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Statistically Convergent Double Sequence Spaces in 2-Normed Spaces Defined by Orlicz Function

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Abstract

The concept of statistical convergence was introduced by Stinhauss [1] in 1951. In this paper, we study convergence of double sequence spaces in 2-normed spaces and obtained a criteria for double sequences in 2-normed spaces to be statistically Cauchy sequence in 2-normed spaces.[1]

Keywords: Double Sequence Spaces, Natural Density, Statistical Convergence, 2-Norm, Orlicz Function

1. Introduction

In order to extend the notion of convergence of sequences, statistical convergence was introduced by Fast [2] and Schoenberg [3] independently. Later on it was further investigated by Fridy and Orhan [4]. The idea depends on the notion of density of subset of $\mathbb{N}$.

The concept of 2-normed spaces was initially introduced by $\ddot{G}$ ahler [5-7] in the mid of 1960's. Since then, many researchers have studied this concept and obtained various results, see for instance [8].

Let X be a real vector space of dimension d, where $2 \le d \le \infty$. A 2-norm on X is a function $\|.,.\| : X \times X \to R$ which satisfies the following four conditions:

1) $\|x_1, x_2\| = 0$ if and only if x_1, x_2 are linearly dependent;

2) $\|x_1, x_2\| = \|x_2, x_1\|$:

3) $\|\alpha x_1, x_2\| = \alpha \|x_1, x_2\|$, for any $\alpha \in R$:

4) $\|x + x', x_2\| \le \|x, x_2\| + \|x', x_2\|$

The pair $(X, \|.,.\|)$ is then called a 2-normed space (see [9]).

Example 1.1. A standard example of a 2-normed space is R^2 equipped with the following 2-norm

$\|x, y\|$:= the area of the triangle having vertices $0, x, y$.

Example 1.2. Let Y be a space of all bounded real-valued functions on R. For f, g in Y, define

$\|f, g\| = 0$, if *f*, *g* are linearly dependent,

$\|f, g\| = \sup_{t \in R} |f(t) \cdot g(t)|$, if *f*, *g* are linearly independent.

Then $\|.,.\|$ is a 2-norm on Y.

We recall some facts connecting with statistical convergence. If K is subset of positive integers $\mathbb{N}$, then K_n denotes the set $\{k \in K : k \le n\}$. The natural density of K is given by $\delta(K) = \lim_{n\to\infty} \frac{|K_n|}{n}$, where $|K_n|$ denotes the number of elements in K_n, provided this limit exists. Finite subsets have natural density zero and $\delta(K^c) = 1 - \delta(K)$ where $K^c = \mathbb{N} \setminus K$, that is the complement of K. If $K_1 \subseteq K_2$ and K_1 and K_2 have natural densities then $\delta(K_1) \le \delta(K_2)$. Moreover, if $\delta(K_1) = \delta(K_2) = 1$, then $\delta(K_1 \cap K_2) = 1$ (see [10]).

A real number sequence $x = (x_j)$ is statistically convergent to L provided that for every $\varepsilon > 0$ the set $\{n \in \mathbb{N} : |x_j - L| \ge \varepsilon\}$ has natural density zero. The sequence $x = (x_j)$ is statustically Cauchy sequence if for each $\varepsilon > 0$ there is positive integer $N = N(\varepsilon)$ such that $\delta(\{n \in \mathbb{N} : |x_j - x_N(\varepsilon)|\}) = 0$ (see [11]).

If $x = (x_j)$ is a sequence that satisfies some property P for all n except a set of natural density zero, then we say that (x_j) satisfies some property P for "almost all *n*".

An *Orlicz Function* is a function $M : [0, \infty) \to [0, \infty)$ which is continuous, nondecreasing and convex with $M(0) = 0$, $M(x) > 0$ for $x > 0$ and $M(x) \to \infty$, as $x \to \infty$.

If convexity of M is replaced by $M(x+y) \le M(x) + M(y)$, then it is called a *Modulus funtion* (see Maddox [12]). An Orlicz function may be bounded or un-

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bounded. For example, $M(x)=x^p\,(0<p\le 1)$ is unbounded and $M(x)=\dfrac{x}{x+1}$ is bounded.

Lindesstrauss and Tzafriri [13] used the idea of Orlicz sequence space;

$$l_M := \left\{ x\in w: \sum_{k=1}^{\infty} M\left(\frac{|x_k|}{\rho}\right) < \infty, \text{ for some } \rho>0 \right\}$$

which is Banach space with the norm

$$\|x\|_M = \inf\left\{ \rho>0: \sum_{k=1}^{\infty} M\left(\frac{|x_k|}{\rho}\right) \le 1 \right\}.$$

The space l_M is closely related to the space l_p, which is an Orlicz sequence space with $M(x)=x^p$ for $1\le p<\infty$.

An Orlicz function M satisfies the $\Delta_2-condition$ ($M\in\Delta_2$ *for short*) if there exist constant $K\ge 2$ and $u_0>0$ such that

$$M(2u)\le KM(u)$$

whenever $|u|\le u_0$.

Note that an Orlicz function satisfies the inequality

$$M(\lambda x)\le \lambda M(x) \text{ for all } \lambda \text{ with } 0<\lambda<1.$$

Orlicz function has been studied by V. A. Khan [14-17] and many others.

Throughout a double sequence $x=(x_{kl})$ is a double infinite array of elements x_{kl} for $k,l\in\mathbb{N}$.

Double sequences have been studied by V. A. Khan [18-20], Moricz and Rhoades [21] and many others.

A double sequence $x=(x_{jk})$ called statistically convergent to L if

$$\lim_{m,n\to\infty}\frac{1}{mn}\left|\left\{(j,k): |x_{jk}-L|\ge\varepsilon,\ j\le m, k\le n\right\}\right| = 0$$

where the vertical bars indicate the number of elements in the set. (see [19])

In this case we write $st_2-\lim x_{jk}=L$.

2. Definitions and Preliminaries

Let (x_j) be a sequence in 2-normed space $(X,\|.,.\|)$. The sequence (x_j) is said to be statistically convergent to L, if for every $\varepsilon>0$, the set

$$\left\{ j\in\mathbb{N}: \|x_j-L,z\|\ge\varepsilon \right\}$$

has natural density zero for each nonzero z in X, in other words (x_j) statistically converges to L in 2-normed space $(X,\|.,.\|)$ if

$$\lim_{n\to\infty}\frac{1}{n}\left|\left\{ j: \|x_j-L,z\| \right\}\right| = 0$$

for each nonzero z in X. It means that for every $z\in X$,

$$\|x_j-L,z\|<\varepsilon \quad a.a.n.$$

In this case we write

$$st-\lim_{n\to\infty}\|x_j-L,z\| := \|L,z\|.$$

Example 2.1 Let $X=R^2$ be equiped with the 2-norm by the formula

$$\|x,y\| = |x_1y_2-x_2y_1|,\ x=(x_1,x_2),\ y=(y_1,y_2).$$

Define the (x_j) in 2-normed space $(X,\|.,.\|)$ by

$$x_j = \begin{cases} (1,n) & \text{if } n=k^2, k\in N, \\ \left(1,\dfrac{n-1}{n}\right) & \text{otherwise.} \end{cases}$$

and let $L=(1,1)$ and $z=(z_1,z_2)$. If $z_1=0$ then

$$K=\left\{ j\in\mathbb{N}: \|x_j-L,z\|\ge\varepsilon \right\}=\theta$$

for each z in X, $\left\{ j\in\mathbb{N}: n\ne k^2, k\ge\dfrac{|z_1|}{\varepsilon} \right\}$ is a finite set, so

$$\begin{aligned}&\left\{ j\in\mathbb{N}: \|x_j-L,z\|\ge\varepsilon \right\}\\ &=\left\{ j\in\mathbb{N}: j=k^2, k\ge\left(\frac{\varepsilon}{|z_1|}+1\right)^{\frac{1}{2}} \right\}\cup\{\text{ finite set}\}.\end{aligned}$$

Therefore,

$$\begin{aligned}&\frac{1}{n}\left|\left\{ j\in\mathbb{N}: \|x_j-L,z\|\ge\varepsilon \right\}\right|\\ &=\left|\left\{ j\in\mathbb{N}: j=k^2, k\ge\left(\frac{\varepsilon}{|z_1|}+1\right)^{\frac{1}{2}} \right\}\right|\cup\frac{1}{n}0(1)\end{aligned}$$

for each z in X. Hence, $\delta\left(\left\{ j\in\mathbb{N}: \|x_n-L,z\|\ge\varepsilon \right\}\right)=0$ for every $\varepsilon>0$ and $z\in X$.

V. A. Khan and Sabiha Tabassum [20] defined a double sequence (x_{jk}) in 2-normed space $(X,\|.,.\|)$ to be Cauchy with respect to the 2-norm if

$$\lim_{j,p\to\infty}\|x_{jk}-x_{pq},z\|=0 \text{ for every } z\in X \text{ and } k,q\in\mathbb{N}.$$

If every Cauchy sequence in X converges to some $L\in X$, then X is said to be complete with respect to the 2-norm. Any complete 2-normed space is said to be 2-Banach space.

Example 2.2 Define the x_i in 2-normed space $(X,\|.,.\|)$ by

$$x_j = \begin{cases} (0,j) & \text{if } j=k^2, k\in N, \\ (0,0) & \text{otherwise.} \end{cases}$$

and let $L=(0,0)$ and $z=(z_1,z_2)$. If $z_1=0$ then

$$\{j\in\mathbb{N}:\|x_j-L,z\|\geq\varepsilon\}\subset\{1,4,9,16,\cdots,j^2;\cdots\}$$

We have that $\delta(\{j\in\mathbb{N}:\|x_j-L,z\|\geq\varepsilon\})=0$ for every $\varepsilon>0$ and $z\in X$. This implies that $st-\lim_{n\to\infty}\|x_j,z\|=\|L,z\|$. But the sequence x_j is not convergent to L.

A sequence which converges statistically need not be bounded. This fact can be seen from Example [2.1] and Example [2.2].

3. Main Results

In this paper we define a double sequence (x_{jk}) in 2-normed space $(X,\|.,.\|)$ to be statistically Cauchy with respect to the 2-norm if for every $\varepsilon>0$ and every nonzero $z\in X$ there exists a number $p=p(\varepsilon,z)$ and $q=q(\varepsilon,z)$ such that

$$\lim_{m,n\to\infty}\frac{1}{mn}\left|(j,k)\in N\times N:\|x_{jk}-x_{pq},z\|\geq\varepsilon, j\leq m,k\leq n\right|=0$$

In this case we write $st_2-\lim\|x_{jk}-L,z\|:=\|L,z\|$.

Theorem 3.1. Let (x_{jk}) be a double sequence in 2-normed space $(X,\|.,.\|)$ and $L,L'\in X$. If $st_2-\lim\|x_{jk},z\|=\|L,z\|$ and $st_2-\lim\|x_{jk},z\|=\|L',z\|$, then $L=L'$.

Proof. Assume $L\neq L'$,. Then $L-L'\neq 0$, so there exists a $z\in X$, such that $L-L'$ and z are linearly independent. Therefore

$$\|L-L',z\|=2\varepsilon,\ \text{with } \varepsilon>0.$$

Now

$$2\varepsilon=\|(L-x_{jk})+(x_{jk}-L'),z\|$$
$$\leq\|x_{jk}-L,z\|+\|x_{jk}-L,z\|.$$

So $\{(j,k):\|x_{jk}-L',z\|<\varepsilon\}\subseteq\{(j,k):\|x_{jk}-L',z\|<\varepsilon\}$. But $\delta(\{(j,k):\|x_{jk}-L',z\|<\varepsilon\})=0$. Contradicting the fact that $x_{jk}\to L'(stat)$.

Theorem 3.2. Let the double sequence (x_{jk}) and (y_{jk}) in 2-normed space $(X,\|.,.\|)$. If (y_{jk}) is a convergent sequence such that $x_{jk}=y_{jk}$ almost all n, then (x_{jk}) is statistically convergent.

Proof. Suppose $\delta(\{(j,k)\in N\times N:x_{jk}\neq y_{jk}\})=0$ and $\lim_{j,k\to\infty}\|y_{jk},z\|=\|L,z\|$. Then for every $\varepsilon>0$ and $z\in X$.

$$\{(j,k)\in N\times N:\|x_{jk}-L,z\|\geq\varepsilon\}$$
$$\subseteq\{(j,k)\in N\times N:x_{jk}\neq y_{jk}\}.$$

Therefore

$$\delta(\{(j,k)\in N\times N:\|x_{jk}-L,z\|\geq\varepsilon\})$$
$$\leq\delta(\{(j,k)\in N\times N:\|y_{jk}-L,z\|\geq\varepsilon\}) \tag{3.1}$$
$$+\delta(\{(j,k)\in N\times N:x_{jk}\neq y_{jk}\}).$$

Since $\lim_{n\to\infty}\|y_{jk},z\|=\|L,z\|$ for every $z\in X$, the set $\{(j,k)\in N\times N:\|y_{jk}-L,z\|\geq\varepsilon\}$ contains finite number of integers. Hence, $\delta(\{(j,k)\in N\times N:\|y_{jk}-L,z\|\geq\varepsilon\})=0$. Using inequality [3.1], we get

$$\delta(\{(j,k)\in N\times N:\|x_{jk}-L,z\|\geq\varepsilon\})=0$$

for every $\varepsilon>0$ and $z\in X$. Consequently,

$$st_2-\lim\|x_{jk}-L,z\|=\|L,z\|.$$

Theorem 3.3. Let the double sequence (x_{jk}) and (y_{jk}) in 2-normed space $(X,\|.,.\|)$ and $L,L'\in X$ and $a\in\mathbb{R}$.

If $st_2-\lim\|x_{jk},z\|=\|L,z\|$

and $st_2-\lim\|y_{jk},z\|=\|L',z\|$,

for every nonzero $z\in X$, then

1) $st_2-\lim\|x_{jk}+y_{jk},z\|=\|L+L',z\|$, for each nonzero $z\in X$ and

2) $st_2-\lim\|ax_{jk},z\|=\|aL,z\|$, for each nonzero $z\in X$.

Proof 1) Assume that $st_2-\lim\|x_{jk},z\|=\|L,z\|$, and $st_2-\lim\|y_{jk},z\|=\|L',z\|$, for every nonzero $z\in X$. Then $\delta(K_1)=0$ and $\delta(K_2)=0$ where

$$K_1=K_1(\varepsilon):=\left\{(j,k)\in N\times N:\|x_{jk}-L,z\|\geq\frac{\varepsilon}{2}\right\}$$

$$K_2=K_2(\varepsilon):=\left\{(j,k)\in N\times N:\|y_{jk}-L',z\|\geq\frac{\varepsilon}{2}\right\}$$

for every $\varepsilon>0$ and $z\in X$. Let

$$K=K(\varepsilon):=\{(j,k)\in N\times N:\|x_{jk}+y_{jk}-(L+L'),z\|\geq\varepsilon\}.$$

To prove that $\delta(K)=0$, it is sufficient to prove that $K\subset K_1\cup K_2$. Suppose $j_0,k_0\in K$. Then

$$\{\|x_{j_0k_o}+y_{j_0k_0}-(L+L'),z\|\geq\varepsilon\} \tag{3.2}$$

Suppose to the contrary that $j_0, k_0 \notin K_1 \cup K_2$. Then $j_0, k_0 \notin K_1$ and $j_0, k_0 \notin K_2$. If $j_0, k_0 \notin K_1$ and $j_0, k_0 \notin K_2$ then

$$\left\|x_{j_0k_0} - L, z\right\| < \frac{\varepsilon}{2} \text{ and } \left\|x_{j_0k_0} - L, z\right\| < \frac{\varepsilon}{2}.$$

Then, we get

$$\left\|x_{j_0k_o} + y_{j_0k_0} - (L + L'), z\right\|$$

$$\le \left\|x_{j_0k_o} - L, z\right\| + \left\|y_{j_0k_0} - L', z\right\| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$$

which contradicts [3.2]. Hence $j_0, k_0 \in K_1 \cup K_2$, that is, $K \subset K_1 \cup K_2$.

2) Let $st_2 - \lim\left\|x_{jk}, z\right\| = \left\|L, z\right\|, a \in \mathbb{R}$ and $a \neq 0$. Then

$$\left(\left\{(j,k) \in N \times N : \left\|x_{jk} - L, z\right\| \ge \frac{\varepsilon}{|a|}\right\}\right) = 0.$$

Then we have

$$\left\{(j,k) \in N \times N : \left\|ax_{jk} - aL, z\right\| \ge \varepsilon\right\}$$

$$= \left\{(j,k) \in N \times N : |a|\left\|x_{jk} - L, z\right\| \ge \varepsilon\right\}$$

$$= \left\{(j,k) \in N \times N : \left\|x_{jk} - L, z\right\| \ge \frac{\varepsilon}{|a|}\right\}.$$

Hence, the right handside of above equality equals 0. Hence, $st_2 - \lim\left\|ax_{jk}, z\right\| = \left\|aL, z\right\|$, for every nonzero $z \in X$.

From Theorem 1 of Fridy [11] we have

Theorem 3.4. Let (x_{jk}) be statistically Cauchy sequence in a finite dimensional 2-normed space $(X, \|.,.\|)$. Then there exists a convergent double sequence (y_{jk}) in $(X, \|.,.\|)$ such that $x_{jk} = y_{jk}$ for almost all *n*.

Proof. See proof of Theorem 2.9 [9].

Theorem 3.5. Let (x_{jk}) be a double sequence in 2-normed space $(X, \|.,.\|)$ The double sequence (x_{jk}) is statistically convergent if and only if (x_{jk}) is a statistically Cauchy sequence.

Proof. Assume that $st_2 - \lim\left\|x_{jk}, z\right\| = \left\|L, z\right\|$ for every nonzero $z \in X$ and $\varepsilon > 0$.

Then, for every $z \in X$,

$$\left\|x_{jk} - L, z\right\| < \frac{\varepsilon}{2} \text{ almost all } n,$$

and if $p = p(\varepsilon, z)$ and $q = q(\varepsilon, z)$ is chosen so that $\left\|x_{pq} - L, z\right\| < \frac{\varepsilon}{2}$, then, we have

$$\left\|x_{jk} - x_{pq}, z\right\| \le \left\|x_{jk} - L, z\right\| + \left\|L - x_{pq}, z\right\|$$

$$< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} \text{ almost all } n.$$

$$= \varepsilon \text{ almost all } n.$$

Hence, (x_{jk}) is statistically Cauchy sequence.

Conversely, assume that x_{jk} is a statistically Cauchy sequence. By Theorem 3.4, we have $st_2 - \lim\left\|x_{jk}, z\right\| = \left\|L, z\right\|$ for each $z \in X$.

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