambda,\mu,(\eta_i)_{i=\overline{1,N}},\ (\theta_j)_{j=\overline{1,k}}$ and for some positive function $\alpha_i,\ \beta_j$ for $i=1,2,\cdots,N$ and $j=1,2,\cdots,k$,

ii) for any $u\in R$,

$$\phi_0(u)\geqq 0\Rightarrow u>0;\ u\geqq 0\Rightarrow\phi_1(u)\geqq 0;$$
$$b_1(x,y)\geqq 0,\ b_0(x,y)>0;$$

c) i) (f,g) is quasi strictly-pseudo d_I-V-univex type I at y with respect to $\lambda,\mu,(\eta_i)_{i=\overline{1,N}},\ (\theta_j)_{j=\overline{1,k}}$ and for some positive functions α_i,β_j for $i=1,2,\cdots,N$ and $j=1,2,\cdots,k$,

ii) for any $u\in R$,

$$\phi_0(u)>0\Rightarrow u>0;\ u>0\Rightarrow\phi_1(u)>0;$$
$$b_0(x,y)>0,\ b_1(x,y)>0.$$

Then $f(x)\nleq f(y)$.

Proof: Since

$$\mu_j g_j(y)\geqq 0, j=1,2,\cdots,k,$$
$$u\geqq 0\Rightarrow\phi_1(u)\geqq 0, b_1(x,y)>0,$$
$$\text{and } \beta_j(x,y)>0, j=1,2,\cdots,k$$

we have

$$b_1(x,y)\phi_1\Big[\sum_{j=1}^{k}\mu_j\beta_j(x,y)g_j(y)\Big]\geqq 0.$$

By Condition a) (in view of definition 16), it follows that

$$\sum_{j=1}^{k}\mu_j\beta_j(x,y)g_j(y;\theta_j(x,y))\leqq 0. \quad (18)$$

Since $\left(y,\lambda,\mu,(\eta_i)_{i=\overline{1,N}},(\theta_j)_{j=\overline{1,k}}\right)$ is a feasible solution for (MD), the first dual constraint with (18) implies that

$$\sum_{i=1}^{N}\lambda_i f_i'(y;\eta_i(x,y))\geqq 0. \quad (19)$$

From (19) and Hypothesis i) of a), we obtain

$$b_0(x,y)\phi_0\left[\sum_{i=1}^{N}\lambda_i\alpha_i(x,y)(f_i(x)-f_i(y))\right]\geqq 0. \quad (20)$$

Condition ii) of a) and Inequality (20) give

$$\sum_{i=1}^{N}\lambda_i\alpha_i(x,y)(f_i(x)-f_i(y))\geqq 0. \quad (21)$$

Assume that $f(x)\leq f(y)$. Since $\alpha_i>0,\ i=1,2,\cdots,N$ and $\lambda>0$, we obtain

$$\sum_{i=1}^{N}\lambda_i\alpha_i(x,y)(f_i(x)-f_i(y))<0, \quad (22)$$

which contradicts (21), Therefore, the conclusion follows:

The proof of part b) and c) are very similar to proof of part a), except that: for part b), the Inequality (21) becomes strict $(>)$ and Inequality (22) becomes non strict $(\leqq)$. For part c), the Inequality (18) becomes strict $(<)$, it follows that the Inequalities (20) and (21) become strict $(>)$. Since $\lambda \geqq 0$, then the Inequality (22) becomes non strict $(\leqq)$. In this cases, the Inequalities (21) and (22) contradicts each other always.

Remark 1: If we omit the assumption $\lambda > 0$ in the condition i) of a) or the word "strictly" in the condition b),we obtain, for this part of theorem, $f(x) \nleq f(y)$.

Theorem 4.2. (Weak Duality). Let x and $\left(y, \lambda, \mu, (\eta_i)_{i=\overline{1,N}}, (\theta_j)_{j=\overline{1,K}}\right)$ be feasible solutions for (MP) and (MD) respectively, Assume that

1) (f, g) is semi-strictly $d_I - V$ -univex type I at y with respect to $\lambda > 0$, μ, $(\eta_i)_{i=\overline{1,N}}$, $(\theta_j)_{j=\overline{1,k}}$ and for some positive functions $(\alpha_i^\star)_{i=\overline{1,N}}$, $(\beta_j^\star)_{j=\overline{1,k}}$,

2) for any $u \in R$,

$$\phi_0(u) > 0 \Rightarrow a > 0,\ (u) \geqq 0 \Rightarrow \phi_1(u) \geqq 0,$$
$$b_0(x, y) > 0,\ b_1(x, y) \geqq 0.$$

Then $f(x) \nleq f(y)$.

Proof: Since $\mu_j g_j(y) \geqq 0, j = 1, 2, \cdots, k,$ which implies that

$$\sum_{j=1}^{k} \mu_j \beta_j^\star(x, y) g_j(y) \geqq 0. \tag{23}$$

By (23) and Hypothesis i) (with $b_1(x, y)/\beta_j(x, y)$) in Definition 12 replaced by $\beta_j^\star(x, y)$ it follows that

$$\sum_{j=1}^{k} \mu_j g_j'(y, \theta_j(x, y)) \leqq 0. \tag{24}$$

The first dual constraint and (24) give

$$\sum_{i=1}^{N} \lambda_i f_i'(y, \eta_i(x, y)) \geqq 0. \tag{25}$$

Dividing both sides of (3) by $\alpha_i(x, y)$ and taking $x \neq y$, by Hypothesis i), we get

$$b_0(x, y)\frac{1}{\alpha_i(x, y)}\phi_0\left[f_i(x) - f_i(y)\right] > f_i'(y, \eta_i(x, y)),$$
$$i = 1, 2, \cdots, N.$$

On Multiplying by λ_i and taking $\alpha_i^\star = \dfrac{1}{\alpha_i(x, y)}$, we get

$$b_0(x, y)\lambda_i\alpha_i^\star\phi_0\left[f_i(x) - f_i(y)\right] > \lambda_i f_i'(y, \eta_i(x, y)),$$
$$i = 1, 2, \cdots, N$$

Adding with respect to *i*, and applying (25) and Hypothesis ii), we have

$$\sum_{i=1}^{N} \lambda_i \alpha_i^\star(x, y)\left(f_i(x) - f_i(y)\right) > 0. \tag{26}$$

Assume that $f(x) \leq f(y)$. Since $\alpha_i^\star > 0$ and $\lambda > 0$, we have

$$\sum_{i=1}^{N} \lambda_i \alpha_i^\star(x, y)\left(f_i(x) - f_i(y)\right) < 0,$$

which contradicts (26).

Theorem 4.3. (Strong Duality).Let x_0 be a weakly efficient solution for (MP). Assume that the function g satisfies the d_I-constraint qualification at x_0 with respect to $(\theta_j)_{j=\overline{1,k}}$. Then there exist $\lambda \in R_>^N$ and $\mu \in R_>^K$ such that $(x_0, \lambda, \mu, \eta_i)_{i=\overline{1,N}}$, $(\theta_j)_{j=\overline{1,k}} \in Y$ and objective functions of (MP) and (MD) have the same values at x_0 and $\left(x_0, \lambda, \mu, (\eta_i)_{i=\overline{1,N}}, (\theta_j)_{j=\overline{1,k}}\right)$, respectively. If, further, the weak duality between (MP) and (MD) in theorem holds with the condition a) without $\lambda > 0$ (resp. with the condition b) or c)), then $\left(x_0, \lambda, \mu, (\eta_i)_{i=\overline{1,N}}, (\theta_j)_{j=\overline{1,k}}\right) \in Y$ is a weakly efficient (resp. an efficient) solutions of (MD).

Proof. By the Theorem 31 [20], there exists $\mu \in \mathbb{R}_{\geq}^{k}$ and $\lambda \in \mathbb{R}_{>}^{|J(x_0)|}$ such that

$$\sum_{i=1}^{N} \lambda_i f_i'(x_0; \eta_i(x, x_0)) + \sum_{j=1}^{k} \mu_j g_j'(x_0; \theta_j(x, x_0)) \geqq 0,$$
$$\forall x \in X.$$

It follows that $\left(x_0, \mu, \lambda, (\eta_i)_{i=\overline{1,N}}, (\theta_j)_{j=\overline{1,k}}\right) \in Y$. Trivially, the objective function values of (MP) and (MD) are equal.

Suppose that $\left(x_0, \lambda, \mu, (\eta_i)_{i=\overline{1,N}}, (\theta_j)_{j=\overline{1,k}}\right) \in Y$ is not a weakly efficient solution of (MD). Then there exists $\left(y^\star, \lambda^\star, \mu^\star, (\eta_i^\star)_{i=\overline{1,N}}, (\theta_j^\star)_{j=\overline{1,k}}\right) \in Y$ such that $f(x_0) < f(y^\star)$ which violates the weak duality theorem. Hence $\left(x_0, \lambda, \mu, (\eta_i)_{i=\overline{1,N}}, (\theta_j)_{j=\overline{1,K}}\right) \in Y$ is indeed a weakly efficient solution of (MD).

Theorem 4.4. (Strict Converse Duality). Let x_0 and $\left(y_0, \lambda, \mu, (\eta_i)_{i=\overline{1,N}}, (\theta_j)_{j=\overline{1,k}}\right)$ be feasible solutions for (MP) and (MD) respectively, such that

$$\sum_{i=1}^{N} \lambda_i f_i(x_0) = \sum_{i=1}^{N} \lambda_i f_i(y_0). \tag{27}$$

Moreover, assume that (f, g) is strictly pseudo quasi $d_I - V -$ type I at y_o with respect to $(\eta_i)_{i=\overline{1,N}}, (\theta_j)_{j=\overline{1,k}}$ and for λ and μ. Then $x_0 = y_0$.

Proof. Since $\mu_j g_j(y_0) \geqq 0\ \ \forall j = 1, 2, \cdots, k$, we have

$$b_1(x_0 y_0)\phi_1\left[\sum_{j=1}^{k}\mu_j\beta_j(x_0,y_0)g_j(y_0)\right]\geqq 0.$$

Using the second part of the hypothesis, we get

$$\sum_{j=1}^{k}\mu_j g_j'\left(y_0;\theta_j(x_0,y_0)\right)\leqq 0. \quad (28)$$

The Inequality (28) and feasibility of $\left(y_0,\lambda,\mu,(\eta_i)_{i=\overline{1,N}},(\theta_j)_{j=\overline{1,k}}\right)$ for (MD) give

$$\sum_{i=1}^{N}\lambda_i f_i'\left(y_0;\eta_i(x_0,y_0)\right)\geqq 0,$$

which by the first part of Hypothesis ii), we obtain

$$b_0(x_0,y_0)\phi_0\left[\sum_{i=1}^{N}\lambda_i\alpha_i(x_0,y_0)\left(f_i(x_0)-f_i(y_0)\right)\right]>0,$$

$\forall x\in X$.

The above inequality along with Hypothesis iii) gives

$$\sum_{i=1}^{N}\lambda_i\alpha_i(x_0,y_0)\left(f_i(x_0)-f_i(y_0)\right)>0. \quad (29)$$

By Hypothesis i), iii) and $\alpha_i(x_0,y_0)>0$, $i=1,2,\cdots,N$ we have

$$\sum_{i=1}^{N}\lambda_i\alpha_i(x_0,y_0)\left(f_i(x_0)-f_i(y_0)\right)=0. \quad (30)$$

Now (29) and (30) contradict each other. Hence the conclusion follows.

5. Conclusion and Future Developments

In this paper, generalized d_I-V-univex functions have been introduced. The sufficient optimality conditions are discussed for a point to be an efficient or properly efficient for (MP) under the introduced functions. Appropriate Mond-Weir type duality relations are established under these assumptions. Sufficiency and duality with generalized d_I-V-univex functions will be studied for nonsmooth variational and nonsmooth control problems, which will orient the future research of the author.

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