Applied Mathematics, 2011, 2, 403-409
doi:10.4236/am.2011.24049 Published Online April 2011 (http://www.SciRP.org/journal/am)

Stationary Characteristics of the Single-Server Queue System with Losses and Immediate Service Quality Control

Aleksey I. Peschansky
Sevastopol National Technical University, Sevastopol, Ukraine
E-mail: *peschansky_sntu@mail.ru*
Received December 27, 2010; *revised January* 31, 2011; *accepted February* 1, 2011

Abstract

Semi-Markovian model of operation of a single-server queue system with losses and immediate service quality control has been built. In case of unsatisfactory request service quality, its re-servicing is carried out. Re-servicing is executed till it is regarded satisfactory. Time between request income, and request service time are assumed to be random values with distribution functions of general kind. An explicit form of the system stationary characteristics has been defined.

Keywords: Queue System, Semi-Markovian Process, System Stationary Characteristics, Request Service Quality

1. Introduction

A large number of works [1-4] are dedicated to the queue systems with losses. In most of them the incoming flux of requests is supposed to be Poisson one or request service time is considered to have exponential distribution. This admission allows efficient modeling of the system operation by means of Markovian processes. But if the incoming flux of requests makes a renewal process and request service time distribution is of general kind, essential difficulties arise when defining system stationary characteristics in explicit form. To overcome them an apparatus of semi-Markovian processes with a common phase field of states is used in the works [5,6]. Thus, in [6] stationary characteristics of the single-server queue system $GI/G/1/0$ with losses were found. In the present article analogical characteristics for the same system have been defined under the assumption that the server admits both satisfactory and unsatisfactory request service. In case of the latter one the re-servicing begins immediately. It is repeated until the service is regarded satisfactory.

In the second chapter of the article the system operation is described, mathematical problem definition and research purpose are stated. In the third section semi-Markovian model of system operation is built, and stationary distribution of embedded Markovian chain is given. In the fourth section system stationary characteristics are defined. These are: final probabilities that the server is free, is in state of service or re-servicing, and mean dwelling times in these states. Besides, formulas of the system stationary characteristics for some subcases are given here. One of them is the exponential distribution of time between request income and request service time.

2. The Problem Definition

Let us investigate the queue system (QS) $GI/G/1/0$ with losses and a single server. Request service time is a random value (RV) α with an absolutely continuous distribution function (DF) $F(t) = P\{\alpha \le t\}$ and density $f(t)$. Time period between requests' income is a RV β with an absolutely continuous DF $G(t) = P\{\beta \le t\}$ and density $g(t)$. If the server is busy with request service, all the incoming requests are lost. With the probability p request service is regarded to be successful, and the queue system passes into a standby state that lasts till next request comes. With the probability $q = 1 - p$ the request service is considered unsatisfactory, and the server begins re-servicing immediately. Time period of such a kind of service is a RV γ with an absolutely continuous DF $\Phi(t) = P\{\gamma \le t\}$ and density $\varphi(t)$. After re-servicing with the probability p the service is regarded satisfactory and with the probability q the request is sent to re-servicing procedure. This process lasts until the service is considered sufficient. It is assumed that RV α, β and γ are independent, have finite mathematical expectations $M\alpha, M\beta, M\gamma$ and variances $D\alpha, D\beta, D\gamma$ respectively. It is necessary to define the following stationary characteristics of the system: the final probabilities that the queue system is in a standby state; that the system is busy with the primary service or re-servicing; mean time periods of system's dwelling in the above-mentioned states.

3. Semi-Markovian Model Building

In order to build the model of the system operation an apparatus of semi-Markovian processes with a discrete-continuous phase field of states [6,7] is used. Let us describe the system operation with the help of semi- Markovian process (SMP) $\xi(t)$ with a phase field

$$E = \{21, 10x,\ 21x,\ 22x,\ 12x\}.$$

Let us write out the codes of the states:

21 – service of the request that has come begins;

$10x$ – request service has been successfully completed, the server passes into a standby state that lasts time x (till the next request income);

$21x$ – the incoming request has been lost, the server is busy with the primary service that will last time x;

$22x$ – the incoming request has been lost, the server is busy with re-servicing that will last time x;

$12x$ – request service has been completed, and its re-servicing has begun, time x is left till the next request income.

In **Figure 1** time diagram of the system operation is shown, and in **Figure 2** there is the system transition graph.

System dwelling times in the states are defined by the formulas:

$$\theta_{21} = \alpha \wedge \beta,\ \theta_{10x} = x,\ \theta_{21x} = \theta_{22x} = \beta \wedge x,\ \theta_{12x} = \gamma \wedge x,$$

where $\wedge$ is a sign of minimum.

Let us define the probabilities and probability densities of the embedded Markovian chain (EMC) $\{\xi_n, n \ge 0\}$ transitions:

$$p_{21}^{21x} = \int_0^\infty g(t) f(x+t)\mathrm{d}t, \quad p_{21}^{10x} = p\int_0^\infty f(t) g(x+t)\mathrm{d}t,$$

$$p_{21}^{12x} = q\int_0^\infty f(t) g(x+t)\mathrm{d}t, \quad P_{10x}^{21} = 1,$$

$$p_{21y}^{21x} = g(y-x),\ x<y; \quad p_{21y}^{10x} = pg(y+x);$$

$$p_{21y}^{12x} = qg(y+x);$$

$$p_{22y}^{22x} = g(y-x),\ x<y; \quad p_{22y}^{10x} = pg(y+x);$$

$$p_{22y}^{12x} = qg(y+x);$$

$$p_{12y}^{10x} = p\varphi(y-x),\ x<y; \quad p_{12y}^{22x} = \varphi(y+x);$$

$$p_{12y}^{12x} = q\varphi(y-x),\ x<y.$$

Now we can proceed to EMC stationary distribution definition, the system of integral equations for its definition is the following one:

$$\begin{cases}
\rho(21x) = \int_x^\infty g(y-x)\rho(21y)\mathrm{d}y + \rho_{21}\int_x^\infty g(y-x) f(y)\mathrm{d}y, \\
\rho(22x) = \int_x^\infty g(y-x)\rho(22y)\mathrm{d}y + \int_0^\infty \varphi(y+x)\rho(12y)\mathrm{d}y, \\
\rho(12x) = q\int_x^\infty \varphi(y-x)\rho(12y)\mathrm{d}y + q\rho_{21}\int_x^\infty f(y-x) g(y)\mathrm{d}y + q\int_0^\infty g(y+x)\big(\rho(21y)+\rho(22y)\big)\mathrm{d}y, \\
\rho(10x) = p\int_x^\infty \varphi(y-x)\rho(12y)\mathrm{d}y + p\rho_{21}\int_x^\infty f(y-x) g(y)\mathrm{d}y + p\int_0^\infty g(y+x)\big(\rho(21y)+\rho(22y)\big)\mathrm{d}y, \\
\rho_{21} = \int_0^\infty \rho(10x)\mathrm{d}x, \quad \rho_{21} + \int_0^\infty \big[\rho(10x)+\rho(12x)+\rho(21x)+\rho(22x)\big]\mathrm{d}x = 1.
\end{cases} \tag{1}$$

Let us introduce the following integral operators:

$$A_g\psi = \int_x^\infty g(y-x)\psi(y)\mathrm{d}y, \quad A_\varphi\psi = \int_x^\infty \varphi(y-x)\psi(y)\mathrm{d}y, \quad A_f\psi = \int_x^\infty f(y-x)\psi(y)\mathrm{d}y,$$

$$\overline{A}_g\psi = \int_0^\infty g(y+x)\psi(y)\mathrm{d}y, \quad \overline{A}_\varphi\psi = \int_0^\infty \varphi(y+x)\psi(y)\mathrm{d}y, \quad \overline{A}_f\psi = \int_0^\infty f(y+x)\psi(y)\mathrm{d}y.$$

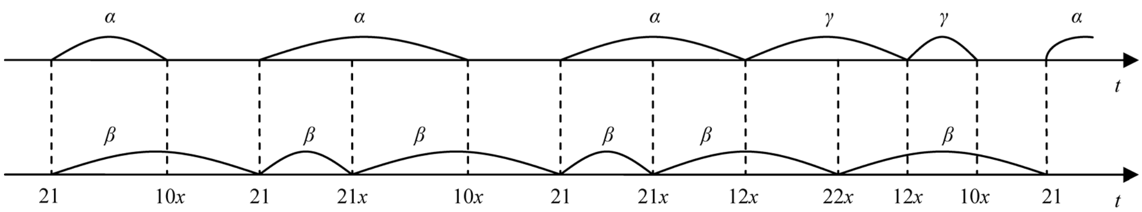

Figure 1. Time diagram of the system operation.

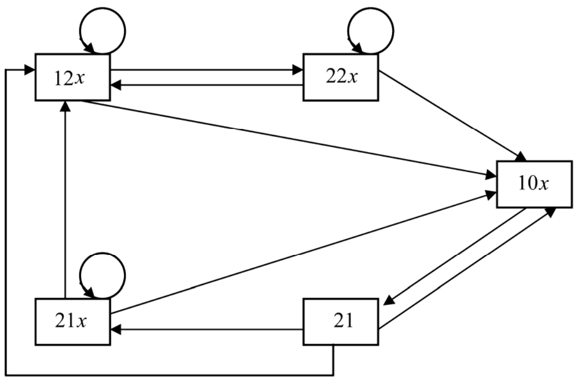

Figure 2. System transition graph.

Then the system of Equations (1) can be rewritten in such a way—Equation (2).

We shall exclude $\rho(21x)$ and $\rho(22x)$ from the first and second equations of the system (2) respectively, and then substitute it in the third equation. The result is

$$\begin{aligned} &\rho(21x) = \rho_{21}\left(I - A_g\right)^{-1} A_g f(x), \\ &\rho(22x) = \left(I - A_g\right)^{-1} \overline{A}_\varphi \rho(12x), \\ &\rho(12x) = q\left[A_\varphi + \overline{A}_g\left(I - A_g\right)^{-1}\overline{A}_\varphi\right]\rho(12x) \\ &\quad + q\rho_{21}\left[A_f + \overline{A}_g\left(I - A_g\right)^{-1}\overline{A}_f\right]g(x). \end{aligned} \tag{3}$$

Here $\left(I - A_g\right)^{-1}\psi(x) = \psi(x) + \int_x^\infty h_g(y - x)\psi(y)\,\mathrm{d}y$, [5] $h_g(x) = \sum_{n=1}^{\infty} g^{*(n)}(x)$ is the density of renewal function $H_g(x)$ generated by DF $G(x)$.

Let us indicate K_φ and K_f integral operators

$$\begin{aligned} K_\varphi\psi &= q\left[A_\varphi + \overline{A}_g\left(I - A_g\right)^{-1}\overline{A}_\varphi\right]\psi \\ &\equiv \int_0^\infty k_{q,\varphi}(x, y)\psi(y)\,\mathrm{d}y, \end{aligned}$$

$$\begin{aligned} K_f\psi &= q\left[A_f + \overline{A}_g\left(I - A_g\right)^{-1}\overline{A}_f\right]\psi \\ &\equiv \int_0^\infty k_{q,f}(x, y)\psi(y)\,\mathrm{d}y. \end{aligned}$$

With regard to the introduced operators the Equation (3) will have the form

$$\rho(12x) = K_\varphi \rho(12x) + \rho_{21} K_f g(x). \tag{4}$$

Let us write down kernels of integral operators K_φ, K_f in explicit forms and single out their probability sense

$$k_{q,\varphi}(x, y) = \begin{cases} q\varphi(y - x) + q\int_0^\infty \varphi(t + y) v_g(t, x)\,\mathrm{d}t, & y > x, \\ q\int_0^\infty \varphi(t + y) v_g(t, x)\,\mathrm{d}t, & y < x, \end{cases}$$

$$k_{q,f}(x, y) = \begin{cases} qf(y - x) + q\int_0^\infty f(t + y) v_g(t, x)\,\mathrm{d}t, & y > x, \\ q\int_0^\infty f(t + y) v_g(t, x)\,\mathrm{d}t, & y < x. \end{cases}$$

$$\begin{cases} \rho(21x) = A_g\rho(21x) + \rho_{21}A_g f(x), \\ \rho(22x) = A_g\rho(22x) + \overline{A}_\varphi\rho(12x), \\ \rho(12x) = qA_\varphi\rho(21x) + q\rho_{21}A_f g(x) + q\overline{A}_g\left[\rho(21x) + \rho(22x)\right], \\ \rho(10x) = pA_\varphi\rho(12x) + p\rho_{21}A_f g(x) + p\overline{A}_g\left[\rho(21x) + \rho(22x)\right], \\ \rho_{21} = \int_0^\infty \rho(10x)\,\mathrm{d}x, \\ \rho_{21} + \int_0^\infty\left[\rho(10x) + \rho(12x) + \rho(21x) + \rho(22x)\right]\mathrm{d}x = 1. \end{cases} \tag{2}$$

Here $v_g(t,x)=g(t+x)+\int_0^t g(t+x-u)h_g(u)\mathrm{d}u$ is the density of the direct residual time β_t for the renewal process generated by RV $\beta:\beta_t=\tau_{\nu(t)+1}-t$ [8]. The functions $k_{q,\varphi}(x,y)$ and $\int_0^\infty k_{q,f}(x,y)g(y)\mathrm{d}y$ are densities of probabilities of system transition from the state 12*y* to the nearest state 12*x* and from the state 21 to the nearest one 12*x* respectively.

As

$$\int_0^\infty k_{q,\varphi}(x,y)\mathrm{d}x=q\int_0^y\varphi(y-x)\mathrm{d}x+q\int_0^\infty \mathrm{d}x\int_0^\infty\varphi(t+y)v_g(t,x)\mathrm{d}t$$
$$=q\left(\Phi(y)+\overline{\Phi}(y)\right)=q<1,$$

it is not difficult to ensure that the operator K_φ is the operator of contraction in the space of summable functions. That is why the solution of Equation (4) can be found by means of successive approximations. This solution with regard to the identity

$$q\overline{A}_g\left(I-A_g\right)^{-1}f(x)=q\int_0^\infty v_g(y,x)f(y)\mathrm{d}y$$

can be represented in the form

$$\rho(12x)=\rho_{21}\sum_{n=1}^\infty q^n l_n(x), \tag{5}$$

where $l_1(x)=\int_0^\infty v_g(y,x)f(y)\mathrm{d}y$ is the density of RV that is time period between the end of primary request service and the moment of the next request income. The function $l_n(x)=\int_0^\infty v_g^{(l_{n-1})}(y,x)\varphi(y)\mathrm{d}y,\ n\ge 2$ is the density of random time period between the end of the $\left(n-1^{st}\right)$ request re-servicing and the moment of the next request income. Here

$$v_g^{(l_{n-1})}(y,x)=l_{n-1}(y+x)+\int_0^y h_g^{(l_{n-1})}(y-u)g(u+x)\mathrm{d}u$$

is the density of the direct residual time for the renewal process $H_g^{(l_{n-1})}(x)$ generated by the functions $L_{n-1}(x)=\int_0^x l_{n-1}(t)\mathrm{d}t$ and $G(x)$. Let us note that the density of renewal function complies with the ratio

$$h_g^{(l_{n-1})}(x)=l_{n-1}(x)+\int_0^x l_{n-1}(x-u)h_g(u)\mathrm{d}u.$$

The rest of EMC stationary distributions are defined by the formulas

$$\begin{aligned}\rho(10x)&=\rho_{21}\frac{p}{q}\sum_{n=1}^\infty q^n l_n(x),\\ \rho(21x)&=\rho_{21}\int_x^\infty h_g(y-x)f(y)\mathrm{d}y,\\ \rho(22x)&=\rho_{21}\sum_{n=1}^\infty q^n\int_0^\infty\varphi(x+y)l_n(y)\mathrm{d}y\\ &\quad+\rho_{21}\sum_{n=1}^\infty q^n\int_0^\infty \mathrm{d}y\int_0^\infty\varphi(x+y+s)l_n(s)\mathrm{d}s.\end{aligned} \tag{6}$$

The stationary probability ρ_{21} of EMC's dwelling in the state 21 is found with the help of normalization requirement:

$$\rho_{21}=\left[2+\int_0^\infty f(x)H_g(x)\mathrm{d}x+\sum_{n=1}^\infty q^n\int_0^\infty\varphi(s)H_g^{(l_n)}(s)\mathrm{d}s\right]^{-1}.$$

4. Definition of System Stationary Characteristics

Mean values of system dwelling times in the states are defined by the formulas

$$\begin{aligned}&M\theta_{21}=\int_0^\infty\overline{F}(t)\overline{G}(t)\mathrm{d}t,\quad M\theta_{10x}=x,\\ &M\theta_{21x}=M\theta_{22x}=\int_0^x\overline{G}(t)\mathrm{d}t,\quad M\theta_{12x}=\int_0^x\overline{\Phi}(t)\mathrm{d}t.\end{aligned} \tag{7}$$

Let us consider the following disjoint subsets of system states: $E_0=\{10x\}$, $E_1=\{21,\ 21x\}$, $E_2=\{12x,\ 22x\}$. Dwelling in the state E_0 means that the server is free and the system is in a standby mode. Dwelling in the state E_1 or E_2 signifies that the server is busy with the primary request service or with re-servicing respectively.

Let us introduce SMP $\xi(t)$ transition probabilities:

$$\Phi(t,e,E_i)=P\left\{\xi(t)\in E_i/\xi(0)=e\right\},$$
$$e\in E=E_0\cup E_1\cup E_2.$$

As it is known [9], under the condition of the unique EMC $\{\xi_n,n\ge 0\}$ of SMP $\xi(t)$ stationary distribution existence the following ratios take place

$$\lim_{t\to\infty}\Phi(t,e,E_i)=\int_{E_i}m(e)\rho(\mathrm{d}e)\left[\int_E m(e)\rho(\mathrm{d}e)\right]^{-1},\ i=\overline{0,2}, \tag{8}$$

where $m(e)$ is mean SMP $\xi(t)$ dwelling time in the state $e\in E$.

With the help of Formulas (5)-(7) integrals contained in the Formula (8) are converted into:

$$\int_{E_1} m(e)\rho(\mathrm{d}e) = \rho_{21}\int_0^\infty \overline{F}(t)\overline{G}(t)\mathrm{d}t + \rho_{21}\int_0^\infty \mathrm{d}x\int_0^x \overline{G}(t)\mathrm{d}t\int_x^\infty h_g(y-x)f(y)\mathrm{d}y = \rho_{21}M\alpha,$$

$$\int_{E_2} m(e)\rho(\mathrm{d}e) = \rho_{21}\sum_{n=1}^\infty q^n\left\{\int_0^\infty l_n(x)\mathrm{d}x\int_0^x \overline{\Phi}(t)\mathrm{d}t + \int_0^\infty dx\int_0^\infty \varphi(x+y)l_n(y)\mathrm{d}y\int_0^x \overline{G}(t)\mathrm{d}t\right.$$
$$\left.+\int_0^\infty \mathrm{d}x\int_0^x \overline{G}(t)\mathrm{d}t\int_0^\infty h_g(y)\mathrm{d}y\int_0^\infty \varphi(x+y+s)l_n(s)\mathrm{d}s\right\} = \rho_{21}\frac{q}{p}M\gamma,$$

$$\int_{E_0} m(e)\rho(\mathrm{d}e) = \frac{p}{q}\rho_{21}\sum_{n=1}^\infty q^n\int_0^\infty l_n(x)x\mathrm{d}x$$
$$= \rho_{21}\left\{M\beta\left[1+\int_0^\infty f(t)H_g(t)\mathrm{d}t + \sum_{n=1}^\infty q^n\int_0^\infty \varphi(t)H_g^{(l_n)}(t)\mathrm{d}t\right] - \frac{q}{p}M\gamma - M\alpha\right\},$$

$$\int_E m(e)\rho(\mathrm{d}e) = \rho_{21}M\beta\left[1+\int_0^\infty f(t)H_g(t)\mathrm{d}t + \sum_{n=1}^\infty q^n\int_0^\infty \varphi(t)H_g^{(l_n)}(t)\mathrm{d}t\right].$$

Consequently, the final probability that the server is free is defined by the formula

$$p_0 = \lim_{t\to\infty}\Phi(t,e,E_0) = 1 - \frac{M\alpha + \frac{q}{p}M\gamma}{M\beta\left[1+\int_0^\infty f(t)H_g(t)\mathrm{d}t + \sum_{n=1}^\infty q^n\int_0^\infty \varphi(t)H_g^{(l_n)}(t)\mathrm{d}t\right]} \tag{9}$$

and the final probabilities p_1 and p_2 that the server is busy either with the primary service or with the re-servicing are:

$$p_1 = \lim_{t\to\infty}\Phi(t,e,E_1) = \frac{M\alpha}{M\beta\left[1+\int_0^\infty f(t)H_g(t)\mathrm{d}t + \sum_{n=1}^\infty q^n\int_0^\infty \varphi(t)H_g^{(l_n)}(t)\mathrm{d}t\right]}, \tag{10}$$

$$p_2 = \lim_{t\to\infty}\Phi(t,e,E_2) = \frac{\frac{q}{p}M\gamma}{M\beta\left[1+\int_0^\infty f(t)H_g(t)\mathrm{d}t + \sum_{n=1}^\infty q^n\int_0^\infty \varphi(t)H_g^{(l_n)}(t)\mathrm{d}t\right]}. \tag{11}$$

It is necessary to note that the ratio

$$\int_0^\infty f(t)H_g(t)\mathrm{d}t + \sum_{n=1}^\infty q^n\int_0^\infty \varphi(t)H_g^{(l_n)}(t)\mathrm{d}t$$

determines an average value of requests lost per time unit of a complete request service.

Let us define mean stationary dwelling times $T(E_i)$ in the extracted subsets E_i, $i=\overline{0.2}$. According to [7] we have

$$T(E_i) = \int_{E_i} m(e)\rho(\mathrm{d}e)\left[\int_{E\setminus E_i}\rho(\mathrm{d}e)P(e,E_i)\right]^{-1}, \quad i=\overline{0.2}. \tag{12}$$

One can define the values of expressions in denominators of Formulas (12). To do it let us take integrals of both parts of the system (1) equations. The result is

$$\int_{E\setminus E_0}\rho(\mathrm{d}e)P(e,E_0) = p\rho_{21}\int_0^\infty f(t)\overline{G}(t)\mathrm{d}t + p\int_0^\infty \overline{G}(x)\rho(21x)\mathrm{d}x + p\int_0^\infty \Phi(x)\rho(12x)\mathrm{d}x + p\int_0^\infty \overline{G}(x)\rho(22x)\mathrm{d}x$$
$$= p\rho_{21}\int_0^\infty f(t)\overline{G}(t)\mathrm{d}t + p\rho_{21}\int_0^\infty \overline{F}(x)g(x)\mathrm{d}x + p\int_0^\infty \Phi(x)\rho(12x)\mathrm{d}x + p\int_0^\infty \overline{\Phi}(x)\rho(12x)\mathrm{d}x$$
$$= p\rho_{21} + p\int_0^\infty \rho(12x)\mathrm{d}x = p\rho_{21} + p\frac{q}{p}\rho_{21} = \rho_{21},$$

$$\int\limits_{E\setminus E_1} \rho(\mathrm{d}e)P(e,E_1) = \int\limits_0^\infty \rho(10x)\mathrm{d}x = \rho_{21},$$

$$\int\limits_{E\setminus E_2} \rho(\mathrm{d}e)P(e,E_2) = q\rho_{21}\int\limits_0^\infty f(t)\overline{G}(t)\mathrm{d}t + q\int\limits_0^\infty \overline{G}(x)\rho(21x)\mathrm{d}x = q\rho_{21}\int\limits_0^\infty f(t)\overline{G}(t)\mathrm{d}t + q\rho_{21}\int\limits_0^\infty f(t)G(t)\mathrm{d}t = q\rho_{21}.$$

Consequently, mean system dwelling times in the extracted subsets are defined by the formulas

$$T(E_0) = M\beta\left[1+\int\limits_0^\infty f(t)H_g(t)dt + \sum_{n=1}^\infty q^n\int\limits_0^\infty \varphi(t)H_g^{(l_n)}(t)\mathrm{d}t\right] - \frac{q}{p}M\gamma - M\alpha,$$

$$T(E_1) = M\alpha,\quad T(E_2) = \frac{1}{p}M\gamma. \qquad (13)$$

It is necessary to note that in case if with the probability equal to 1 satisfactory request service is carried out the system characteristics defined by the Formulas (9), (10) and (13) coincide with ones found in [6].

To illustrate some subcases of the results gained let us write down QS $M/M/1/0$, $M/G/1/0$, $GI/M/1/0$ characteristics.

The system $M/M/1/0$ stationary characteristics. If the RV α, β, γ densities have the form $f(t) = \lambda e^{-\lambda t}$, $g(t) = \mu e^{-\mu t}$, $\varphi(t) = \nu e^{-\nu t}$, $t \ge 0$ system stationary characteristics are defined by the formulas

$$p_0 = \left[1+\frac{\mu}{\lambda}+\frac{q}{p}\frac{\mu}{\nu}\right]^{-1},\quad p_1 = \left[1+\frac{\lambda}{\mu}+\frac{q}{p}\frac{\lambda}{\nu}\right]^{-1},$$

$$p_2 = \left[1+\frac{p}{q}\frac{\nu}{\lambda}+\frac{p}{q}\frac{\nu}{\mu}\right]^{-1},$$

$$T(E_0) = \frac{1}{\mu},\quad T(E_1) = \frac{1}{\lambda},\quad T(E_2) = \frac{1}{p\nu}.$$

The system $M/G/1/0$ stationary characteristics. In case when $g(t) = \mu e^{-\mu t}$, $t \ge 0$, and densities $f(t)$, $\varphi(t)$ are of general kind we have $l_n(x) = \mu e^{-\mu t}$, $H_g^{(l_n)}(x) = \mu x$,

$$\int\limits_0^\infty f(t)H_g(t)\mathrm{d}t + \sum_{n=1}^\infty q^n\int\limits_0^\infty \varphi(t)H_g^{(l_n)}(t)\mathrm{d}t$$
$$= \mu M\alpha + \mu M\gamma\sum_{n=1}^\infty q^n = \mu M\alpha + \mu M\gamma\frac{q}{p}.$$

The Formulas (9)-(11) and (13) for defining system stationary characteristics convert into

$$p_0 = \frac{1}{1+\mu M\alpha + \frac{q}{p}\mu M\gamma},\quad p_1 = \frac{\mu M\alpha}{1+\mu M\alpha+\frac{q}{p}\mu M\gamma},$$

$$p_2 = \frac{\frac{q}{p}\mu M\gamma}{1+\mu M\alpha+\frac{q}{p}\mu M\gamma},$$

$$T(E_0) = \frac{1}{\mu},\quad T(E_1) = M\alpha,\quad T(E_2) = \frac{1}{p}M\gamma.$$

The system $GI/M/1/0$ stationary characteristics. For this system the incoming flux of requests is generated by RV β with density $g(t)$ of a general kind, and $f(t) = \lambda e^{-\lambda t}$, $\varphi(t) = \nu e^{-\nu t}$, $t \ge 0$. Let us find the functionals contained in the formulas for defining stationary characteristics. We have

$$\int\limits_0^\infty f(t)H_g(t)\mathrm{d}t = \int\limits_0^\infty \overline{F}(t)h_g(t)\mathrm{d}t$$
$$= \int\limits_0^\infty e^{-\lambda t}h_g(t)\mathrm{d}t = \frac{\tilde{g}(\lambda)}{1-\tilde{g}(\lambda)},$$

where $\tilde{g}(\lambda) = \int\limits_0^\infty g(t)e^{-\lambda t}\mathrm{d}t$ is Laplace transformation of the density $g(t)$;

$$\sum_{n=1}^\infty q^n\int\limits_0^\infty \varphi(t)H_g^{(l_n)}(t)\mathrm{d}t = \sum_{n=1}^\infty q^n\int\limits_0^\infty \overline{\Phi}(t)h_g^{(l_n)}(t)\mathrm{d}t$$
$$= \sum_{n=1}^\infty q^n\int\limits_0^\infty e^{-\nu t}h_g^{(l_n)}(t)\mathrm{d}t = \frac{1}{1-\tilde{g}(\nu)}\sum_{n=1}^\infty q^n\tilde{l}_n(\nu).$$

From the recurrence formula

$$l_n(t) = \int\limits_0^\infty \varphi(s)l_{n-1}(s+t)\mathrm{d}s$$
$$+\int\limits_0^\infty l_{n-1}(u)\mathrm{d}u\int\limits_0^\infty v_g(y,t)\varphi(y+u)\mathrm{d}y$$

the recurrence formula for defining $\tilde{l}_n(\nu)$ can be concluded:

$$\tilde{l}_n(\nu) = \nu\int\limits_0^\infty t\,l_{n-1}(t)e^{-\nu t}\mathrm{d}t + \nu\tilde{l}_{n-1}(\nu)\int\limits_0^\infty \mathrm{d}t\int\limits_0^\infty v_g(y,t)e^{-\nu(t+y)}\mathrm{d}y$$
$$= -\nu\frac{\mathrm{d}}{\mathrm{d}\nu}\tilde{l}_{n-1}(\nu) - \nu\frac{\tilde{g}'(\nu)}{1-\tilde{g}(\nu)}\tilde{l}_{n-1}(\nu)$$

That is why the sum of series

$S(\nu)=\frac{1}{1-\tilde g(\nu)}\sum_{n=1}^{\infty} q^n \tilde l_n(\nu)$ is the solution of differential equation

$$S(\nu)=\frac{q\tilde l_1(\nu)}{1-\tilde g(\nu)}-q\nu S'(\nu).$$

This solution obeys the condition $\lim_{\nu\to 0}(1-\tilde g(\nu))S(\nu)=\frac{q}{p}$. This equation solution is defined by the formula $S(\nu)=\nu^{-\frac{1}{q}}\int_0^{\nu}\frac{x^{\frac{p}{q}}\tilde l_1(x)}{1-\tilde g(x)}\mathrm{d}x$. As $\tilde l_1(x)=\frac{\lambda}{\lambda-x}\frac{\tilde g(x)-\tilde g(\lambda)}{1-\tilde g(\lambda)}$ we find out that

$$S(\nu)=\frac{\lambda\nu^{-\frac{1}{q}}}{(1-\tilde g(\lambda))}\int_0^{\nu}\frac{x^{\frac{p}{q}}\left[\tilde g(x)-\tilde g(\lambda)\right]}{(\lambda-x)(1-\tilde g(x))}\mathrm{d}x.$$

Consequently, system stationary characteristics are defined by the following ratios

$$p_0=1-\frac{\left[\frac{1}{\lambda}+\frac{q}{p\nu}\right](1-\tilde g(\lambda))}{M\beta\left[1+\lambda\nu^{-\frac{1}{q}}\int_0^{\nu}\frac{x^{\frac{p}{q}}\left[\tilde g(x)-\tilde g(\lambda)\right]}{(\lambda-x)(1-\tilde g(x))}\mathrm{d}x\right]},$$

$$p_1=\frac{1-\tilde g(\lambda)}{\lambda M\beta\left[1+\lambda\nu^{-\frac{1}{q}}\int_0^{\nu}\frac{x^{\frac{p}{q}}\left[\tilde g(x)-\tilde g(\lambda)\right]}{(\lambda-x)(1-\tilde g(x))}\mathrm{d}x\right]},$$

$$p_2=\frac{\frac{q}{p}(1-\tilde g(\lambda))}{\nu M\beta\left[1+\lambda\nu^{-\frac{1}{q}}\int_0^{\nu}\frac{x^{\frac{p}{q}}\left[\tilde g(x)-\tilde g(\lambda)\right]}{(\lambda-x)(1-\tilde g(x))}\mathrm{d}x\right]},$$

$$T(E_1)=\frac{1}{\lambda},\ T(E_2)=\frac{1}{p\nu},$$

$$T(E_0)=M\beta\left[\frac{1}{1-\tilde g(\lambda)}+\frac{\lambda\nu^{-\frac{1}{q}}}{1-\tilde g(\lambda)}\int_0^{\nu}\frac{x^{\frac{p}{q}}\left[\tilde g(x)-\tilde g(\lambda)\right]}{(\lambda-x)(1-\tilde g(x))}\mathrm{d}x\right]-\frac{q}{\nu p}-\frac{1}{\lambda}.$$

5. Conclusions

In the present work semi-Markovian model of the operation of the single-server queue system $GI/G/1/0$ with losses in which unsatisfactory request service quality is admitted has been built. Service quality control is carried out immediately, and in case of unsatisfactory service quality re-servicing is executed until the service is regarded as satisfactory. With the help of this model stationary characteristics in explicit form have been defined. These are: the final probabilities of system's dwelling in a standby state, in the states of primary service and re-servicing, and, besides, mean stationary dwelling times in these states. These characteristics depend on mean time periods between requests' income, mean service time, average value of requests lost per time unit of a complete request service and the probability of unsatisfactory service. In case of satisfactory service only the characteristics found coincide with the formerly defined ones.

6. References

[1] J. Riordan, "Stochastic Service Systems," Member Technical Staff Bell Telephone Laboratories, Inc., John Wiley & Sons, Inc., New York, 1962.

[2] B. V. Gnedenko and I. N. Kovalenko, "An Introduction to Queuing Theory," Nauka, Moscow, 1987.

[3] V. V. Anisimov, O. K. Zakusilov and V. S. Donchenko, "Elements of Queuing Theory and System Asymptotic Analysis," Higher School, Kiev, 1987.

[4] B. V. Gnedenko and D. Konig, "Handbuch der Bedienungsteorie II (Formeen und Andere Ergebnisse)," Verlag, Berlin, 1984.

[5] Y. Y. Obzherin and A. I. Peschansky, "Stationary Characteristics of a Single-Server Queue System with a Single Waiting Space," *Cybernetics and System Analysis*, Vol. 42, No. 5, 2006, pp. 51-62.

[6] A. N. Korlat, V. N. Kuznetsov, M. I. Novikov and A. F. Turbin, "Semi-Markovian Models of Restorable and Service Systems," Shtiintsa, Kishinev, 1991.

[7] V. S. Korolyuk and A. F. Turbin, "Markovian Restoration Processes in the Problems of System Reliability," Naukova Dumka, Kiev, 1982.

[8] F. Beichelt and P Franken, "Zuverlässigkeit und Instanphaltung. Mathematische Methoden," VEB Verlag Technik, Berlin, 1983.

[9] I. N. Kovalenko, N. Y. Kuznetsov and V. M. Shurenkov, "Stochastic Processes. Guidance," Naukova Dumka, Kiev, 1983.

