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Semi-Markovian Model of Control of Restorable System with Latent Failures

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Abstract

Mathematical model of control of restorable system with latent failures has been built. Failures are assumed to be detected after control execution only. Stationary characteristics of system operation reliability and efficiency have been defined. The problem of control execution periodicity optimization has been solved. The model of control has been built by means of apparatus of semi-Markovian processes with a discrete-continuous field of states.

Keywords: Control, Latent Failure, Semi-Markovian Process, System Stationary Characteristics

1. Introduction

An important factor providing reliability, high quality, and efficiency of modern technological complexes is the presence of control systems in them. The review of the results concerning model building of control systems can be found in [1,2]. Some mathematical models of control systems are represented in [3,4].

In the present article the model of control execution and restoration of a single-unit system with latent failures has been investigated. The latent failure is the one that does not show up till the control is executed. Defective goods are produced up to the failure detection.

The problems of technological complexes' control are closely connected with their maintenance. In the work [5] maintenance models were built by means of semi-Markovian processes with a common phase field of states [6]. In the present article this apparatus is used to build the model of control under the condition of latent failures occurrence.

In the second section of the article the system operation is described, its semi-Markovian model is built. Besides, stationary distribution of embedded Markovian chain is given. In the third section main stationary characteristics of the system operation reliability and efficiency are defined. These are: mean stationary operation time, mean stationary restoration time, availability function, mean income and expenses per time unit. In the fourth section the problem of control execution periodicity optimization is solved.

2. The Problem Definition and Mathematical Model Building

Let us investigate the system operating in the following way. At the time zero the system begins operating, and the control is on. System failure-free operation time is a random value (RV) α with distribution function (DF) $F(t)=P\{\alpha\le t\}$ and distribution density (DD) $f(t)$. The control is executed in random time δ with DF $R(t)=P\{\delta\le t\}$ and DD $r(t)$. The failure is detected only when control is carried out. Control duration is RV γ with DF $V(t)=P\{\gamma\le t\}$ and DD $v(t)$. After failure detection system restoration begins immediately and the control is deactivated. System restoration time is RV β with DF $G(t)=P\{\beta\le t\}$ and DD $g(t)$. After the system restoration all its properties are completely restored. All the RV are supposed to be independent, have finite assembly averages and variances. Time diagram of the system operation and the system transition graph are shown in **Figure 1** and **Figure 2** respectively.

The purpose of the present article is to find stationary reliability and economical characteristics of the single-unit restorable system with regard to control under the condition of latent failures occurrence, and to define control execution optimal periodicity.

To describe the system operation let us use semi-Markovian process $\xi(t)$ with the following field of states:

$$E=\{111,\ 211x,\ 210x,\ 101x,\ 200,\ 100x,\ 201\}.$$

The meaning of state codes is the following:

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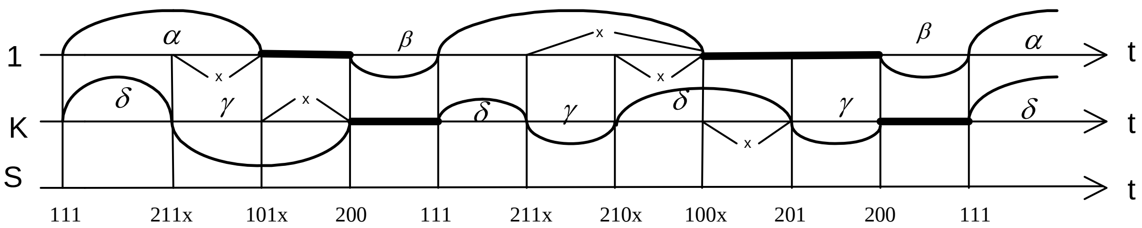

Figure 1. Time diagram of the system operation.

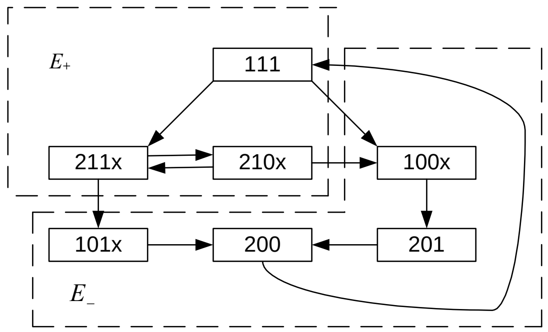

Figure 2. System transition graph.

111—the system begins operating, the control is activated;

211 x—control has begun, the system is in up state, time x is left till the latent failure;

210 x—control has ended, the system is in up state, time x is left till the latent failure;

101 x—latent failure has occurred, control is executed, time x is left till the failure detection;

200—the failure has been detected, control has been deactivated, the system restoration has begun;

100 x—the system has failed, time x is left till the beginning of control;

201—the system is in down state, control has begun.

Let us define the probabilities and probability densities of the embedded Markovian chain (EMC) $\{\xi_n,\ n \ge 0\}$ transitions:

$$p_{111}^{211y} = \int_0^\infty f(y+t)r(t)dt\ ;\ p_{111}^{100y} = \int_0^\infty r(y+t)f(t)dt\ ;$$

$$p_{210x}^{211y} = r(x-y),\quad 0<y<x\ ;$$

$$p_{210x}^{100y} = r(y+x),\quad y>0\ ;\quad P_{101x}^{200} = P_{100x}^{201} = P_{201}^{200} = P_{200}^{111} = 1;$$

$$p_{211x}^{101y} = v(y+x),\quad y>0\ ;\quad p_{211x}^{210y} = v(x-y),\quad 0<y<x\ . \tag{1}$$

Let us indicate $\rho(111)$, $\rho(200)$, $\rho(201)$ the values of EMC $\{\xi_n,\ n\ge 0\}$ stationary distribution for the states 111, 200, 201 and assume the existence of stationary densities $\rho(211x), \rho(210x), \rho(101x)$, $\rho(100x)$ for the states 211x, 210 x, 101 x, 100 x respectively. The system of integral equations for them is the following:

$$\begin{cases}
\rho_0 = \rho(111) = \rho(200), \\
\rho(211x) = \rho(111)\int_0^\infty f(x+t)r(t)dt + \int_x^\infty \rho(210y)r(y-x)dy, \\
\rho(101x) = \int_0^\infty \rho(211y)v(x+y)dy, \\
\rho(210x) = \int_x^\infty \rho(211y)v(y-x)dy, \\
\rho(200) = \int_0^\infty \rho(101y)dy + \rho(201), \\
\rho(100x) = \rho(111)\int_0^\infty r(x+t)f(t)dt + \int_0^\infty \rho(210y)r(y+x)dy, \\
\rho(201) = \int_0^\infty \rho(100y)dy, \\
2\rho_0 + \int_0^\infty \rho(100x)dx + \int_0^\infty \rho(211x)dx + \int_0^\infty \rho(101x)dx \\
\quad + \int_0^\infty \rho(210x)dx + \rho(201) = 1
\end{cases} \tag{2}$$

The last equation of the system (2) is a normalization requirement.

With the help of method of successive approximations one can prove that the solution of the system of Equations (2) is:

$$\begin{cases}
\rho(111) = \rho(200) = \rho_0, \\
\rho(211x) = \rho_0 \int_x^\infty h_0(t-x)f(t)dt, \\
\rho(101x) = \rho_0 \int_0^\infty f(t)v^{(0)}(t,x)dt, \\
\rho(210x) = \rho_0 \int_x^\infty h_1(t-x)f(t)dt, \\
\rho(100x) = \rho_0 \int_0^\infty f(t)v^{(1)}(t,x)dt, \\
\rho(201) = \rho_0 \int_0^\infty \left[\hat{H}_1(t) - H_0(t)\right] f(t)dt.
\end{cases} \tag{3}$$

Here the constant ρ_0 is found from the normalization requirement; $h_0(t)=\sum_{n=1}^{\infty} r*(v*r)^{*(n-1)}(t)$ is the density of 0th restoration function $H_0(t)=\sum_{n=1}^{\infty} R*(V*R)^{*(n-1)}(t)$ of alternating process generated by RV δ and γ; $h_1(t)=\sum_{n=1}^{\infty}(r*v)^{*(n)}(t)$ is the density of the 1st restoration function $H_1(t)=\sum_{n=1}^{\infty}(R*V)^{*(n)}(t)$ of the same alternating process, $\hat{H}_1(t)=1+H_1(t)$;

$v^{(0)}(t,x)=\int_0^t v(x+t-y)h_0(y)dy$ is the density of residual time of control;

$v^{(1)}(t,x)=r(t+x)+\int_0^t r(x+t-y)h_1(y)dy$ is the density of the direct residual time left till the beginning of control [3].

3. Definition of System Stationary Characteristics

Let us define system stationary characteristics: mean stationary operation time T_+, mean stationary restoration time T_-, stationary availability function $K_г$.

For the initial system the sets of up states E_+. and down states E_- are as following:

$E_+=\{111,\ 211x,\ 210x\}$, $E_-=\{101x,\ 200,\ 100x,\ 201\}$.

Mean stationary operation time T_+ and mean stationary restoration time T_- can be found with the help of formulas [6]:

$$T_+=\frac{\int_{E_+} m(e)\rho(de)}{\int_{E_+} P(e,E_-)\rho(de)},\quad T_-=\frac{\int_{E_-} m(e)\rho(de)}{\int_{E_+} P(e,E_-)\rho(de)} \tag{4}$$

where $\rho(de)$ is the EMC $\{\xi_n, n\ge 0\}$ stationary distribution; $m(e)$ are Mean values of system dwelling times in its states; $P(e,E_-)$ are the probabilities of EMC $\{\xi_n, n\ge 0\}$ transitions from up into down states.

Mean values of system dwelling times in the states are:

$$m(111)=\int_0^{\infty}\bar{F}(t)\bar{R}(t)dt;$$

$$m(211x)=\int_0^{x}\bar{V}(t)dt;\quad m(101x)=x;$$

$$m(201)=M\gamma;\quad m(100x)=x;$$

$$m(200)=M\beta;\quad m(210x)=\int_0^{x}\bar{R}(t)dt \tag{5}$$

Taking into account the Formulas (1), (3) and (5) we can define the expressions included in (4):

$$\begin{aligned}\int_{E_+} m(e)\rho(de)&=m(111)\rho(111)+\int_0^{\infty}m(211x)\rho(211x)dx+\int_0^{\infty}m(210x)\rho(210x)dx\\&=\rho_0\int_0^{\infty}\bar{F}(t)\bar{R}(t)dt+\rho_0\int_0^{\infty}dx\int_0^{x}\bar{R}(t)dt\int_x^{\infty}h_1(y-x)f(y)dy+\rho_0\int_0^{\infty}dx\int_0^{x}\bar{V}(t)dt\int_x^{\infty}f(y)h_0(y-x)dy.\end{aligned} \tag{6}$$

Transforming the right part of ratio (6) with regard to the formula

$$\int_0^{y}\bar{R}(t)+\int_0^{y}h_0(y-x)dx\int_0^{x}\bar{V}(t)dt+\int_0^{y}h_1(y-x)dx\int_0^{x}\bar{R}(t)dt=y$$

we get that

$$\int_{E_+} m(e)\rho(de)=\rho_0 M\alpha.$$

Hereafter

$$\begin{aligned}\int_{E_-} m(e)\rho(de)&=m(200)\rho(200)+m(201)\rho(201)+\int_0^{\infty}m(100x)\rho(100x)dx+\int_0^{\infty}m(101x)\rho(101x)dx\\&=\rho_0M\beta+\rho_0M\gamma\int_0^{\infty}\left[\hat{H}_1(t)-H_0(t)\right]f(t)dt+\rho_0\int_0^{\infty}f(t)dt\int_0^{\infty}x\left[v^{(0)}(t,x)+v^{(1)}(t,x)\right]dx\\&=\rho_0M\beta+\rho_0M\gamma\int_0^{\infty}\left[\hat{H}_1(t)-H_0(t)\right]f(t)dt+\rho_0\int_0^{\infty}f(t)dt\int_0^{\infty}\left[\bar{V}^{(0)}(t,y)+\bar{V}^{(1)}(t,y)\right]dy\\&=\rho_0M\beta+\rho_0M\gamma\int_0^{\infty}\left[\hat{H}_1(t)-H_0(t)\right]f(t)dt+\rho_0M\gamma\int_0^{\infty}f(t)H_0(t)dt+\rho_0M\delta\int_0^{\infty}f(t)\hat{H}_1(t)dt-\rho_0M\alpha\\&=\rho_0M\beta+\rho_0(M\gamma+M\delta)\int_0^{\infty}f(t)\hat{H}_1(t)dt-\rho_0M\alpha.\end{aligned} \tag{7}$$

The identity

$$\int_0^\infty \left[\overline{V}^{(0)}(t,y) + \overline{V}^{(1)}(t,y) \right] dy = M\gamma H_0(t) + M\delta \hat{H}_1(t) - t$$

has been used while transforming the expression (7).

Then

$$\int_E (m)(e)\rho(de) = \rho_0 \left((M\gamma + M\delta) \int_0^\infty f(z) \hat{H}_1(z) dz + \mathrm{M}\beta \right). \quad (8)$$

Hereafter

$$\int_{E_+} P(e, E_-)\rho(de) = \rho(111) P(111, E_-) + \int_0^\infty \rho(211x) P(211x, E_-) dx + \int_0^\infty \rho(210x) P(210x, E_-) dx$$
$$= \rho_0 \int_0^\infty \overline{R}(t) f(t) dt + \int_0^\infty \overline{V}(x) \rho(211x) dx + \int_0^\infty \overline{R}(x) \rho(210x) dx = \rho_0.$$

Thus, mean stationary operation time T_+ and mean stationary restoration time T_- are defined with the help of formulas

$$T_+ = M\alpha,$$

$$T_- = M\beta - M\alpha + (M\delta + M\gamma) \int_0^\infty f(t) \hat{H}_1(t) dt .$$

Stationary availability function is defined by the ratio $K_г = \frac{T_+}{T_+ + T_-}$. We get

$$K_г = \frac{M\alpha}{M\beta + (M\delta + M\gamma) \int_0^\infty f(t) \hat{H}_1(t) dt}. \quad (9)$$

It is necessary to note the probability essence of the functional in the Formula (9): $\int_0^\infty f(t) \hat{H}_1(t) dt$ is an average value of controls executed before the latent failure detection.

Important characteristics for system operation quality testing are economical criteria, such as mean income S per unit of calendar time and mean expenses C per time unit of system's up state. To define them let us use the formulas [7]:

$$S = \frac{\int_E m(e) f_s(e) \rho(de)}{\int_E m(e) \rho(de)}, \quad C = \frac{\int_E m(e) f_c(e) \rho(de)}{\int_{E_+} m(e) \rho(de)}. \quad (10)$$

Here $f_s(e)$, $f_c(e)$ are the functions defining income and expenses in each state respectively.

Let c_1 be the income received per time unit of system's up state; c_2–expenses per time unit of restoration; c_3–expenses per time unit of control; c_4 are wastes caused by defective goods per time unit of latent failure. For the given system the functions $f_s(e)$, $f_c(e)$ are the following:

$$f_s(e) = \begin{cases} c_1,\ e \in \{111,\ 210x\}, \\ c_1 - c_3,\ e \in \{211x\}, \\ -c_2,\ e = 220, \\ -c_3 - c_4,\ e \in \{101x,\ 201\}, \\ -c_4, e \in \{100x\}, \end{cases} \quad f_c(e) = \begin{cases} 0,\ e \in \{111,\ 210x\}, \\ c_3,\ e \in \{211x\}, \\ c_2,\ e = 220, \\ c_3 + c_4,\ e \in \{101x,\ 201\}, \\ c_4, e \in \{100x\}. \end{cases} \quad (11)$$

Using Formulas (3), (5), (8) and (11) we will define the functionals included into the expressions (10):

$$\int_E m(e) f_s(e) \rho(de) = c_1 \rho(111) m(111) + (c_1 - c_3) \int_0^\infty \rho(211x) m(211x) dx + c_1 \int_0^\infty \rho(210x) m(210x) dx$$
$$- c_2 \rho(200) m(200) - (c_3 + c_4) \rho(201) M\gamma - (c_3 + c_4) \int_0^\infty x \rho(101x) dx - c_4 \int_0^\infty x \rho(100x) dx$$
$$= c_1 \rho_0 M\alpha - c_2 \rho_0 M\beta - c_4 \rho_0 \left[(M\gamma + M\delta) \int_0^\infty f(t) \hat{H}_1(t) dt - M\alpha \right] \quad (12)$$
$$- c_3 \rho_0 \left[M\gamma \int_0^\infty \left[\hat{H}_1(t) - H_0(t) \right] f(t) dt + \int_0^\infty f(t) dt \int_0^t h_0(t-x) dx \int_0^x \overline{V}(t) dt + \int_0^\infty f(t) dt \int_0^\infty \overline{V}^{(0)}(t,x) dx \right]$$
$$= \rho_0 \left[(c_1 + c_4) M\alpha - c_2 M\beta - (c_3 M\gamma + c_4 (M\gamma + M\delta)) \int_0^\infty f(t) \hat{H}_1(t) dt \right]$$

$$\int_E m(e) f_c(e) \rho(de) = c_2 \rho(200) M\beta + c_3 \int_0^\infty \rho(211x) m(211x)\, dx + c_3 \rho(201) M\gamma$$

$$+ c_3 \int_0^\infty x\rho(101x)\, dx + c_4 \int_0^\infty x\rho(101x)\, dx + c_4 \int_0^\infty x\rho(100x)\, dx + c_4 \rho(201) M\gamma$$

$$= c_4 \rho_0 \left[(M\gamma + M\delta) \int_0^\infty f(t) \hat{H}_1(t)\, dt - M\alpha \right] + c_2 \rho_0 M\beta \tag{13}$$

$$+ c_3 \rho_0 \left[\int_0^\infty f(t)\, dt \int_0^t h_0(t-x)\, dx \int_0^x \bar{V}(t)\, dt + \int_0^\infty f(t)\, dt \int_0^\infty \bar{V}^{(0)}(t,x)\, dx + M\gamma \int_0^\infty \left[\hat{H}_1(t) - H_0(t) \right] f(t)\, dt \right]$$

$$= \rho_0 \left[c_2 M\beta - c_4 M\alpha + \left[(c_3 + c_4) M\gamma + c_4 M\delta \right] \int_0^\infty f(t) \hat{H}_1(t)\, dt \right].$$

When transforming the ratios (12), (13), the identity

$$\int_0^\infty f(t)\, dt \int_0^t h_0(t-x)\, dx \int_0^x \bar{V}(t)\, dt + \int_0^\infty f(t)\, dt \int_0^\infty \bar{V}^{(0)}(t,x)\, dx = M\gamma \int_0^\infty f(t) H_0(t)\, dt$$

was taken into account. Consequently, income per calendar time unit can be calculated by means of the formula:

$$S = \frac{(c_1 + c_4) M\alpha - c_2 M\beta - \left(c_3 M\gamma + c_4 (M\gamma + M\delta) \right) \int_0^\infty f(t) \hat{H}_1(t)\, dt}{M\beta + (M\delta + M\gamma) \int_0^\infty f(t) \hat{H}_1(t)\, dt} \tag{14}$$

Expenses per time unit of system's good state are defined by the formula:

$$C = c_2 \frac{M\beta}{M\alpha} + \left[(c_3 + c_4) \frac{M\gamma}{M\alpha} + c_4 \frac{M\delta}{M\alpha} \right] \int_0^\infty f(t) \hat{H}_1(t)\, dt - c_4. \tag{15}$$

Let us write down the formulas for the definition of stationary reliability and economical characteristics of the system investigated under the condition that time periods between control execution are non-random values $\tau > 0$. Taking into account that in this case $R(t) = 1(t-\tau)$, where $\tau = const$, the ratio (9) transforms into:

$$K_г = \frac{M\alpha}{M\beta + (\tau + M\gamma)\left(1 + \sum_{n=1}^\infty \int_0^\infty \bar{F}(z + n\tau) v^{*(n)} dz \right)}. \tag{16}$$

Under the assumption that the control duration is non-random as well: $V(t) = 1(t-h)$, where $h = const$. Then Formulas (9), (14) and (15) look like this:

$$K_г = \frac{M\alpha}{M\beta + (\tau + h) \sum_{n=0}^\infty \bar{F}(n(\tau + h))}, \tag{17}$$

$$S = \frac{(c_1 + c_4) M\alpha - c_2 M\beta - \left((c_3 + c_4) h + c_4 \tau \right) \sum_{n=0}^\infty \bar{F}(n(\tau + h))}{M\beta + (\tau + h) \sum_{n=0}^\infty \bar{F}(n(\tau + h))} \tag{18}$$

$$C = c_2 \frac{M\beta}{M\alpha} + \left[(c_3 + c_4) \frac{h}{M\alpha} + c_4 \frac{\tau}{M\alpha} \right] \sum_{n=0}^\infty \bar{F}(n(\tau + h)) - c_4 \tag{19}$$

One should note that if $h = 0$ and $\tau \to 0$ we get characteristics for the system with continuous control [6]:

$$K_г = \frac{M\alpha}{M\alpha + M\beta}, \quad S = \frac{c_1 M\alpha - c_2 M\beta}{M\alpha + M\beta}, \quad C = c_2 \frac{M\alpha}{M\beta}. \tag{20}$$

Table 1. Optimal control execution period definition.

Distribution laws of random values	Initial data			Results			
	$M\alpha$, h	$M\beta$, h	h, h	τ_{opt}^{s}, h	$S\left(\tau_{opt}^{s}\right)$, c.u./h	τ_{opt}^{c}, h	$C\left(\tau_{opt}^{c}\right)$, c.u./h
Exponential	70	0,2	0,5	3,525	4,477	5,334	0,356
Erlangian of the 4th order	70	0,2	0,5	3,563	4,479	5,416	0,354
Erlangian of the 8th order	70	0,2	0,5	3,563	4,479	5,416	0,354

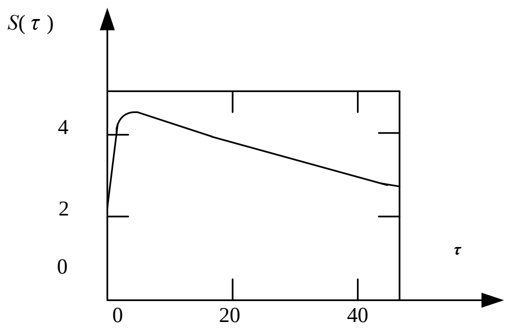

Figure 3. Graph of mean income $S(\tau)$ against control periodicity τ.

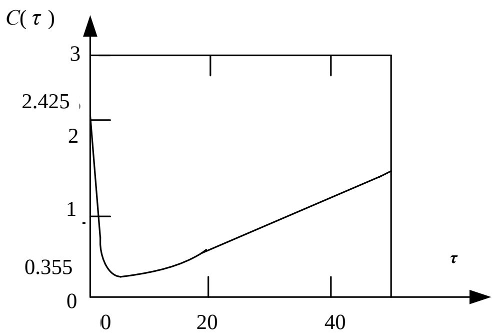

Figure 4. Graph of mean expenses $C(\tau)$ against control periodicity τ.

4. Optimization of Control Execution Periodicity

The problem of control execution periodicity optimization is reduced to analysis of extremums of the system characteristics $K_{г}$, S, C as functions of a single variable τ. Using Formulas (17)-(19) one can find an optimal period of control of the system investigated for different distribution laws of random values. The initial data for calculations of optimal values of control periodicity are: mean time of failure-free operation $M\alpha$, mean restoration time $M\beta$, control duration h. Let us suppose RV α and β to have Erlangian distribution. For the calculation of optimal value τ_{opt}^{s} providing maximal mean income $S\left(\tau_{opt}^{s}\right)$ per calendar time unit and of optimal value τ_{opt}^{c} providing minimal mean expenses $C\left(\tau_{opt}^{c}\right)$ per time unit of system's good state the following initial data have been taken: $c_1 = 5$ c.u./h; $c_2 = 3$ c.u./h; $c_3 = 2$ c.u./h; $c_4 = 4$ c.u./h. The results of these calculations are represented in the **Table 1**. The graphs of functions $S(\tau), C(\tau)$ for the case of Erlangian distribution of the 8th order are shown in **Figures 3** and **4**.

5. Conclusions

Using an apparatus of semi-Markovian processes with a common phase field it is possible to define reliability and economical stationary performance indexes of restorable system, the latent failures of which can be detected while control execution only. It allows solving the problems of control execution periodicity optimization for gaining best system economical indexes.

Later on it is planned to use the method suggested in the present article to build and investigate mathematical models of multicomponent automatized systems and of different kinds of control.

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