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Pointwise Approximation Theorems for Combinations of Bernstein Polynomials with Inner Singularities

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Abstract

It is well-known that Bernstein polynomials are very important in studying the characters of smoothness in theory of approximation. A new type of combinations of Bernstein operators are given in [1]. In this paper, we give the Bernstein-Markov inequalities with step-weight functions $\bar{w}(x)$ for combinations of Bernstein polynomials with inner singularities as well as direct and inverse theorems.

Keywords: Bernstein Polynomials, Inner Singularities, Pointwise Approximation, Bernstein-Markov Inequalities, Direct and Inverse Theorems

1. Introduction

The set of all continuous functions, defined on the interval I, is denoted by $C(I)$. For any $f \in C([0,1])$, the corresponding Bernstein operators are defined as follows:

$$B(f,x) := \sum_{k=0}^{n} f\left(\frac{k}{n}\right) p_{n,k}(x),$$

where

$$p_{n,k}(x) := \mathrm{C}_n^k x^k (1-x)^{n-k}, \quad k = 0,1,\cdots,n, \quad x \in [0,1].$$

Approximation properties of Bernstein operators have been studied very well (see [2-7], for example). In order to approximate the functions with singularities, Della Vecchia *et al*. [8] introduced some kinds of modified Bernstein operators. Throughout the paper, C denotes a positive constant independent of n and x, which may be different in different cases. Ditzian and Totik extended the method of combinations and defined the following combinations of Bernstein operators:

$$B_{n,r}(f,x) := \sum_{i=0}^{r-1} C_i(n) B_{n_i}(f,x),$$

with the conditions:

a) $n = n_0 < n_1 < \cdots n_{r-1} \le Cn$,

b) $\sum_{i=0}^{r-1} |C_i(n)| \le C$,

c) $\sum_{i=0}^{r-1} C_i(n) = 1$,

d) $\sum_{i=0}^{r-1} C_i(n) n_i^{-k} = 0$, for $k = 1,\cdots,r-1$.

For any positive integer r, we consider the determinant

$$A_r := \begin{vmatrix} 1 & 1 & 1 & \cdots & 1 \\ \frac{2r+1}{(2r)(2r+1)} & \frac{2r+2}{(2r+1)(2r+1)} & \frac{2r+3}{(2r+2)(2r+3)} & \cdots & \frac{4r+1}{(4r)(4r+1)} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 2\cdots(2r+1) & 3\cdots(2r+2) & 4\cdots(2r+3) & \cdots & (2r+2)\cdots(4r+1) \end{vmatrix}$$

We obtain $A_r = \prod_{j=2}^{2r} j! \ne 0$. Thus, there is a unique solution for the system of nonhomogeneous linear equations:

$$\begin{cases} a_1 + a_2 + \cdots + a_{2r+1} = 1, \\ (2r+1)a_1 + (2r+2)a_2 + \cdots + (4r+1)a_{2r+1} = 0, \\ (2r+1)(2r)a_1 + (2r+2)(2r+1)a_2 + \cdots + (4r)(4r+1)a_{2r+1} = 0, \\ \cdots \\ (2r+1)!a_1 + 3\cdots(2r+1)a_2 + \cdots + (2r+2)\cdots(4r+1)a_{2r+1} = 0. \end{cases} \tag{1.1}$$

Let

$$\psi(x) := \begin{cases} a_1 x^{2r+1} + a_2 x^{2r+2} + \cdots + a_{2r+1} x^{4r+1} = 1, & 0 < x < 1, \\ 0, & x < 0, \\ 1, & x = 1. \end{cases}$$

with the coefficients $a_1, a_2, \cdots, a_{2r+1}$ satisfying (1.1). From (1.1), we see that $\psi(x) \in C^{(2r)}_{(-\infty,+\infty)}$, $0 \le \psi(x) \le 1$ for $0 \le x \le 1$. Moreover, it holds that $\psi(1) = 1$, $\psi^{(i)}(0) = 0$, $i = 0,1,\cdots,2r$ and $\psi^{(i)}(1) = 0$, $i = 1,\cdots,2r$.

Let

$$H(f,x) := \sum_{i=1}^{r+1} f(x_i) l_i(x),$$

and

$$l_i(x) := \frac{\prod_{j=1,j\ne i}^{r+1} (x - x_j)}{\prod_{j=1,j\ne i}^{r+1} (x_i - x_j)},$$

$$x_i = \frac{\left[n\xi - ((r-1)/2 + i)\right]}{n}, \quad i = 1,2,\cdots,r+1.$$

Further, let

$$x_1' = \frac{\left[n\xi - 2\sqrt{n}\right]}{n}, \quad x_2' = \frac{\left[n\xi - \sqrt{n}\right]}{n},$$

$$x_3' = \frac{\left[n\xi + \sqrt{n}\right]}{n}, \quad x_4' = \frac{\left[n\xi + 2\sqrt{n}\right]}{n}$$

and

$$\bar{\psi}_1(x) = \psi\left(\frac{x - x_1'}{x_2' - x_1'}\right), \quad \bar{\psi}_2(x) = \psi\left(\frac{x - x_3'}{x_4' - x_3'}\right).$$

Set

$$\bar{F}_n(f,x) := \bar{F}(x) = f(x)\left(1 - \bar{\psi}_1(x) + \bar{\psi}_2(x)\right) + \bar{\psi}_1(x)\left(1 - \bar{\psi}_2(x)\right)H(x).$$

We have

$$\bar{F}_n(f,x) = \begin{cases} f(x), & x \in \left[0, x_{r-5/2}\right] \cup \left[x_{r-3/2}, 1\right], \\ f(x)\left(1 - \bar{\psi}_1(x)\right) + \bar{\psi}_1(x) H(x), & x \in \left[x_{r-5/2}, x_{r-3/2}\right], \\ H(x), & x \in \left[x_{r-3/2}, x_{r+1/2}\right], \\ H(x)\left(1 - \bar{\psi}_2(x)\right) + \bar{\psi}_2(x) f(x), & x \in \left[x_{r+1/2}, x_{r+3/2}\right]. \end{cases}$$

Obviously, $\bar{F}_n(f,x)$ is linear, reproduces polynomials of degree r, and $\bar{F}_n(f,x) \in C^{(2r)}([0,1])$, provided that $f \in C^{(2r)}([0,1])$. Now, we can define our new combinations of Bernstein operators as follows:

$$\bar{B}_{n,r}(f,x) = B_{n,r}(\bar{F}_n, x) = \sum_{i=0}^{r-1} C_i(n) B_{n_i}(\bar{F}_n, x) \tag{1.2}$$

where $C_i(n)$ satisfy the conditions (a)-(d).

2. The Main Results

Let $\phi:[0,1] \to R$, $\phi \ne 0$ be an admissible step-weight function of the Ditzian-Totik modulus of smoothness, that is, ϕ satisfies the following conditions:

1) For every proper subinterval $[a,b] \subseteq [0,1]$ there exists a constant $C_1 \equiv C(a,b) > 0$ such that $C_1^{-1} \le \phi(x) \le C_1$ for $x \in [a,b]$.

2) There are two numbers $\beta(0) \ge 0$ and $\beta(1) \ge 0$ for which

$$\phi(x) \sim \begin{cases} x^{\beta(0)}, & \text{as } x \to 0_+, \\ (1-x)^{\beta(1)}, & \text{as } x \to 1_-. \end{cases}$$

($X \sim Y$ $\Leftrightarrow$ means $C^{-1}Y \le X \le CY$ for some C).

Combining conditions (I) and (II) on ϕ, we can deduce that

$$C^{-1}\phi_2(x) \le \phi(x) \le C\phi_2(x), \quad x \in [0,1],$$

where $\phi_2(x) = x^{\beta(0)}(1-x)^{\beta(1)}$.

Let $\overline{w}(x)=|x-\xi|^{\alpha}$, $0<\xi<1$, $\alpha>0$ and

$$C_{\overline{w}}=\left\{f\in C([0,1]\setminus\{\xi\}):\lim_{x\to\xi}(\overline{w}f)(x)=0\right\}.$$

The norm in $C_{\overline{w}}$ is defined as

$\|f\|_{\overline{w}}:=\|\overline{w}f\|_{C_{\overline{w}}}=\sup_{0\le x\le 1}|(\overline{w}f)(x)|$. Define

$$W_{\phi}^{r}:=\left\{f\in C_{\overline{w}}:f^{(r-1)}\in AC((0,1)),\left\|\overline{w}\phi^{r}f^{(r)}\right\|<\infty\right\},$$

$$W_{\varphi,\lambda}^{r}:=\left\{f\in C_{\overline{w}}:f^{(r-1)}\in AC((0,1)),\left\|\overline{w}\varphi^{r\lambda}f^{(r)}\right\|<\infty\right\}.$$

For $f\in C_{\overline{w}}$, we define the weighted modulus of smoothness by

$$W_{\phi}^{r}(f,t)_{\overline{w}}:=\sup_{0<h\le t}\sup_{0\le x\le 1}\left|\overline{w}(x)\Delta_{h\phi(x)}^{r}f(x)\right|,$$

where

$$\Delta_{h\phi}^{r}f(x)=\sum_{k=0}^{r}(-1)^{k}C_{r}^{k}f\left(x+\left(\frac{r}{2}-k\right)h\phi(x)\right),$$

$$\Delta_{h}^{r}f(x)=\sum_{k=0}^{r}(-1)^{k}C_{r}^{k}f(x+(r-k)h).$$

Recently Felten showed the following two theorems in [4]:

Theorem A. Let $\varphi(x)=\sqrt{x(1-x)}$ and let $\phi:[0,1]\to R$, $\phi\ne 0$ be an admissible step-weight function of the Ditzian-Totik modulus of smoothness ([3]) such that ϕ^{2} and φ^{2}/ϕ^{2} are concave. Then, for $f\in C[0,1]$ and $0<\alpha<2$,

$$\left|B_{n}(f,x)-f(x)\right|\le C\omega_{\phi}^{2}\left(f,n^{-1/2}\frac{\varphi(x)}{\phi(x)}\right).$$

Theorem B. Let $\varphi(x)=\sqrt{x(1-x)}$ and let $\phi:[0,1]\to R$, $\phi\ne 0$ be an admissible step-weight function of the Ditzian-Totik modulus of smoothness such that ϕ^{2} and φ^{2}/ϕ^{2} are concave. Then, for $f\in C[0,1]$ and $0<\alpha<2$,

$$\left|B_{n}(f,x)-f(x)\right|\le C\left(\left(n^{-1/2}\frac{\varphi(x)}{\phi(x)}\right)^{\alpha}\right)$$

implies $\omega_{\phi}^{2}(f,t)=O(t^{\alpha})$.

Our main results are the following:

Theorem 2.1. For any $\alpha>0$, $\min\{\beta(0),\beta(1)\}\ge\frac{1}{2}$, $f\in C_{\overline{w}}$, we have

$$\left|\overline{w}(x)\phi^{r}(x)\overline{B}_{n,r-1}^{(r)}(f,x)\right|\le Cn^{\frac{r}{2}}\|\overline{w}f\|. \tag{2.1}$$

Theorem 2.2. For any $\alpha>0$, $f\in W_{\phi}^{r}$, we have

$$\left|\overline{w}(x)\phi^{r}(x)\overline{B}_{n,r-1}^{(r)}(f,x)\right|\le C\left\|\overline{w}\phi^{r}f^{(r)}\right\|. \tag{2.2}$$

Theorem 2.3. For $f\in C_{\overline{w}}$, $\alpha>0$, $\min\{\beta(0),\beta(1)\}\ge\frac{1}{2}$, $\alpha_{0}\in(0,r)$, $\delta_{n}(x)=\varphi(x)+\frac{1}{\sqrt{n}}$, we have

$$\begin{aligned}&\overline{w}(x)\left|f(x)-\overline{B}_{n,r-1}(f,x)\right|\\&=O\left(\left(\frac{\delta_{n}(x)}{n^{1/2}\phi(x)}\right)^{\alpha_{0}}\right)\Leftrightarrow\omega_{\phi}^{r}(f,t)_{\overline{w}}=O(t^{\alpha_{0}})\end{aligned} \tag{2.3}$$

3. Lemmas

Lemma 3.1. For any non-negative real *u* and *v*, we have

$$\sum_{k=1}^{n-1}p_{n,k}(x)\left(\frac{n}{k}\right)^{-u}\left(\frac{n}{n-k}\right)^{-v}\le Cx^{-u}(1-x)^{-v}. \tag{3.1}$$

Lemma 3.2. If $\gamma\in R$, then

$$\sum_{k=0}^{n}p_{n,k}(x)|k-nx|^{\gamma}\le Cn^{\frac{\gamma}{2}}\varphi^{\gamma}(x). \tag{3.2}$$

Lemma 3.3. For any $f\in W_{\phi}^{r}$, $\alpha>0$, we have

$$\left\|\overline{w}\phi^{r}\overline{F}_{n}^{(r)}\right\|\le C\left\|\overline{w}\phi^{r}f^{(r)}\right\|. \tag{3.3}$$

Proof. We first prove $x\in\left[x_{r-5/2},x_{r-3/2}\right]$ (The same as the others), we have

$$\begin{aligned}\left|\overline{w}(x)\phi^{r}(x)\overline{F}_{n}^{(r)}(x)\right|&\le\left|\overline{w}(x)\phi^{r}(x)f^{(r)}(x)\right|\\&+\left|\overline{w}(x)\phi^{r}(x)\left(f(x)-\overline{F}_{n}(x)\right)^{(r)}\right|:\le I_{1}+I_{2}\end{aligned}$$

Obviously

$$I_{1}\le C\left\|\overline{w}\phi^{r}f^{(r)}\right\|$$

For I_{2}, we have

$$\begin{aligned}I_{2}&=\left|\overline{w}(x)\phi^{r}(x)\left(f(x)-\overline{F}_{n}(x)\right)^{(r)}\right|\\&=\overline{w}(x)\phi^{r}(x)\sum_{i=0}^{r}n^{\frac{i}{2}}\left|\left(f(x)-\overline{F}_{n}(x)\right)^{(r-i)}\right|\end{aligned}$$

By [3], we have

$$\begin{aligned}&\left|\left(f(x)-\overline{F}_{n}(x)\right)^{(r-i)}\right|_{\left[x_{r-5/2},x_{r-3/2}\right]}\\&\le C\left(n^{\frac{r-i}{2}}\|f-H\|_{\left[x_{r-5/2},x_{r-3/2}\right]}+n^{-\frac{i}{2}}\left\|f^{(r)}\right\|_{\left[x_{r-5/2},x_{r-3/2}\right]}\right).\end{aligned}$$

So

$$I_2 \le Cn^{\frac{r}{2}}\bar{w}(x)\phi^r(x)\|f-H\|_{[x_{r-5/2},x_{r-3/2}]} + C\bar{w}(x)\phi^r(x)\left\|f^{(r)}\right\|_{[x_{r-5/2},x_{r-3/2}]} := T_1+T_2$$

By Taylor expansion, we have

$$f(x_i)=\sum_{u=0}^{r-1}\frac{(x_i-x)^u}{u!}f^{(u)}(x) + \frac{1}{(r-1)!}\int_x^{x_i}(x_i-s)^{r-1}f^{(r)}(s)\,\mathrm{d}s. \tag{3.4}$$

It follows from (3.4) and the identity

$$\sum_{i=1}^{r}x_i^v l_i(x)=Cx^v, \quad v=0,1,\cdots,r.$$

we have

$$\begin{aligned}H(f,x)&=\sum_{i=1}^{r}\sum_{u=0}^{r}\frac{(x_i-x)^u}{u!}f^{(u)}(x)l_i(x) + \frac{1}{(r-1)!}\sum_{i=1}^{r}l_i(x)\int_x^{x_i}(x_i-s)^{r-1}f^{(r)}(s)\,\mathrm{d}s\\ &=f(x)+C\sum_{u=1}^{r}f^{(u)}(x)\left(\sum_{v=0}^{u}C_u^v(-x)^{u-v}\sum_{i=1}^{r}x_i^v l_i(x)\right) + \frac{1}{(r-1)!}\sum_{i=1}^{r}l_i(x)\int_x^{x_i}(x_i-s)^{r-1}f^{(r)}(s)\,\mathrm{d}s,\end{aligned}$$

which implies that

$$\bar{w}(x)\phi^r(x)|f(x)-H(f,x)| = \frac{1}{(r-1)!}\bar{w}(x)\phi^r(x)\cdot\sum_{i=1}^{r}l_i(x)\int_x^{x_i}(x_i-s)^{r-1}f^{(r)}(s)\,\mathrm{d}s,$$

Since

$$|l_i(x)|\le C, \text{ for } x\in[x_{r-5/2},x_{r-3/2}], \quad i=1,\cdots,r.$$

It follows from

$$\frac{|x_i-s|^{r-1}}{\bar{w}(s)} \le \frac{|x_i-x|^{r-1}}{\bar{w}(x)},$$

s between x_i and x, then

$$\begin{aligned}&\bar{w}(x)\phi^r(x)|f(x)-H(f,x)|\\ &= C\bar{w}(x)\phi^r(x)\sum_{i=1}^{r}\int_x^{x_i}(x_i-s)^{r-1}\left|f^{(r)}(s)\right|\mathrm{d}s\\ &\le \frac{C}{n^{r/2}}\left\|\bar{w}\phi^r f^{(r)}\right\|.\end{aligned}$$

So

$$I_2 \le C\left\|\bar{w}\phi^r f^{(r)}\right\|.$$

Then, the lemma is proved.

According to methods of Lemma 3.3, we can easily get:

Lemma 3.4. If $f\in W_\phi^r$, $\alpha>0$, then

$$\bar{w}(x)|g(x)-H(g,x)| \le C\left(\frac{\delta_n(x)}{\sqrt{n}\phi(x)}\right)^r\left\|\bar{w}\phi^r g^{(r)}\right\| \tag{3.5}$$

Lemma 3.5. For any $\alpha>0$, $f\in C_{\bar{w}}$, we have

$$\left\|\bar{w}\bar{B}_{n,r-1}(f)\right\| \le C\|\bar{w}f\|. \tag{3.6}$$

Proof. By (1.2), we have

$$\begin{aligned}&\left|\bar{w}(x)\bar{B}_{n,r-1}(f,x)\right| = \left|\bar{w}(x)B_{n,r-1}(\bar{F}_n,x)\right|\\ &\le \bar{w}(x)\sum_{i=0}^{r-1}\sum_{k=1}^{n_i-1}C_i(n)\left|\bar{F}_n\left(\frac{k}{n_i}\right)\right|p_{n_i,k}(x)\\ &\quad+\bar{w}(x)\sum_{i=0}^{r-1}C_i(n)\left|\bar{F}_n(0)\right|p_{n_i,0}(x)\\ &\quad+\bar{w}(x)\sum_{i=0}^{r-1}C_i(n)\left|\bar{F}_n(1)\right|p_{n_i,n_i}(x)\\ &:= I_1+I_2+I_3.\end{aligned}$$

Now, the theorem can be proved easily.

Lemma 3.6. Let $\min\{\beta(0),\beta(1)\}\ge\frac{1}{2}$, then for $r\in N$, $0<t<\frac{1}{8r}$ and $\frac{rt}{2}<x<1-\frac{rt}{2}$, we have

$$\int_{-\frac{t}{2}}^{\frac{t}{2}}\cdots\int_{-\frac{t}{2}}^{\frac{t}{2}}\phi^{-r}\left(x+\sum_{k=1}^{r}u_k\right)\mathrm{d}u_1\cdots\mathrm{d}u_r \le Ct^r\phi^{-r}(x) \tag{3.7}$$

Lemma 3.7. Let $A_n(x) := \bar{w}(x)\sum_{|k-n\xi|\le\sqrt{n}}p_{n,k}(x)$, then $A_n(x)\le Cn^{-\alpha/2}$ for $0<\xi<1$ and $\alpha>0$.

Proof. If $|x-\xi|\le\frac{3}{\sqrt{n}}$, then the statement is trivial. Hence assume $0\le x\le\xi-\frac{3}{\sqrt{n}}$ (the case $\xi+\frac{3}{\sqrt{n}}\le x\le 1$ can be treated similarly). Then for a fixed x the maximum of $p_{n,k}(x)$ is attained for $k=k_n:=\left[n\xi-\sqrt{n}\right]$. By using Stirling's formula, we get

$$\begin{aligned}p_{n,k_n}(x) &\le C\frac{\left(\frac{n}{e}\right)^n\sqrt{n}x^{k_n}(1-x)^{n-k_n}}{\left(\frac{k_n}{e}\right)^{k_n}\sqrt{k_n}\left(\frac{n-k_n}{e}\right)^{n-k_n}\sqrt{n-k_n}}\\ &= \frac{C}{\sqrt{n}}\left(1-\frac{k_n-nx}{k_n}\right)^{k_n}\left(1+\frac{k_n-nx}{n-k_n}\right)^{n-k_n}.\end{aligned}$$

Now from the inequalities

$$k_n - nx = \left[n\xi - \sqrt{n}\right] - nx > n(\xi - x) - \sqrt{n} - 1 \geq \frac{1}{2}n(\xi - x),$$

and $1-u \leq e^{-u-\frac{1}{2}u^2}$, $1+u \leq e^u$, $u \geq 0$.

We have that the second inequality is valid. To prove the first one we consider the function $\lambda(u) = e^{-u-\frac{1}{2}u^2} + u - 1$. Here $\lambda(0) = 0$, $\lambda'(u) = -(1+u)e^{-u-\frac{1}{2}u^2} + 1$, $\lambda'(0) = 0$, $\lambda''(u) = u(u+2)e^{-u-\frac{1}{2}u^2} \geq 0$, whence $\lambda(u) \geq 0$ for $u \geq 0$.

Hence

$$p_{n,k_n}(x) \leq \frac{C}{\sqrt{n}} \exp\left\{k_n\left[-\frac{k_n - nx}{k_n} - \frac{1}{2}\left(\frac{k_n - nx}{k_n}\right)^2 + k_n - nx\right]\right\}$$

$$= \frac{C}{\sqrt{n}} \exp\left\{\frac{(k_n - nx)^2}{2k_n}\right\} \leq e^{-Cn(\xi - x)^2}.$$

Thus $A_n(x) \leq C(\xi - x)^\alpha e^{-Cn(\xi-x)^2}$. An easy calculation shows that here the maximum is attained when $\xi - x = \frac{C}{\sqrt{n}}$ and the lemma follows.

Lemma 3.8. For $0 < \xi < 1$, α, $\beta > 0$, we have

$$\overline{w}(x) \sum_{|k - n\xi| \leq \sqrt{n}} p_{n,k}(x) |k - nx|^\beta \leq Cn^{\frac{\beta - \alpha}{2}} \varphi^\beta(x) \qquad (3.8)$$

Proof. By (3.2) and the lemma 3.7, we have

$$\overline{w}(x)^{\frac{1}{2n}} \left(\overline{w}(x) \sum_{|k - n\xi| \leq \sqrt{n}} p_{n,k}(x)\right)^{\frac{2n-1}{2n}} \left(\sum_{|k - n\xi| \leq \sqrt{n}} p_{n,k}(x) \cdot |k - nx|^{2n\beta}\right)^{\frac{1}{2n}} \leq Cn^{\frac{\beta - \alpha}{2}} \varphi^\beta(x).$$

Lemma 3.9. For any $\alpha > 0$, $f \in W_\phi^r$, we have

$$\left\|\overline{w}\overline{B}_{n,r-1}^{(r)}(f)\right\| \leq Cn^r \left\|\overline{w}f\right\|. \qquad (3.9)$$

Proof. We first prove $x \in \left[0, \frac{1}{n}\right)$ (The same as $x \in \left(1 - \frac{1}{n}, 1\right]$, now

$$\left|\overline{w}(x)\overline{B}_{n,r-1}^{(r)}(f,x)\right|$$

$$\leq \overline{w}(x) \sum_{i=0}^{r-2} \frac{n_i!}{(n_i - r)!} \sum_{k=0}^{n_i - r} C_i(n) \left|\vec{\Delta}_{\frac{1}{n_i}}^r \overline{F}_n\left(\frac{k}{n_i}\right)\right| p_{n_i - r,k}(x)$$

$$\leq C\overline{w}(x) \sum_{i=0}^{r-2} n_i^r \sum_{k=0}^{n_i - r} \sum_{j=0}^{r} C_r^j \left|\overline{F}_n\left(\frac{k + r - j}{n_i}\right)\right| p_{n_i - r,k}(x)$$

$$\leq C\overline{w}(x) \sum_{i=0}^{r-2} n_i^r \sum_{j=0}^{r} C_r^j \left|\overline{F}_n\left(\frac{r - j}{n_i}\right)\right| p_{n_i - r,0}(x)$$

$$+ C\overline{w}(x) \sum_{i=0}^{r-2} n_i^r \sum_{j=0}^{r} C_r^j \left|\overline{F}_n\left(\frac{n_i - j}{n_i}\right)\right| p_{n_i - r,n_i - r}(x)$$

$$+ C\overline{w}(x) \sum_{i=0}^{r-2} n_i^r \sum_{k=1}^{n_i - r - 1} \sum_{j=0}^{r} C_r^j \left|\overline{F}_n\left(\frac{k + r - j}{n_i}\right)\right| p_{n_i - r,k}(x)$$

$$:= H_1 + H_2 + H_3.$$

We have

$$H_1 \leq C\overline{w}(x) \sum_{i=0}^{r-2} n_i^r \sum_{j=0}^{r} \left|\overline{F}_n\left(\frac{r - j}{n_i}\right)\right| p_{n_i - r,0}(x)$$

$$\leq Cn^r \left\|\overline{w}f\right\| \sum_{i=0}^{r-2} \sum_{j=0}^{r} \left(\frac{n_i|x - \xi|}{r - j - n_i\xi}\right)^\alpha (1 - x)^{n_i - r}$$

$$\leq Cn^r \left\|\overline{w}f\right\|.$$

Similarly, we can get $H_2 \leq Cn^r \left\|\overline{w}f\right\|$, and $H_3 \leq Cn^r \left\|\overline{w}f\right\|$.

When $x \in \left[\frac{1}{n}, 1 - \frac{1}{n}\right]$, according to [3], we have

$$\left|\overline{w}(x)\overline{B}_{n,r-1}^{(r)}(f,x)\right| = \left|\overline{w}(x)B_{n,r-1}^{(r)}(\overline{F}_n, x)\right|$$

$$\leq \overline{w}(x)\left(\varphi^2(x)\right)^{-r} \sum_{i=0}^{r-2} \sum_{j=0}^{r} Q_j(x, n_i) C_i(n) n_i^j \cdot \sum_{k/n_i \in A} \left|\left(x - \frac{k}{n_i}\right)^j \overline{F}_n\left(\frac{k}{n_i}\right)\right| p_{n_i,k}(x)$$

$$+ \overline{w}(x)\left(\varphi^2(x)\right)^{-r} \sum_{i=0}^{r-2} \sum_{j=0}^{r} Q_j(x, n_i) C_i(n) n_i^j \cdot \sum_{x_2' \leq k/n_i \leq x_3'} \left|\left(x - \frac{k}{n_i}\right)^j H\left(\frac{k}{n_i}\right)\right| p_{n_i,k}(x) := \sigma_1 + \sigma_2$$

where $A := [0, x_2'] \cup [x_3', 1]$, H is a linear function. If $\frac{k}{n_i} \in A$, when $\frac{\overline{w}(x)}{\overline{w}(k/n_i)} \leq C\left(1 + n_i^{-\frac{\alpha}{2}} |k - n_i x|^\alpha\right)$, we have $|k - n_i\xi| \geq \frac{\sqrt{n_i}}{2}$, also $Q_j(x, n_i) = (n_i x(1 - x))^{\left[\frac{r-j}{2}\right]}$, and $\left(\varphi^2(x)\right)^{-r} Q_j(x, n_i) n_i^j \leq C\left(n_i / \varphi^2(x)\right)^{\frac{r+j}{2}}$. By (3.2), then

$$\sigma_1 \le C\bar{w}(x)\sum_{i=0}^{r-2}\sum_{j=0}^{r}\left(\frac{n_i}{\varphi^2(x)}\right)^{\frac{r+j}{2}}\sum_{k=0}^{n_i}\left|\left(x-\frac{k}{n_i}\right)^j \bar{F}_n\left(\frac{k}{n_i}\right)\right|p_{n_i,k}(x)$$

$$\le C\|\bar{w}f\|\sum_{i=0}^{r-2}\sum_{j=0}^{r}\left(\frac{n_i}{\varphi^2(x)}\right)^{\frac{r+j}{2}}\sum_{k=0}^{n_i}\left[1+n_i^{-\frac{\alpha}{2}}|k-n_ix|^{\alpha}\right]\left|x-\frac{k}{n_i}\right|^j p_{n_i,k}(x) := I_1+I_2.$$

By a simple calculation, we have $I_1 \le Cn^r\|\bar{w}f\|$. By (3.2), then

$$I_2 \le C\|\bar{w}f\|\sum_{i=0}^{r-2}\sum_{j=0}^{r}n_i^{-\left(\frac{\alpha}{2}+j\right)}\left(\frac{n_i}{\varphi^2(x)}\right)^{\frac{r+j}{2}}\sum_{k=0}^{n_i}|k-n_ix|^{\alpha+j}p_{n_i,k}(x) \le Cn^r\|\bar{w}f\|.$$

We note that

$$\left|H\left(\frac{k}{n_i}\right)\right| \le \max\left(|H(x_1')|,|H(x_4')|\right) := H(a).$$

if $x\in[x_1',x_4']$,we have $\bar{w}(x)\le\bar{w}(a)$ .So, if $x\in[x_1',x_4']$, then $\sigma_2 \le Cn^r\bar{w}(a)H(a)\le Cn^r\|\bar{w}f\|$.

If $x\notin[x_1',x_4']$, then $\bar{w}(a) > n_i^{-\frac{\alpha}{2}}$, by (3.8), we have

$$\sigma_2 \le C\bar{w}(a)H(a)\bar{w}(x)\sum_{i=0}^{r-2}\sum_{j=0}^{r}n_i^{\frac{\alpha}{2}}\left(\frac{n_i}{\varphi^2(x)}\right)^{\frac{r+j}{2}}$$

$$\cdot\sum_{x_2'\le k/n_i\le x_3'}\left|x-\frac{k}{n_i}\right|^j p_{n_i,k}(x) \le Cn^r\|\bar{w}f\|.$$

It follows from combining the above inequalities that the lemma is proved.

4. Proof of Theorems

4.1. Proof of Theorem 2.1

When $f\in C_{\bar{w}}$, $\min\{\beta(0),\beta(1)\}\ge 1/2$, we discuss it as follows:

Case 1. If $0\le\varphi(x)\le\frac{1}{\sqrt{n}}$, by (3.9), we have

$$\left|\bar{w}(x)\phi^r(x)\bar{B}_{n,r-1}^{(r)}(f,x)\right|$$
$$= C\varphi^r(x)\cdot\frac{\phi^r(x)}{\varphi^r(x)}\left|\bar{w}(x)\bar{B}_{n,r-1}^{(r)}(f,x)\right| \le Cn^{\frac{r}{2}}\|\bar{w}f\| \quad (4.1)$$

Case 2. If $\varphi(x)>\frac{1}{\sqrt{n}}$, we have

$$\bar{B}_{n,r-1}^{(r)}(f,x) = B_{n,r-1}^{(r)}(\bar{F}_n,x)$$
$$\le \left(\varphi^2(x)\right)^{-r}\sum_{i=0}^{r-2}\sum_{j=0}^{r}Q_j(x,n_i)C_i(n)n_i^j.$$

$$\sum_{k=0}^{n_i}\left|\left(x-\frac{k}{n_i}\right)^j\bar{F}_n\left(\frac{k}{n_i}\right)\right|p_{n_i,k}(x),$$

where

$$Q_j(x,n_i) = \left(n_ix(1-x)\right)^{\left[\frac{r-j}{2}\right]}, \text{ and}$$

$$\left(\varphi^2(x)\right)^{-r}Q_j(x,n_i)n_i^j \le C\left(n_i/\varphi^2(x)\right)^{\frac{r+j}{2}}$$

So

$$\left|\bar{w}(x)\phi^r(x)\bar{B}_{n,r-1}^{(r)}(f,x)\right| \le C\bar{w}(x)\phi^r(x)$$
$$\cdot\sum_{i=0}^{r-2}\sum_{j=0}^{r}\left(\frac{n_i}{\varphi^2(x)}\right)^{\frac{r+j}{2}}\sum_{k=0}^{n_i}\left|\left(x-\frac{k}{n_i}\right)^j\bar{F}_n\left(\frac{k}{n_i}\right)\right|p_{n_i,k}(x)$$
$$= C\bar{w}(x)\phi^r(x)\sum_{i=0}^{r-2}\sum_{j=0}^{r}\left(\frac{n_i}{\varphi^2(x)}\right)^{\frac{r+j}{2}}\sum_{k/n_i\in A}\left|\left(x-\frac{k}{n_i}\right)^j\bar{F}_n\left(\frac{k}{n_i}\right)\right|$$
$$\cdot p_{n_i,k}(x) + C\bar{w}(x)\phi^r(x)\sum_{i=0}^{r-2}\sum_{j=0}^{r}\left(\frac{n_i}{\varphi^2(x)}\right)^{\frac{r+j}{2}}$$
$$\cdot\sum_{x_2'\le k/n_i\le x_3'}\left|\left(x-\frac{k}{n_i}\right)^j H\left(\frac{k}{n_i}\right)\right|p_{n_i,k}(x) := \sigma_1+\sigma_2. \quad (4.2)$$

where $A := [0,x_2']\cup[x_3',1]$, we can easily get $\sigma_1 \le Cn^{\frac{r}{2}}\|\bar{w}f\|$, and $\sigma_2 \le Cn^{\frac{r}{2}}\|\bar{w}f\|$. By bringing these facts together, the theorem is proved.

4.2. Proof of Theorem 2.2

When $f\in W_\phi^r$, by [3], we have

$$B_{n,r-1}^{(r)}(\bar{F}_n,x) = \sum_{i=0}^{r-2}C_i(n)n_i^r\sum_{k=0}^{n_i-r}\Delta_{\frac{1}{n_i}}^r\bar{F}_n\left(\frac{k}{n_i}\right)p_{n_i-r,k}(x). \quad (4.3)$$

If $0<k<n_i-r$, we have

$$\left|\Delta^r_{\frac{1}{n_i}}\bar{F}_n\left(\frac{k}{n_i}\right)\right|\le Cn_i^{-r+1}\int_0^{\frac{r}{n_i}}\left|\bar{F}_n^{(r)}\left(\frac{k}{n_i}+u\right)\right|\mathrm{d}u\,. \quad (4.4)$$

If $k=0$, we have

$$\left|\Delta^r_{\frac{1}{n_i}}\bar{F}_n(0)\right|\le C\int_0^{\frac{r}{n_i}}u^{r-1}\left|F_n^{(r)}(u)\right|\mathrm{d}u\,. \quad (4.5)$$

Similarly

$$\left|\Delta^r_{\frac{1}{n_i}}\bar{F}_n\left(\frac{n_i-r}{n_i}\right)\right|\le Cn_i^{-r+1}\int_{1-\frac{r}{n_i}}^{1}(1-u)^{\frac{r}{2}}\left|\bar{F}_n^{(r)}(u)\right|\mathrm{d}u. \quad (4.6)$$

By (4.3), we have

$$\begin{aligned}&\left|\bar{w}(x)\phi^r(x)\bar{B}^{(r)}_{n,r-1}(f,x)\right|\\ &\le C\bar{w}(x)\phi^r(x)\sum_{i=0}^{r-2}n_i^r\sum_{k=0}^{n_i-r}\left|\vec{\Delta}^r_{\frac{1}{n_i}}\bar{F}_n\left(\frac{k}{n_i}\right)\right|p_{n_i-r,k}(x)\\ &\le C\bar{w}(x)\phi^r(x)\sum_{i=0}^{r-2}n_i^r\sum_{k=1}^{n_i-r-1}\left|\vec{\Delta}^r_{\frac{1}{n_i}}\bar{F}_n\left(\frac{k}{n_i}\right)\right|p_{n_i-r,k}(x) \quad (4.7)\\ &+C\bar{w}(x)\phi^r(x)\sum_{i=0}^{r-2}n_i^r\left|\vec{\Delta}^r_{\frac{1}{n_i}}\bar{F}_n(0)\right|p_{n_i-r,0}(x)\\ &+C\bar{w}(x)\phi^r(x)\sum_{i=0}^{r-2}n_i^r\left|\vec{\Delta}^r_{\frac{1}{n_i}}\bar{F}_n(1)\right|p_{n_i-r,n_i-r}(x).\end{aligned}$$

which combining with (4.4)-(4.6) give

$$\left|\bar{w}(x)\phi^r(x)\bar{B}^{(r)}_{n,r-1}(f,x)\right|\le C\left\|\bar{w}\phi^r f^{(r)}\right\|.$$

Combining with the theorem 2.1 and theorem 2.2, we can obtain

Corollary For any $\alpha>0$, $0\le\lambda\le1$, we have

$$\left|\bar{w}(x)\varphi^{r\lambda}(x)\bar{B}^{(r)}_{n,r-1}(f,x)\right|\le \begin{cases}Cn^{r/2}\left\{\max\left\{n^{r(1-\lambda)/2},\varphi^{r(\lambda-1)}(x)\right\}\right\}\left\|\bar{w}f\right\|, & f\in C_{\bar{w}}, \quad (4.8)\\ C\left\|\bar{w}\varphi^{r\lambda}f^{(r)}\right\|, & f\in W^r_{\bar{w},\lambda}.\end{cases}$$

4.3. Proof of Theorem 2.3

4.3.1. The Direct Theorem

We know

$$\bar{F}_n(t)=\bar{F}_n(x)+\bar{F}_n'(t-x)+\cdots+\frac{1}{(r-1)!}\int_x^t(t-u)^{r-1}\bar{F}_n^{(r)}(u)\mathrm{d}u \quad (4.9)$$

$$B_{n,r-1}\left((\cdot-x)^k,x\right)=0,\quad k=1,2,\cdots,r-1 \quad (4.10)$$

According to the definition of W^r_ϕ, for any $g\in W^r_\phi$, we have $\bar{B}^{(r)}_{n,r-1}(g,x)=B^{(r)}_{n,r-1}\left(\bar{G}_n(g),x\right)$, and

$$\bar{w}(x)\left|\bar{G}_n(x)-B_{n,r-1}\left(\bar{G}_n,x\right)\right|=\bar{w}(x)\left|B_{n,r-1}\left(R_r\left(\bar{G}_n,t,x\right),x\right)\right|,$$

there of $R_r\left(\bar{G}_n,t,x\right)=\int_x^t(t-u)^{r-1}\bar{G}_n^{(r)}(u)\mathrm{d}u$, we have

$$\begin{aligned}&\bar{w}(x)\left|\bar{G}_n(x)-B_{n,r-1}\left(\bar{G}_n,x\right)\right|\\ &\le C\left\|\bar{w}\phi^r\bar{G}_n^{(r)}\right\|\bar{w}(x)B_{n,r-1}\left(\int_x^t\frac{|t-u|^{r-1}}{\bar{w}(u)\phi^r(u)}du,x\right)\\ &\le C\left\|\bar{w}\phi^{r\lambda}\bar{G}_n^{(r)}\right\|\bar{w}(x)\left(B_{n,r-1}\left(\int_x^t\frac{|t-u|^{r-1}}{\phi^{2r}(u)}\mathrm{d}u,x\right)\right)^{\frac{1}{2}} \quad (4.11)\\ &\cdot\left(B_{n,r-1}\left(\int_x^t\frac{|t-u|^{r-1}}{\bar{w}^2(u)}\mathrm{d}u,x\right)\right)^{\frac{1}{2}}.\end{aligned}$$

also

$$\int_x^t\frac{|t-u|^{r-1}}{\phi^{2r}(u)}\mathrm{d}u\le C\frac{|t-x|^r}{\phi^{2r}(x)},\ \int_x^t\frac{|t-u|^{r-1}}{\bar{w}^2(u)}\mathrm{d}u\le C\frac{|t-x|^r}{\bar{w}^2(x)}. \quad (4.12)$$

By (3.2), (3.3) and (4.12), we have

$$\begin{aligned}&\bar{w}(x)\left|\bar{G}_n(x)-B_{n,r-1}\left(\bar{G}_n,x\right)\right|\\ &\le C\left\|\bar{w}\phi^r\bar{G}_n^{(r)}\right\|\phi^{-r}(x)B_{n,r-1}\left(|t-x|^r,x\right). \quad (4.13)\\ &\le C\left(\frac{\delta_n(x)}{\sqrt{n}\phi(x)}\right)^r\left\|\bar{w}\phi^r\bar{G}_n^{(r)}\right\|\end{aligned}$$

By (3.3), (3.5) and (4.13), when $g\in W^r_\phi$, then

$$\begin{aligned}&\bar{w}(x)\left|g(x)-\bar{B}_{n,r-1}(g,x)\right|\le\bar{w}(x)\left|g(x)-\bar{G}_n(g,x)\right|\\ &+\bar{w}(x)\left|\bar{G}_n(g,x)-\bar{B}_{n,r-1}(g,x)\right|\le\bar{w}(x)\left|g(x)-H(g,x)\right|\\ &+C\left(\frac{\delta_n(x)}{\sqrt{n}\phi(x)}\right)^r\left\|\bar{w}\phi^r\bar{G}_n^{(r)}\right\|\le C\left(\frac{\delta_n(x)}{\sqrt{n}\phi(x)}\right)^r\left\|\bar{w}\phi^r g^{(r)}\right\|.\end{aligned} \quad (4.14)$$

For $f\in C_{\bar{w}}$, we choose proper $g\in W^r_\phi$, by (3.6) and (4.14), then

$$\begin{aligned}&\bar{w}(x)\left|f(x)-\bar{B}_{n,r-1}(f,x)\right|\le\bar{w}(x)\left|f(x)-g(x)\right|\\ &+\bar{w}(x)\left|\bar{B}_{n,r-1}(f-g,x)\right|+\bar{w}(x)\left|g(x)-\bar{B}_{n,r-1}(g,x)\right|\\ &\le C\omega^r_\phi\left(f,\frac{\delta_n(x)}{\sqrt{n}\phi(x)}\right)_{\bar{w}}.\end{aligned}$$

4.3.2. The Inverse Theorem

The weighted *K*-function is given by

$$K_{r,\phi}\left(f,t^r\right)_{\bar{w}}=\inf_g\left\{\left\|\bar{w}(f-g)\right\|+t^r\left\|\bar{w}\phi^r g^{(r)}\right\|:g\in W_\phi^r\right\}.$$

By [3], we have

$$C^{-1}\omega_\phi^r(f,t)_{\bar{w}}\le K_{r,\phi}\left(f,t^r\right)_{\bar{w}}\le C\omega_\phi^r(f,t)_{\bar{w}}. \quad (4.15)$$

Proof. Let $\delta>0$, by (4.15), we choose proper g so that

$$\left\|\bar{w}(f-g)\right\|\le C\omega_\phi^r(f,\delta)_{\bar{w}},\quad \left\|\bar{w}\phi^r g^{(r)}\right\|\le C\delta^{-r}\omega_\phi^r(f,\delta)_{\bar{w}}. \quad (4.16)$$

For $r\in N$, $0<t<\frac{1}{8r}$ and $\frac{rt}{2}<x<1-\frac{rt}{2}$, we have

$$\begin{aligned}\left|\bar{w}(x)\Delta_{h\phi}^r f(x)\right|&\le\left|\bar{w}(x)\Delta_{h\phi}^r\left(f(x)-\bar{B}_{n,r-1}(f,x)\right)\right|+\left|\bar{w}(x)\Delta_{h\phi}^r\bar{B}_{n,r-1}(f-g,x)\right|+\left|\bar{w}(x)\Delta_{h\phi}^r\bar{B}_{n,r-1}(g,x)\right|\\&\le\sum_{j=0}^r C_r^j\left(n^{-\frac{1}{2}}\frac{\delta_n\left(x+\left(\frac{r}{2}-j\right)h\phi(x)\right)}{\phi\left(x+\left(\frac{r}{2}-j\right)h\phi(x)\right)}\right)^{\alpha_0}+\int_{-\frac{h\phi(x)}{2}}^{\frac{h\phi(x)}{2}}\cdots\int_{-\frac{h\phi(x)}{2}}^{\frac{h\phi(x)}{2}}\bar{w}(x)\bar{B}_{n,r-1}^{(r)}\left(f-g,x+\sum_{k=1}^r u_k\right)\mathrm{d}u_1\cdots\mathrm{d}u_r \quad (4.17)\\&\quad+\int_{-\frac{h\phi(x)}{2}}^{\frac{h\phi(x)}{2}}\cdots\int_{-\frac{h\phi(x)}{2}}^{\frac{h\phi(x)}{2}}\bar{w}(x)\bar{B}_{n,r-1}^{(r)}\left(g,x+\sum_{k=1}^r u_k\right)\mathrm{d}u_1\cdots\mathrm{d}u_r:=J_1+J_2+J_3\end{aligned}$$

Obviously

$$J_1\le C\left(\frac{\delta_n(x)}{n^{1/2}\phi(x)}\right)^{\alpha_0}. \quad (4.18)$$

By (3.9) and (4.16), we have

$$J_2\le Cn^r\left\|\bar{w}(f-g)\right\|\int_{-\frac{h\phi(x)}{2}}^{\frac{h\phi(x)}{2}}\cdots\int_{-\frac{h\phi(x)}{2}}^{\frac{h\phi(x)}{2}}\mathrm{d}u_1\cdots\mathrm{d}u_r\le Cn^r h^r\phi^r(x)\omega_\phi^r(f,\delta)_{\bar{w}} \quad (4.19)$$

By the first inequality of (4.8) and (4.16), we let $\lambda=1$, then

$$J_2\le Cn^{\frac{r}{2}}\left\|\bar{w}(f-g)\right\|\int_{-\frac{h\phi(x)}{2}}^{\frac{h\phi(x)}{2}}\cdots\int_{-\frac{h\phi(x)}{2}}^{\frac{h\phi(x)}{2}}\varphi^{-r}\left(x+\sum_{k=1}^r u_k\right)\mathrm{d}u_1\cdots\mathrm{d}u_r\le Cn^{\frac{r}{2}}h^r\phi^r(x)\varphi^{-r}(x)\omega_\phi^r(f,\delta)_{\bar{w}}. \quad (4.20)$$

By (3.7) and (4.16), we have

$$J_3\le C\left\|\bar{w}\phi^r g^{(r)}\right\|\bar{w}(x)\int_{-\frac{h\phi(x)}{2}}^{\frac{h\phi(x)}{2}}\cdots\int_{-\frac{h\phi(x)}{2}}^{\frac{h\phi(x)}{2}}\bar{w}^{-1}\left(x+\sum_{k=1}^r u_k\right)\phi^{-r}\left(x+\sum_{k=1}^r u_k\right)\mathrm{d}u_1\cdots\mathrm{d}u_r\le Ch^r\delta^{-r}\omega_\phi^r(f,\delta)_{\bar{w}} \quad (4.21)$$

Now, by (4.17)-(4.21), there exists a constant $M>0$ so that

$$\begin{aligned}\left|\bar{w}(x)\Delta_{h\phi}^r f(x)\right|&\le C\left(\left(\frac{\delta_n(x)}{n^{1/2}\phi(x)}\right)^{\alpha_0}+\min\left\{n^{\frac{r}{2}}\frac{\phi^r(x)}{\varphi^r(x)},n^r\phi^r(x)\right\}h^r\omega_\phi^r(f,\delta)_{\bar{w}}+h^r\delta^{-r}\omega_\phi^r(f,\delta)_{\bar{w}}\right)\\&\le C\left(\left(\frac{\delta_n(x)}{n^{1/2}\phi(x)}\right)^{\alpha_0}+h^rM^r\left(n^{-\frac{1}{2}}\frac{\delta_n(x)}{\phi(x)}\right)^{-r}\omega_\phi^r(f,\delta)_{\bar{w}}+h^r\delta^{-r}\omega_\phi^r(f,\delta)_{\bar{w}}\right).\end{aligned}$$

When $n\ge 2$, we have

$$n^{-\frac{1}{2}}\delta_n(x)<(n-1)^{-\frac{1}{2}}\delta_{n-1}(x)\le\sqrt{2}n^{-\frac{1}{2}}\delta_n(x),$$

Choosing proper x, δ, $n\in N$, so that

$$n^{-\frac{1}{2}}\frac{\delta_n(x)}{\phi(x)}\le\delta<(n-1)^{-\frac{1}{2}}\frac{\delta_{n-1}(x)}{\phi(x)},$$

Therefore

$$\left|\bar{w}(x)\Delta_{h\phi}^r f(x)\right|\le C\left\{\delta^{\alpha_0}+h^r\delta^{-r}\omega_\phi^r(f,\delta)_{\bar{w}}\right\}$$

which implies

$$\omega_\phi^r(f,t)_{\bar{w}}\le C\left\{\delta^{\alpha_0}+h^r\delta^{-r}\omega_\phi^r(f,\delta)_{\bar{w}}\right\}.$$

So, by Berens-Lorentz lemma in [3], we get

$$\omega_{\phi}^{r}\left(f,t\right)_{\bar{w}}\leq Ct^{\alpha_0}\ .$$

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