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Moments of Discounted Dividend Payments in the Sparre Andersen Model with a Constant Dividend Barrier*

Jiyang Tan[1], Lin Xiao[2], Shaoyue Liu[1], Xiangqun Yang[2]
[1]*Department of Statistics, Xiangtan University, Xiangtan, China*
[2]*Department of Mathematics, Hunan Normal University, Changsha, China*
E-mail: tanjiyang15@163.com
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Abstract

We consider the Sparre Andersen risk process in the presence of a constant dividend barrier, and propose a new expected discounted penalty function which is different from that of Gerber and Shiu. We find that iteration mothed can be used to compute the values of expected discounted dividends until ruin and the new penalty function. Applying the new function and the recursion method proposed in Section 5, we obtain the arbitrary moments of discounted dividend payments until ruin.

Keywords: Sparre Andersen Model, Expected Discounted Penalty Function, Constant Dividend Barrier, Recursion, Iteration

1. Introduction

The dividend problem in risk theory was brought out initially by De Finetti [1] and has been studied extensively in many literatures by now. Much of the literature on dividend theory is concentrated on the classical risk model, in which claims occur as a Poisson process. For the classical risk model with a barrier strategy, Lin *et al*. [2] studied the Gerber-Shiu discounted penalty function at ruin; Dickson and Waters [3] studied arbitrary moments of the discounted sum of dividend payments until ruin; Gerber *et al*. [4] recently developed methods for estimating the optimal dividend barrier.

The surplus process is not necessarily a compound Poisson process. Andersen [5] lets claims occur according to a more general renewal process. Since then, Sparre Andersen risk model was studied extensively. For some recent contributions to Sparre Andersen risk models with a dividend barrier, see [6-8]. It is worth mentioning that Albrecher *et al*. [8] studied a class of Sparre Andersen risk models with generalized Erlang(n) waiting times in the presence of a constant dividend barrier *b*, and gained some results on the distribution of dividend payments until ruin. It is natural to ask for developing some methods to get the distribution or moments of discounted dividend payments in an arbitrary Sparre Andersen model.

In this paper, we consider the Sparre Andersen model with arbitrary distributed waiting times in the presence of a constant dividend barrier *b*. The analysis is focused on the evaluation of the new expected discounted penalty function defined in Section 2, which will permit us to obtain arbitrary moments of discounted dividend payments by applying the proposed recursion method.

2. The Model

Consider the Sparre Andersen risk model, which is given by

$$U(t) = u + ct - S(t) = u + ct - \sum_{i=1}^{N(t)} X_i \tag{2.1}$$

where $u \ge 0$ is the initial surplus, c is a constant premium rate, $S(t)$ is the aggregate claim up to time *t*, $N(t)$ is the number of claims occurring in $(0,t]$, and X_i is the *i*th claim. Let $M_1, M_2, \cdots$ denote the inter-claim times, and assume that

$$L_n = \sum_{k=1}^{n} M_k .$$

We assume that $\{X_n, n \ge 1\}$ and $\{M_n, n \ge 1\}$ are independent sequences of i.i.d. non-negative random variables. $\{X_n, n \ge 1\}$ have a common distribution

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$F(x)=\Pr\{X_1\le x\}$ with $F(0)=0$, and $\{M_n, n\ge 1\}$ have a common distribution $G(x)=\Pr\{M_1\le x\}$ with $G(0)=0$. Assume $EM_1<\infty$.

The risk process (2.1) is now modified by introducing a constant dividend barrier *b* $(b\ge 0)$, *i.e.* whenever the surplus process reaches the level *b*, the premium income is paid out as dividends to shareholders and the modified surplus process remains at level *b* until the occurrence of the next claim. Let $\{U_b(t)\}$ denote the modified surplus process, and the random variable *D* (*u*,*b*) denote the sum of the discounted dividend payments until ruin (with force of interest $\delta_0\ge 0$). In the sequel we will be interested in the *k*th moment of the sum of discounted dividend payments

$$W_b^{(k)}(u)=E\left[D^k(u,b)\,|\,U(0)=u\right],\quad k=1,2,\cdots.$$

Let $W_b^{(1)}(u)=W_b(u)$, which is the expectation. We will always assume that $0\le u\le b$.

Let the time of ruin for this modified surplus process $\{U_b(t)\}$ be

$$T=T_b(u)=\inf\{t\ge 0:U_b(t)<0\}(\infty\ \text{otherwise})$$

Obviously, *T* must be some L_i. Define the stochastic time τ by

$$\tau=\tau_b(u)=\inf\{L_i:U(L_i-)>b, i\ge 1\}.$$

For $x\ge 0$, let *v*(*x*) be a non-negative measurable function. For $\delta_0\ge 0$ we define a new expected discounted penalty function by

$$\varphi_b(u)=E\left[e^{-\delta\tau}v(U(\tau-))I(\tau\le T)\middle|U(0)=u\right], 0\le u\le b \tag{2.2}$$

where $I(\cdot)$ is the indicator function. The function $\varphi_b(u)$ is similar to (but different from) Gerber-Shiu expected discounted penalty function.

Let *S* denote the space of real-valued measurable functions on [0,*b*]. Choosing the metric defined by

$$\mathrm{d}(x,y)=\sup_{u\in[0,b]}|x(u)-y(u)|,\ \forall x,y\in S,$$

$S=(S,d)$ is complete metric space. Obviously, $W_b(u)$ and $\varphi_b(u)$ are contained within *S* because of the measurable property of $v(x)$ and the monotone property of $W_b(u)$.

3. The Expectation of Discounted Dividends Until Ruin

Define an operator $\hat{T}:S\to S$ by

$$\begin{aligned}\hat{T}f=&\int_0^{(b-u)/c}\int_0^{u+ct}f(u+ct-x)e^{-\delta_0 t}\mathrm{d}F(x)\mathrm{d}G(t)\\&+\int_{(b-u)/c}^{\infty}\int_0^b f(b-x)e^{-\delta_0 t}\mathrm{d}F(x)\mathrm{d}G(t)+D_1,\end{aligned}\tag{3.1}$$

where *f* = *f*(*u*) is an arbitrary function in *S*, and

$$D_1=\begin{cases}\dfrac{c}{\delta_0}e^{\frac{\delta_0}{c}(u-b)}\left[1-G\left(\dfrac{b-u}{c}\right)\right]-\dfrac{c}{\delta_0}\displaystyle\int_{(b-u)/c}^{\infty}e^{-\delta_0 t}\mathrm{d}G(t), & \delta_0>0\\ (u-b)\left[1-G\left(\dfrac{b-u}{c}\right)\right]+c\displaystyle\int_{(b-u)/c}^{\infty}t\mathrm{d}G(t), & \delta_0=0\end{cases}\tag{3.2}$$

Theorem 1. Under the assumption that $F(b)<1$ or $\delta_0>0$, the function $W_b(u)$ is equal to

$$W_b(u)=\left(\lim_{n\to\infty}\hat{T}^n f\right)(u),\quad \forall b\ge 0.\tag{3.3}$$

As an approximation of $W_b(u)$, $\left(\hat{T}^n f\right)(u)$ satisfies

$$\sup_{u\in[0,b]}\left|\left(\hat{T}^n f\right)(u)-W_b(u)\right|\le\frac{\alpha^n}{1-\alpha}\sup_{u\in[0,b]}\left|f(u)-\left(\hat{T}f\right)(u)\right|,\tag{3.4}$$

where $\alpha=F(b)\int_0^{\infty}e^{-\delta_0 t}\mathrm{d}G(t)$.

Proof. Ruin can not occur in (0, L_1) and the expectation of the discounted dividends paid out in this period is D_1. By the renewal argument we have

$$W_b(u)=E\left[e^{-\delta_0 L_1}W_b((u+cL_1)\wedge b-X_1)\right]+D_1,$$

namely,

$$\begin{aligned}W_b(u)=&\int_0^{(b-u)/c}\int_0^{u+ct}W_b(u+ct-x)e^{-\delta_0 t}\mathrm{d}F(x)\mathrm{d}G(t)\\&+\int_{(b-u)/c}^{\infty}\int_0^b W_b(b-x)e^{-\delta_0 t}dF(x)\mathrm{d}G(t)+D_1,\end{aligned}\tag{3.5}$$

which is an integral equation for $W_b(u)$. Since $W_b(u)\in S$, the Equation (3.5) can be rewritten as

$$W_b(u)=\left(\hat{T}W_b\right)(u).\tag{3.6}$$

For arbitrary $f,g\in S$, we have

$$\begin{aligned}\mathrm{d}\left(\hat{T}f,\hat{T}g\right)\le&\sup_{u\in[0,b]}\\&\left\{\int_0^{(b-u)/c}\int_0^{u+ct}|f(u+ct-x)-g(u+ct-x)|e^{-\delta_0 t}\mathrm{d}F(x)\mathrm{d}G(t)\right.\\&\left.+\int_{(b-u)/c}^{\infty}\int_0^b|f(b-x)-g(b-x)|e^{-\delta_0 t}\mathrm{d}F(x)\mathrm{d}G(t)\right\}\\\le&\sup_{u\in[0,b]}|f(u)-g(u)|\cdot\sup_{u\in[0,b]}\left\{\int_0^{(b-u)/c}\int_0^{u+ct}e^{-\delta_0 t}\mathrm{d}F(x)\mathrm{d}G(t)\right.\\&\left.+\int_{(b-u)/c}^{\infty}\int_0^b e^{-\delta_0 t}\mathrm{d}F(x)\mathrm{d}G(t)\right\}\\\le&\mathrm{d}(f,g)\cdot\int_0^{\infty}\int_0^b e^{-\delta_0 t}\mathrm{d}F(x)\mathrm{d}G(t)\end{aligned}$$

which leads to

$$d(\hat{T}f,\hat{T}g)\le d(f,g)F(b)\int_0^{\infty}e^{-\delta_0 t}\mathrm{d}G(t).\tag{3.7}$$

Thus, $\hat{T}$ is a contraction on S if $F(b) < 1$ or $\delta_0 > 0$ (see [9]). (3.3) and (3.4) follow.

Note that when $b = 0$ we have, for $\delta_0 \geq 0$,

$$W_0(0) = \frac{c}{\delta_0}\left\{1 - \int_0^{\infty} e^{-\delta_0 t} \mathrm{d}G(t)\right\}, \tag{3.8}$$

and for $\delta_0 = 0$, $W_0(0) = cE[M_1]$.

Theorem 1 gives an iteration procedure by which we can obtain approximations to $W_b(u)$ and error bounds. In order to gain more information about the sum of discounted dividend payments until ruin, we discuss the new expected discounted penalty function in Section 4.

4. Expected Discounted Penalty Function

Define an integral operator as follows:

$$\begin{aligned}\tilde{T} f = &\int_0^{(b-u)/c} \int_0^{u+ct} f(u+ct-x) e^{-\delta t} \mathrm{d}F(x) \mathrm{d}G(t) \\ &+ \int_{(b-u)/c}^{\infty} e^{-\delta t} v(u+ct) \mathrm{d}G(t), \forall f \in S.\end{aligned} \tag{4.1}$$

Theorem 2. The penalty function $\varphi_b(u)$ is the solution of the integral equation

$$\varphi_b(u) = (\tilde{T}\varphi_b)(u). \tag{4.2}$$

And under the assumption that $F(b) < 1$ or $G(b/c) < 1$ or $\delta > 0$, the penalty function $\varphi_b(u)$ is the unique solution and equal to

$$\varphi_b(u) = \left(\lim_{n\to\infty} \tilde{T}^n f\right)(u), \tag{4.3}$$

where f is an arbitrary real-valued measurable function.

Proof. The discrete time process $\{U_b(L_n), n = 0,1,\cdots\}$ has stationary and independent increments. By the renewal argument, we have the integral Equation (4.2). The uniqueness and the result (4.3) are due to the fact that $\tilde{T}$ is a contraction under the conditions $F(b) < 1$ or $G(b/c) < 1$ or $\delta > 0$. In fact, for arbitrary real-valued measurable functions $y(u)$ and $z(u)$ on $[0,b]$, we have

$$\begin{aligned}&\sup_{u\in[0,b]} \left|\tilde{T}y - \tilde{T}z\right| \leq \sup_{u\in[0,b]} \\ &\left\{\int_0^{(b-u)/c} \int_0^{u+ct} \left|y(u+ct-x) - z(u+ct-x)\right| e^{-\delta t} \mathrm{d}F(x) \mathrm{d}G(t)\right\} \\ &\leq \sup_{u\in[0,b]} \left|y(u) - z(u)\right| \cdot \sup_{u\in[0,b]} \int_0^{(b-u)/c} e^{-\delta t} F(u+ct) \mathrm{d}G(t) \\ &\leq \sup_{u\in[0,b]} \left|y(u) - z(u)\right| \cdot F(b) \int_0^{b/c} e^{-\delta t} \mathrm{d}G(t),\end{aligned}$$

where $F(b)\int_0^{b/c} e^{-\delta t} \mathrm{d}G(t) < 1$. The results are proven.

Remark. 1) Obviously, when $u = b$ we can obtain the explicit expression

$$\varphi_b(b) = \int_0^{\infty} e^{-\delta t} v(b+ct) \mathrm{d}G(t). \tag{4.4}$$

2) According to Theorem 2, we can obtain the approximation of $\varphi_b(u)$ by the iteration method. As an approximation of $\varphi_b(u)$, $\tilde{T}^n f$ satisfy

$$\sup_{u\in[0,b]} \left|\left(\tilde{T}^n f\right)(u) - \varphi_b(u)\right| \leq \frac{\alpha^n}{1-\alpha} \sup_{u\in[0,b]} \left|f(u) - \left(\tilde{T} f\right)(u)\right|, \tag{4.5}$$

where $\alpha = \sup_{u\in[0,b]} \int_0^{(b-u)/c} e^{-\delta t} F(u+ct) \mathrm{d}G(t)$ .The error bound (4.5) can be used for estimating the number of steps necessary to reach a given accuracy.

Now, we give some examples of *dividend-related quantities* (such as the probability of the event $\tau_b(u) \leq T_b(u)$, the kth moment of the discounted dividends paid out in time period $(0,\tau)$, and the distribution function of the sum of the dividends paid out in time period $(0,\tau)$, etc.) to illustrate applications of Theorem 2.

***Example* 4.1.** Letting $v(x) = 1$ and $\delta = 0$, we have

$$\varphi_b(u) = \Pr(\tau_b(u) \leq T_b(u)).$$

The contraction $\tilde{T}$ is defined by

$$\begin{aligned}\left(\tilde{T} f\right)(u) = &\int_0^{(b-u)/c} \int_0^{u+ct} f(u+ct-x) \mathrm{d}F(x) \mathrm{d}G(t) \\ &+ 1 - G\left(\frac{b-u}{c}\right), \quad \forall f \in S.\end{aligned} \tag{4.6}$$

For any real-valued measurable function f on $[0,b]$, we have

$$\Pr(\tau_b(u) \leq T_b(u)) = \lim_{n\to\infty} \tilde{T}^n f. \tag{4.7}$$

From (4.4), it is easily seen that

$$\Pr(\tau_b(b) \leq T_b(b)) = 1. \tag{4.8}$$

***Example* 4.2.** Letting $v(x) = 1$ and $\delta \geq 0$, we have

$$\varphi_b(u) = E\left[e^{-\delta\tau} I(\tau_b(u) \leq T_b(u))\right].$$

Let $R_\delta(u)$ denote this function. Note that

$$R_0(u) = \Pr(\tau_b(u) \leq T_b(u)).$$

Here, the contraction $\tilde{T}$ is defined by

$$\begin{aligned}\left(\tilde{T} f\right)(u) = &\int_0^{(b-u)/c} \int_0^{u+ct} e^{-\delta t} f(u+ct-x) \mathrm{d}F(x) \mathrm{d}G(t) \\ &+ \int_{(b-u)/c}^{\infty} e^{-\delta t} \mathrm{d}G(t).\end{aligned} \tag{4.9}$$

Choosing $f = 0$, we have

$$R_\delta(u) = \lim_{n\to\infty} \tilde{T}^n \left[\int_{(b-u)/c}^{\infty} e^{-\delta t} \mathrm{d}G(t)\right]. \tag{4.10}$$

The error estimate is

$$\sup_{u\in[0,b]}\left|\left(\tilde{T}^n f\right)(u)-R_\delta(u)\right|\le\frac{\alpha^n}{1-\alpha}\int_0^\infty e^{-\delta t}\mathrm{d}G(t). \quad (4.11)$$

By (4.4), we have

$$R_\delta(b)=\int_0^\infty e^{-\delta t}\mathrm{d}G(t). \quad (4.12)$$

Example **4.3.** The insurer will continuously pay dividends in time period $\left(\tau-\left(U(\tau-)-b\right)/c,\tau\right)$ at rate c when $\tau\le T$. Set

$$w(x)=\frac{c}{\delta_0}\left[e^{\frac{\delta_0}{c}(x-b)}-1\right].$$

Then, the present value of these dividends is $e^{-\delta_0\tau}w(U(\tau-))$. For arbitrary $i=0,1,\cdots$, letting $v(x)=w^i(x)$ and $\delta\ge0$, yields

$$\varphi_b(u)=E\left[e^{-\delta\tau}w^i\left(U(\tau-)\right)I\left(\tau_b(u)\le T_b(u)\right)\right],$$

which we denote by $A_{\delta,i}(u)$. The contraction $\tilde{T}$ is defined by

$$\left(\tilde{T}f\right)(u)=\int_0^{(b-u)/c}\int_0^{u+ct}e^{-\delta t}f(u+ct-x)\mathrm{d}F(x)\mathrm{d}G(t)+\int_{(b-u)/c}^\infty e^{-\delta t}w^i(u+ct)\mathrm{d}G(t). \quad (4.13)$$

Choosing $f(u)=0$, we have

$$A_{\delta,i}(u)=\lim_{n\to\infty}\tilde{T}^n\left[\int_{(b-u)/c}^\infty e^{-\delta t}w^i(u+ct)\mathrm{d}G(t)\right]. \quad (4.14)$$

The error estimate is

$$\sup_{u\in[0,b]}\left|\left(\tilde{T}^n f\right)(u)-A_{\delta,i}(u)\right|\le\frac{c^i\alpha^n}{\delta_0^i(1-\alpha)}\int_0^\infty e^{-\delta t}\left[e^{\delta_0 t}-1\right]^i\mathrm{d}G(t). \quad (4.15)$$

Note that if $\delta=i\delta_0$, then $A_{\delta,i}(u)=A_{i\delta_0,i}(u)$ is the ith moment of the discounted dividends paid out in the period $(0,\tau)$; if $i=0$, *i.e.* $v(x)=w^0(x)$, then $A_{\delta,0}(u)=R_\delta(u)$. In addition, it should be pointed out that

$$A_{\delta,i}(b)=\int_0^\infty e^{-\delta t}w^i(u+ct)\mathrm{d}G(t). \quad (4.16)$$

Example **4.4.** Letting $\delta=0$, and $v(x)=I(x-b\le z)$ for arbitrary $z\ge0$, we have

$$\varphi_b(u)=\Pr\left[U(\tau-)-b\le z,\tau\le T\right],$$

which is the distribution function of the cumulative dividends in time period $\left(\tau-\left(U(\tau-)-b\right)/c,\tau\right)$. We denote this distribution function by $\Phi(u,z)$. The contraction $\tilde{T}$ is defined by

$$\left(\tilde{T}f\right)(u)=\int_0^{(b-u)/c}\int_0^{u+ct}f(u+ct-x)\mathrm{d}F(x)\mathrm{d}G(t)+G\left(\frac{b-u+z}{c}\right)-G\left(\frac{b-u}{c}\right). \quad (4.17)$$

Choosing $f(u)=0$, we have

$$\Phi(u,z)=\lim_{n\to\infty}\tilde{T}^n\left[G\left(\frac{b-u+z}{c}\right)-G\left(\frac{b-u}{c}\right)\right]. \quad (4.18)$$

The error estimate is

$$\sup_{u\in[0,b]}\left|\left(\tilde{T}^n f\right)(u)-\Phi(u,z)\right|\le\frac{\alpha^n}{1-\alpha}G\left(\frac{b+z}{c}\right), \quad (4.19)$$

where

$$\alpha=\sup_{u\in[0,b]}\int_0^{(b-u)/c}F(u+ct)\mathrm{d}G(t).$$

5. The *k*th Moment of Discounted Dividend Payments Until Ruin

In this section, we use $\tau_1(u)$ (or τ_1) instead of $\tau_b(u)$ and use $T_1(u)$ (or T_1) instead of $T_b(u)$.

If ruin doesn't occur at time $\tau_1(u)$ and $U_b(\tau_1)=u_2$ ($0\le u_2<b$), we view the process as "starting again" with "initial surplus" u_2, and similarly to the definitions of $T_b(u)$ and $\tau_b(u)$, define the stochastic times $T_2=T_2(u_2)$ and $\tau_2=\tau_2(u_2)$ respectively for the new pro cess $U_b(u_2)$. If ruin doesn't occur at τ_2 either, similarly we define $T_3=T_3(u_3)$ and $\tau_3=\tau_3(u_3)$. Applying repeatedly the idea of "starting again", we can define two sequences of mutually independent random variables $\{T_i,i\ge1\}$ and $\{\tau_i,i\ge1\}$. Suppose that the claim amount at τ_i is $X_{(i+1)}$. Then, $u_i=b-X_{(i)}$ ($i\ge2$).

Further, suppose that

$$\xi_1=e^{-\delta_0\tau_1}I(\tau_1\le T_1),$$

$$\xi_i=I\left(X_{(i)}\le b\right)\cdot e^{-\delta_0\tau_i}I(\tau_i\le T_i),\quad i\ge2.$$

Then, ξ_i ($i\ge2$) are i.i.d. random variables and independent of ξ_1. According to Example 2 in Section 4, we have $E\xi_1^k=R_{k\delta_0}(u)$ and

$$E\xi_2^k=E\left[I\left(X_{(2)}\le b\right)R_{k\delta_0}\left(b-X_{(2)}\right)\right]=\int_0^b R_{k\delta_0}(b-x)\mathrm{d}F(x). \quad (5.1)$$

We denote $E\xi_2^k$ by $\bar{R}_k$. Assume that

$$\eta_1=e^{-\delta_0\tau_1}w\left(U(\tau_1-)\right)I(\tau_1\le T_1),$$

$$\eta_i=I\left(X_{(i)}\le b\right)\cdot e^{-\delta_0\tau_i}w\left(U(\tau_i-)\right)I(\tau_i\le T_i),\quad i\ge2,$$

which is the "present value" of the dividends paid in time period $(0,\tau_i)$ in the $(i-1)$th "starting again" process. Obviously, η_i $(i\ge2)$ are i.i.d. random variables and random vectors (ξ_1,η_1), $(\xi_2,\eta_2),\cdots$ are mutually independent. According to Example 3 in Section 4, we have

$$E\left(\xi_1^i\eta_1^j\right)=A_{(i+j)\delta_0,j}\left(u\right),\quad i,j\geq 0, \tag{5.2}$$

$$E\left(\xi_2^i\eta_2^j\right)=E\left(I\left(X_{(2)}\leq b\right)A_{(i+j)\delta_0,j}\left(b-X_{(2)}\right)\right)=\int_0^b A_{(i+j)\delta_0,j}\left(b-x\right)\mathrm{d}F\left(x\right)\left(i,j\geq 0\right). \tag{5.3}$$

We denote $E\left(\xi_2^i\eta_2^j\right)$ by $\overline{A}_{i+j,j}$.

Theorem 3. The *k*th moment of the sum of discounted dividend payments until ruin is equal to

$$W_b^{(k)}\left(u\right)=\sum_{i=0}^{k}\frac{k!}{i!\left(k-i\right)!}A_{k\delta_0,i}\left(u\right)x_{k-i},\ k=1.2,\cdots, \tag{5.4}$$

where $x_j\left(j=0,1,2,\cdots\right)$ satisfy the following recursive formulas:

$$x_0=1;\quad x_j=\sum_{i=1}^{j}\frac{j!}{i!\left(j-i\right)!}\overline{A}_{j,i}x_{j-i}\Big/\left(1-\overline{R}_j\right),\forall j>0. \tag{5.5}$$

Proof. It is easily seen that the sum of the discounted dividend payments until ruin is equal to

$$D\left(u,b\right)=\eta_1+\xi_1\eta_2+\xi_1\xi_2\eta_3+\cdots=\sum_{i=1}^{\infty}\left(\eta_i\prod_{k=1}^{i-1}\xi_k\right), \tag{5.6}$$

where we adopt the convention that $\prod_{k=1}^{0}\xi_k=1$. Thus, we have

$$D^k\left(u,b\right)=\sum_{i=0}^{k}\frac{k!}{i!\left(k-i\right)!}\eta_1^i\left(\xi_1\eta_2+\xi_1\xi_2\eta_3+\xi_1\xi_2\xi_3\eta_4+\cdots\right)^{k-i}=\sum_{i=0}^{k}\frac{k!}{i!\left(k-i\right)!}\eta_1^i\xi_1^{k-i}\left(\eta_2+\xi_2\eta_3+\xi_2\xi_3\eta_4+\cdots\right)^{k-i}. \tag{5.7}$$

Taking expectation of (5.7) yields

$$W_b^{(k)}\left(u\right)=\sum_{i=0}^{k}\frac{k!}{i!\left(k-i\right)!}A_{k\delta_0,i}\left(u\right)E\left[\eta_2+\xi_2\eta_3+\xi_2\xi_3\eta_4+\cdots\right]^{k-i},\ k=1,2,\cdots. \tag{5.8}$$

Note that

$$\begin{aligned}&E\left[\eta_2+\xi_2\eta_3+\xi_2\xi_3\eta_4+\cdots\right]^j\\&=E\sum_{i=0}^{j}\frac{j!}{i!\left(j-i\right)!}\eta_2^i\left(\xi_2\eta_3+\xi_2\xi_3\eta_4+\xi_2\xi_3\xi_4\eta_5+\cdots\right)^{j-i}\\&=E\sum_{i=0}^{j}\frac{j!}{i!\left(j-i\right)!}\xi_2^{j-i}\eta_2^i\left(\eta_3+\xi_3\eta_4+\xi_3\xi_4\eta_5+\cdots\right)^{j-i}\\&=\sum_{i=0}^{j}\frac{j!}{i!\left(j-i\right)!}E\left[\xi_2^{j-i}\eta_2^i\right]\cdot E\left(\eta_3+\xi_3\eta_4+\xi_3\xi_4\eta_5+\cdots\right)^{j-i}\\&=\sum_{i=0}^{j}\frac{j!}{i!\left(j-i\right)!}\overline{A}_{j,i}\cdot E\left(\eta_2+\xi_2\eta_3+\xi_2\xi_3\eta_4+\cdots\right)^{j-i}.\end{aligned}$$

Since $\overline{A}_{j,0}=\overline{R}_j$, it follows that, for $j=1,2,\cdots$,

$$E\left(\eta_2+\xi_2\eta_3+\xi_2\xi_3\eta_4+\cdots\right)^j=\sum_{i=1}^{j}\frac{j!}{i!\left(j-i\right)!}\overline{A}_{j,i}\frac{E\left(\eta_2+\xi_2\eta_3+\xi_2\xi_3\eta_4+\cdots\right)^{j-i}}{1-\overline{R}_j}, \tag{5.9}$$

which leads to

$$x_j=E\left[\eta_2+\xi_2\eta_3+\xi_2\xi_3\eta_4+\cdots\right]^j,\quad j=0,1,2,\cdots. \tag{5.10}$$

From (5.8) and (5.10), we get (5.4).

6. Numerical Illustration

As an illustration of the results in Sections 3 and 5, consider the case of a Sparre Andersen model with Erlang (2) interclaim times and Erlang (2) claim amounts, *i.e.*

$$G\left(t\right)=F\left(t\right)=1-\left(\lambda t+1\right)e^{-\lambda t}\quad\left(t\geq 0\right).$$

Let $\lambda=2, c=1.1$, and $\delta_0=0.03$. These accord with the assumptions in the Example 4.1 in [8].

Let us first consider the expectation of discounted dividend payments until ruin. Given $b=0,1,2,\cdots,10$ respectively, according to Theorem 1 we choose the function $f\left(u\right)=0$ and determine a number of steps *n* necessary to obtain $\left(\hat{T}^n f\right)\left(u\right)$ as an approximation for $W_b\left(u\right)$ such that

$$\sup_{u\in[0,b]}\left|\left(\hat{T}^n f\right)\left(u\right)-W_b\left(u\right)\right|<0.0001,$$

see **Table 1**. Using the iteration procedure, we get some approximate values of $W_b\left(u\right)$ in **Table 2**. Comparing with the exact values given by Albrecher *et al.* [8], it can be seen that the approximate values in **Table 2** are fairly good. Note that, when *b* = 0, the numerical value 1.076 is obtained by (3.8).

For the *k*th moment of discounted dividend payments until ruin, we need compute $A_{j\delta_0,i}\left(u\right)$, $\overline{A}_{j,i}$ (j = 1,2, $\cdots$, *k*; $i=1,2,\cdots,j$) and $\overline{R}_j$ (j = 1,2, $\cdots$, *k*). In this example, we only consider three cases: *k* = 1, 2, 3. In order to reach an accuracy of 0.00001, the necessary numbers of steps for iteration are given in **Table 3**. By formula (5.4), we obtain the approximate values for $W_b\left(u\right)$ again, see **Table 4**. In **Table 5**, the approximate values for the standard deviation $\sqrt{W_b^{(2)}\left(u\right)-W_b^2\left(u\right)}$ are given. Comparing with the **Table 2** in [8], one can find the approximate values are very excellent too. In **Table 6**, approximations for the third moment are displayed. In addition, we point out that when *b* = 0 the number of steps is not offered in **Table 3**, and the corresponding approximations in **Tables 4-6** can be obtained by (4.8) (4.12) and (4.16).

7. Summary

As shown in Section 6, the iteration method and the re-

Table 1. Numbers of steps for computing the expectation $W_b(u)$ by Theorem 1.

b	1	2	3	4	5	6	7	8	9	10
n	19	76	197	283	307	311	312	312	312	312

Table 2. Approximations for the expectation $W_b(u)$ by Theorem 1.

b \ *u*	0	1	2	3	4	5	6	7	8	9	10
0	1.076\1.076										
1	0.836\0.836	1.808\1.808									
2	0.856\0.856	1.847\1.847	2.846\2.846								
3	0.848\0.848	1.829\1.828	2.815\2.815	3.803\3.803							
4	0.802\0.801	1.728\1.728	2.661\2.661	3.597\3.597	4.574\4.574	5.143\5.143					
5	0.730\0.730	1.575\1.575	2.424\2.424	3.277\3.277	4.175\4.174	4.575\4.575					
6	0.648\0.648	1.397\1.397	2.151\2.151	2.908\2.908	3.705\3.705	3.988\3.988	5.538\5.538				
7	0.565\0.565	1.218\1.218	1.875\1.875	2.535\2.535	3.229\3.229	3.436\3.436	4.840\4.840	5.799\5.799			
8	0.487\0.486	1.049\1.049	1.615\1.615	2.184\2.184	2.782\2.782	2.938\2.9	4.170\4.170	5.010\5.010	5.967\5.967		
9	0.416\0.416	0.897\0.897	1.381\1.381	1.867\1.867	2.379\2.379	38	3.566\3.566	4.285\4.285	5.118\5.118	6.073\6.073	
10	0.354	0.763	1.175	1.589	2.025	2.500	3.035	3.647	4.357	5.185	6.139

a. The exact values given by Albrecher *et al.* (2005) are in smaller size after \.

Table 3. Numbers of steps for computing $A_{\delta,i}(u)$ (i = 0, 1, 2, 3).

δ \ n \ u	1	2	3	4	5	6	7	8	9	10
0.03	8	18	39	78	141	220	292	340	366	379
0.06	8	18	36	67	106	143	169	184	191	194
0.09	8	17	34	58	85	107	120	127	130	131

Table 4. Approximations for the expectation $W_b(u)$ by Formula (5.4).

b \ *u*	0	1	2	3	4	5	6	7	8	9	10
0	1.0757										
1	0.8357	1.8082									
2	0.8564	1.8469	2.8462								
3	0.8480	1.8285	2.8146	3.8027							
4	0.8015	1.7283	2.6605	3.5969	4.5740						
5	0.7302	1.5745	2.4239	3.2774	4.1745	5.1433					
6	0.6479	1.3971	2.1507	2.9081	3.7048	4.5745	5.5376				
7	0.5647	1.2177	1.8746	2.5347	3.2292	3.9881	4.8396	5.7989			
8	0.4865	1.0490	1.6149	2.1836	2.7819	3.4357	4.1703	5.0101	5.9670		
9	0.4160	0.8970	1.3809	1.8672	2.3788	2.9379	3.5661	4.2853	5.1178	6.0731	
10	0.3541	0.7635	1.1753	1.5893	2.0246	2.5005	3.0352	3.6474	4.3570	5.1849	6.1393

cursion method proposed in this paper give good approximations for the arbitrary moments of discounted dividend payments until ruin. The exact mothed presented by Albrecher *et al.* [8] can only be used in the model with generalied Erlang(n)-distributed inter-claim times. The purpose of this paper is to find an approach which can be

Table 5. Approximations for the standard deviation $\sqrt{W_b^{(2)}(u)-W_b^2(u)}$.

b \ u	0	1	2	3	4	5	6	7	8	9
0	0.7440\0.744									
1	1.2397\1.240	1.3988\1.399								
2	1.6667\1.667	2.1105\2.110	2.1930\2.193							
3	1.8637\1.864	2.4562\2.456	2.6948\2.695	2.7416\2.742						
4	1.8841\1.884	2.5275\2.528	2.8457\2.846	2.9887\2.989	3.0201\3.020					
5	1.7972\1.797	2.4365\2.436	2.7834\2.783	2.9813\2.981	3.0855\3.085	3.1112\3.111				
6	1.6564\1.656	2.2634\2.263	2.6128\2.613	2.8365\2.836	2.9884\2.988	3.0796\3.080	3.1040\3.104			
7	1.4958\1.496	2.0577\2.058	2.3959\2.396	2.6288\2.629	2.8072\2.807	2.9450\2.945	3.0348\3.035	3.0599\3.060		
8	1.3343\1.334	1.8467\1.847	2.1666\2.167	2.3987\2.399	2.5897\2.590	2.7548\2.755	2.8918\2.892	2.9844\2.984	3.0106\3.011	
9	1.1815\1.181	1.6444\1.644	1.9424\1.942	2.1675\2.167	2.3617\2.362	2.5404\2.540	2.7048\2.705	2.8454\2.845	2.9416\2.942	2.9690\2.969
10	1.0415	1.4571	1.7317	1.9458	2.1369	2.3197	2.4974	2.6652	2.8100	2.9095

a. The exact values given by Albrecher *et al.* (2005) are in smaller size after \.

Table 6. Approximations for the third moment $W_b^{(3)}(u)$.

b \ u	0	1	2	3	4	5	6	7	8	9	10
0	3.5755										
1	8.4888	20.770									
2	19.421	45.111	77.832								
3	26.504	61.464	104.56	160.74							
4	26.994	62.598	106.45	163.03	239.93						
5	23.379	54.214	92.189	141.20	207.79	300.33					
6	18.484	42.863	72.888	111.63	164.32	237.87	341.11				
7	13.877	32.179	54.720	83.809	123.36	178.62	256.75	366.92			
8	10.113	23.453	39.881	61.081	89.909	130.19	187.17	268.21	382.71		
9	7.2458	16.803	28.573	43.762	64.415	93.273	134.11	192.22	275.06	392.21	
10	5.1400	11.920	20.269	31.044	45.694	66.166	95.133	136.36	195.18	279.13	397.90

used in an arbitrary Sparre Andersen model. The iteration and the recursion prove helpful in achieving the goal. I think that more extensive applications about these methods can be found. Obviously, the iteration method can also be used to compute the Gerber-Shiu penalty function in Sparre Andersen model, even more complicated model.

8. References

[1] B. De Finetti, "Su un'Impostazione Alternativa Della Teoria Collettiva del Rischio," *Transactions of the XVth International Congress of Actuaries*, Vol. 2, No. 1, 1957, pp. 433-443.

[2] X. S. Lin, G. E. Willmot and S. Drekic, "The Classical Risk Models with a Constant Dividend Barrier: Analysis of the GERBER-Shiu Discounted Penalty Function," *Insurance*: *Mathematics and Economics*, Vol. 33, No. 3, 2003, pp. 551-566.

[3] D. C. M. Dickson and H. R. Waters, "Some Optimal Dividend Problems," *Astin Bulletin*, Vol. 34, No. 1, 2004, pp. 49-74. doi:10.2143/AST.34.1.504954

[4] H. U. Gerber, E. S. W. Shiu and N. Smith, "Methods for Estimating the Optimal Dividend Barrier and the Probability of Ruin," *Insurance*: *Mathematics and Economics*, Vol. 42, No. 2, 2008, pp. 243-254. doi:10.1016/j.insmatheco.2007.02.002

[5] E. S. Anderson, "On the Collective Theory of Risk in Case of Contagion between Claims," *Bulletin of the Institute of Mathematics and Its Applications*, Vol. 12, No. 2, 1957, pp. 275-279.

[6] S. Li and J. Garrido, "On a Class of Renewal Risk Models with a Constant Dividend Barrier," *Insurance*: *Mathematics and Economics*, Vol. 35, No. 3, 2004, pp. 691-701.

doi:10.1016/j.insmatheco.2004.08.004

[7] M. M. Claramunt, M. Marmol and R. Lacayo, "On the Probability of Reaching a Barrier in an Erlang(2) Risk Process," Working Paper No. 24, Universitat Autonoma de Barcelona, Cerdanyola del Vallès, 2004.

[8] H. Albrecher, M. Mercgravee Claramunt and M. Marmol, "On the Distribution of Dividend Payments in a Sparre Andersen Model with Generalized Erlang(n) Interclaim Times," *Insurance*: *Mathematics and Economics*, Vol. 37, No. 2, 2005, pp.324-334. doi:10.1016/j.insmatheco.2005.05.004

[9] Erwin Kreyszig, "Introductory Functional Analysis with Applications," John Wiley & Sons, Hoboken, 1978, pp. 299-302.

