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On Embedding of *m*-Sequential *k*-ary Trees into Hypercubes*

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Abstract

In this paper, we present an algorithm for embedding an *m*-sequential *k*-ary tree into its optimal hypercube with dilation at most 2 and prove its correctness.

Keywords: Hypercube, Embedding, Dilation, Pre-order Labeling, Hamiltonian Cycle, *k*-ary Tree

1. Introduction

Let *G* and *H* be finite graphs with *n* vertices. $V(G)$ and $V(H)$ denote the vertex sets of *G* and *H*, respectively. $E(G)$ and $E(H)$ denote the edge sets of *G* and *H*, respectively. An embedding f of *G* into *H* is defined [1] as follows:

1) *f* is a bijective map from $V(G) \to V(H)$

2) *f* is a one-to-one map from $E(G)$ to $\{P_f(f(u), f(v)): P_f(f(u), f(v))$ is a path in *H* between $f(u)$ and $f(v)\}$.

The *dilation* of an embedding *f* of *G* into *H* is given by

$$dil(f) = \max\left\{\left|P_f(f(u), f(v))\right| : (u,v) \in E(G)\right\}$$

where $\left|P_f(f(u), f(v))\right|$ denotes the length of the path P_f. Then, the dilation of *G* into *H* is defined as

$$dil(G,H) = \min dil(f)$$

where the minimum is taken over all embeddings *f* of *G* into *H*. Embedding *G* into *H* with minimum dilation is important for network design and for the simulation of one computer architecture by another [2].

Embeddings as mathematical models of parallel computing have been discussed extensively in the literature [3,4]. In these models, both the algorithm to be implemented and the interconnection network of the parallel computing system are represented by graphs. The implementation details are then studied through the embedding.

A tree is a connected graph that contains no cycles. Trees are the most fundamental graph-theoretic models used in many fields: information theory, automatics classification, data structure and analysis, artificial intelligence, design of algorithms, operation research, combinatorial optimization, theory of electrical networks and design of network [5]. Trees are ubiquitous in computer science. A rooted tree represents a data structure with a hierarchical relationship among its various elements [6].

The most common type of tree is the binary tree. It is so named because each node can have at most two descendents. Binary trees are widely used in data structures because they are easily stored, easily manipulated and easily retrieved [5]. A *k*-ary tree is a rooted tree in which each node has no more than *k* children. It is also known as a *k*-way tree.

From a computing perspective, trees form an important class of computational structures. Many operations such as searching and storing can be easily performed on tree data structures. Hence, there is a large literature on embeddings of various kinds of trees into the graphs of interconnection networks [3,7-25]. In particular, embeddings of binary trees into hypercubes have received special attention since they naturally arise as the computational structures of algorithms that employ dvide-and-conquer paradigm [12].

Hypercubes are very popular models for paralled computation because of their regularity and the relatively small number of interprocessor connections. The hypercube embedding problem is the problem of mapping a communication graph into a hypercube multiprocessor. Hypercubes are known to be able to simulate other structures such as grids and binary trees [3,26]. It has been shown that an arbitrary binary tree can be embed-

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ded into a hypercube with constant dilation [26].

In 1984, Havel [9,27] conjectured that a binary tree can be embedded into a *k*-dimensional hypercube Q^k with dilation one if and only if each of its partite sets contains at most 2^{k-1} vertices. In 1985, Bhatt and Ipsen [7] conjectured that a binary tree can be embedded into its optimal hypercube with dilation at most two as well as into its next-to-optimal hypercube with dilation one. As observed in [13], the conjecture of Havel is stronger than those of Bhatt and Ipsen. Though all these problems have been resolved in several special cases, they still remain open for the general case.

Monien and Sudborough [14] proved that every binary tree can be embedded into a hypercube with dilation 3 and $O(1)$ expansion. Chen and Stallmann [26] proved that a simple linear-time heuristic embeds an arbitrary binary tree into a hypercube with expansion 1 and average dilation no more than 2. Dvořák [15] constructed the embedding of certain classes of binary trees into hypercubes based on an iterative embedding into their subgraphs induced by dense sets. Heun and Mayr [16] proved that arbitrary binary trees can be embedded into hypercubes with dialtion 8. Further, they constructed an embedding of double-rooted complete binary trees into hypercubes [17]. Sunitha [4] constructed an embedding of some hierarchical caterpillars into their optimal hypercube with dilation 2. Choudum *et al*. [28] proved that subclasses of height-balanced trees can be embedded into hypercubes. Further, they proved that the height-balanced trees and Fibonacci trees can be embedded into hypercubes [6]. Eğecioğlu and Ibel [18] proved that the dynamic *k*-ary tree can be embedded into its asymptotic hypercube.

Thus there is a vast literature on embedding trees into hypercubes. In this paper, we define an *m*-sequential *k*-ary tree and prove that it can be embedded into its optimal hypercube with dilation at most 2.

2. Preliminaries

In this section, we introduce the labeling ***hal*** to label the vertices of the hypercube architecture. We obtain results that enable us to prove the main result of this paper.

Definition 1 [5] *An r-dimensional hypercube* Q^r *has nodes respesented by all the binary r-tuples with two nodes being adjacent if and only if their corresponding r-tuples differ in exactly one position.*

The decimal representation of the vertices is the set $\{0,1,2,\cdots,2^n-1\}$. For convenience, we use the symbol $x+1$ instead of *x* and therefore the set of labels of the vertices is $\{1,2,\cdots,2^n\}$.

Remark 1 *Let* G *be graph with* m *vertices. The hypercube of dimension* $\lceil \log_2(m) \rceil$ *is called its optimal hypercube, and one of dimension* $\lceil \log_2(m) \rceil + 1$ *is called next-to-optimal.*

Definition 2 *Let* Q^r *be an r-dimensional hypercube. A partial ordering "*$\le$*" on* Q^r *is defined by* $Q^i \le Q^j$ *if and only if* Q^i *is a subcube of* Q^j, $1 \le i, j \le r$. *The notation* $Q^i < Q^j$ *shall mean* $Q^i \le Q^j$ *and* $Q^i \ne Q^j$.

Definition 3 *A hamiltonian labeling of hypercube* Q^r, *denoted by **hal**, is the labeling of the vertices of* Q^r *defined inductively as follows: Consider the ordering* $Q^1 < Q^2 < \cdots < Q^i < \cdots < Q^r$ *on* Q^r. *Label vertices of* Q^1 *as* 0 *and* 1. *If* $u \in V(Q^i)$ *is labeled* x, *then label the unique vertex* $v \in \left(V\left(Q^{i+1}\right) \setminus V\left(Q^i\right)\right)$ *adjacent to* u *as* $2^{i+1}+1-x$.

Remark 2 *The labeling hal is the decimal equivalent of the binary gray code sequence* [26].

Definition 4 *A hamiltonian cycle is a cycle that visits each node of the graph exactly once. By convention, the trivial graph on a single node is considered to possess a hamiltonian cycle. A graph possessing a hamiltonian cycle is said to be a hamiltonian graph.*

Theorem 1 *The hamiltonian labeling hal of* Q^r *determines a hamiltonian cycle* $C_r = \left(1,2,\cdots,2^r,1\right)$ *in* Q^r, $r \ge 2$.

Proof We prove that *hal* determines the hamiltonian cycle $\left(1,2,\cdots,2^l,1\right)$ in Q^l for all l, $2 \le l \le r$. We prove the result by induction on r. For $r = 2$, $C_2 = (1,2,3,4,1)$ is a hamiltonian cycle in Q^2. Assume that $C_{k-1} = \left(1,2,3,\cdots,2^{k-1},1\right)$ is a hamiltonian cycle in Q^{k-1}. Consider Q^k. We observe that Q^{k-1} is a subcube of Q^k. Let F^{k-1} denote the other $(k-1)$-dimensional subcube contained in Q^k. By definition of Q^k, (u,v) is an edge in Q^{k-1} if and only if $\left(u',v'\right)$ is an edge in F^{k-1}, where $\left(u,u'\right)$ and $\left(v,v'\right)$ are edges in Q^k. Thus $P = \left(1,2,3,\cdots,2^{k-1}\right)$ is a hamiltonian path in Q^{k-1} implying that

$$P' = \left(2^k+1\right)-1, \left(2^k+1\right)-2, \cdots, \left(2^k+1\right)-2^{k-1}$$
$$= \left(2^k, 2^k-1, \cdots, 2^{k-1}+1\right)$$

is a hamiltonian path in F^{k-1}. Moreover, the vertex labeled 2^{k-1} is adjacent to the vertex labeled $\left(2^k+1\right)-2^{k-1} = 2^{k-1}+1$. Again the vertex labeled 1 is adjacent to the vertex labeled $\left(2^k+1\right)-1 = 2^k$. Thus $P \circ \left(2^{k-1}, 2^{k-1}+1\right) \circ \left(P'\right)^{-1} \circ \left(2^k,1\right) = \left(1,2,\cdots,2^{k-1},2^{k-1}+1,\cdots,2^k,1\right)$ is a hamiltonian cycle in Q^k.

The following results are easy consequences of Theorem 1 and the *hal* labeling of vertices of the hypercube.

Lemma 1 *Let* x *and* y *be the hal labeling of vertices* u *and* v *in* Q^r. *If* $x-y=2^m$ *for some m,* $1 \le m \le r$, *then* $d(u,v)=2$ *in* Q^r.

Proof Without loss of generality let $x > y$. Now $x-y=2^m \Rightarrow x-y=2^{m+1}-2^m = \left(2^{m+1}+1-z\right)-\left(2^m+1-z\right)$, where z is the label of some vertex w in Q^r. Thus $\{x = 2^{m+1} + 1 - z, y = 2^m + 1 - z\}$ is a solution to the equation $x - y = 2^m$. This implies that w is adjacent to both u and v. In other words, $d(u,v) = 2$ in Q^r.

Lemma 2 *Let x and y be the hal labeling of vertices u and v in* Q^r. *If* $y - x = 2^m - 2x + 1$ *for some m,* $1 \le m \le r$, *then* $d(u,v) = 1$ *in* Q^r.

Lemma 3 *Let* 1 *and y be the hal labeling of vertices u and v in* Q^r. *If* $y - 1 = 2^{m+i} - 2^m$ *for some m and i,* $1 \le m, i \le r$, *then* $d(u,v) = 2$ *in* Q^r.

Proof It is straightforward to see that $y - 1 = 2^{m+i} - 2^m$ implies $y = 2^{m+i} - z + 1$, where $z = 2^m$ is the label of some vertex w in Q^r. Thus, $\{y = 2^{m+i} + 1 - z, z = 2^m\}$ is a solution to the equation $y - 1 = 2^{m+i} - 2^m$. This implies that w is adjacent to both u and v. In other words, $d(u,v) = 2$ in Q^r.

3. Embedding Algorithm

Embeddings with dilation more than one become significant when trees with branch lengths representing time are considered. In this section, we obtain an embedding algorithm and prove its correctness.

Definition 5 *A rooted tree T is said to be a* ***step-up tree*** *if for every internal node u of T, any two subtrees* T_i *and* T_j *rooted at children of u are ordered in T in such a way that if* $|V(T_i)| \le |V(T_j)|$, *then* T_i *lies to the left of* T_j. *See* ***Figure*** **1**.

Definition 6 *For* $m \ge 2$ *and* $k \ge 1$, *an* ***m-sequential k-ary tree T(m,k)*** *is a step-up tree obtained by recursively growing the root u with k children under the following conditions.*

(*a*) The subtrees rooted at the children of u are of sequential order $(2^m - 2, 2^m, 2^{m+1}, 2^{m+2}, \cdots, 2^{m+k-2})$ or $(2^m - 1, 2^{m+1}, 2^{m+2}, \cdots, 2^{m+k-1})$.

(*b*) The root of a subtree of order $2^s - t$, $0 \le t \le 2$, $s \ge 3$ has $s - 1$ children which in turn are the roots of subtrees of sequential order $(3, 2^2, 2^3, \cdots, 2^{s-2}, 2^{s-1} - t)$ except when $s = 3, t = 2$ and in this case the sequential order is (2, 3).

(*c*) A subtree of order 3 or 4 with v as a root node is isomorphic to one of the graphs shown in **Figure 2**.

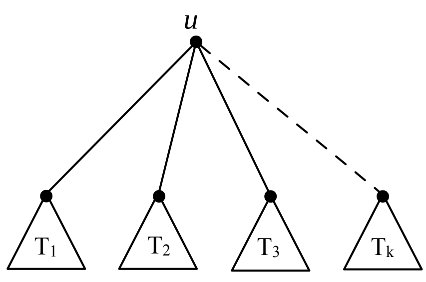

Figure 1. Step-up tree *T* with $|\mathbf{V}(\mathbf{T}_1)| \le |\mathbf{V}(\mathbf{T}_2)| \le \cdots \le |\mathbf{V}(\mathbf{T}_k)|$.

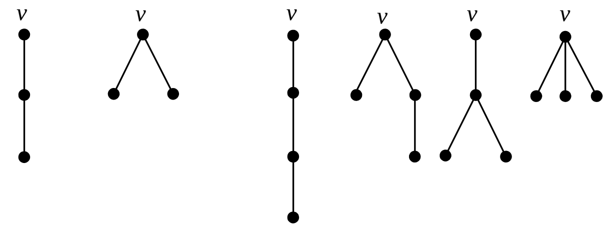

Figure 2. A subtree of order 3 or 4 with *v* as a root node.

Remark 3 $T(m,k)$ *rooted at u is said to be of Type A or Type B depending whether the subtrees rooted at the children of u are of sequential order* $(2^m - 2, 2^m, 2^{m+1}, 2^{m+2}, \cdots, 2^{m+k-2})$ *or* $(2^m - 1, 2^{m+1}, 2^{m+2}, \cdots, 2^{m+k-1})$, *respectively. See* ***Figure*** **3**.

Remark 4 $T(m,k)$ *of Type A has* $2^{m+k-3} - 1$ *nodes and* $T(m,k)$ *of Type B has* $2^m(2^k - 1)$ *nodes.*

In the sequel, we need the following definition of pre-order tree traversal.

Definition 7 *Let T be a rooted tree. A pre-order labeling of nodes of T is the labeling of its nodes the first time we visit the nodes in the traversal. See* ***Figure*** **4**.

Embedding Algorithm

Input: $T(m,k)$ and its optimal hypercube Q^r.

Algorithm: Label the nodes of T using pre-order labeling and the vertices of the hypercube using *hal* labeling.

Output: Embedding of T into Q^r with dilation at most 2. See **Figure 5**.

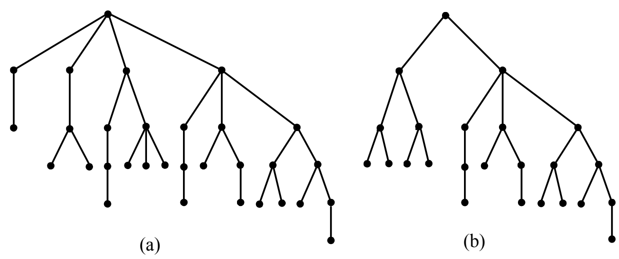

Figure 3. (a) *T* (2, 4) of Type A; (b) *T* (3, 2) of Type B.

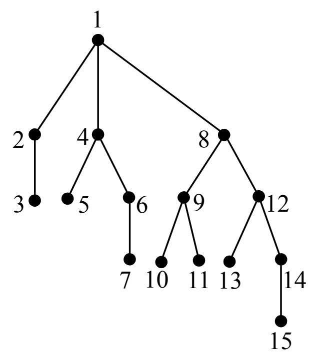

Figure 4. A pre-order labeling of nodes of *T*.

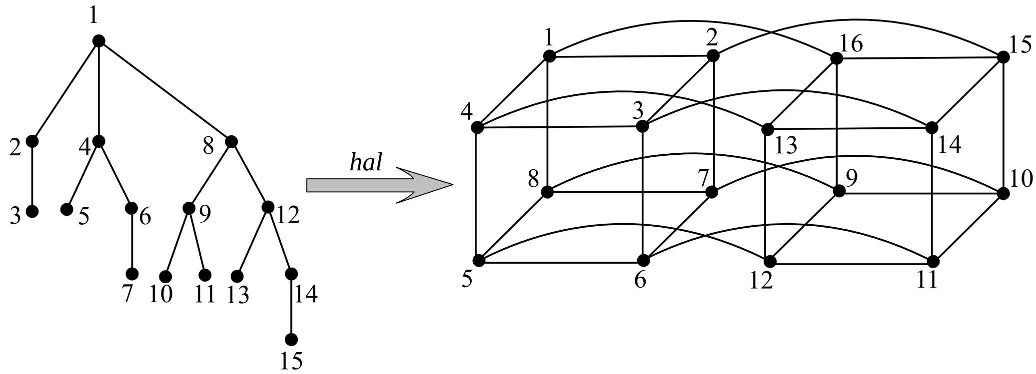

Figure 5. Embedding of $T(2, 3)$ of Type A into Q^4.

Theorem 2 *An m-sequential k-ary tree can be embedded into its optimal hypercube with dilation at most* 2.

Proof Consider an embedding f of T into Q^r using the embedding algorithm, labeling the nodes of T using pre-order labeling and the vertices of the hypercube using *hal* labeling such that $f(x) = x$.

Let v be an internal node of T with children $v_1, v_2, \cdots, v_p$. Let $T_1, T_2, \cdots, T_p$ be the subtrees rooted at $v_1, v_2, \cdots, v_p$, respectively. Since T is a step-up tree, we have $|V(T_1)| \leq |V(T_2)| \leq \cdots \leq |V(T_p)|$. Suppose that the root of T is u. Then the label of u is 1.

Case 1 (v is a child of u):

Let $e_i = (1, y_i)$, $1 \leq i \leq k$.

By definition of pre-order traversal,

$$y_i = |V(T_1)| + |V(T_2)| + \cdots + |V(T_{i-1})| + 2.$$

By definition of m-sequential k-ary tree of Type A,

$$\begin{aligned}&|V(T_1)| + |V(T_2)| + \cdots + |V(T_{i-1})| \\ &= (2^m - 2) + 2^m + 2^{m+1} + 2^{m+2} + \cdots + 2^{m+(i-3)} \\ &= 2^m[1 + 2 + \cdots + 2^{i-3}] - 2 = 2^{m+i-4} - 2.\end{aligned}$$

Therefore $y_i - 1 = 2^{m+i-4} - 1$ and hence by Lemma 2, the edge e_i is of dilation 1.

Again by definition of m-sequential k-ary tree of Type B**,**

$$\begin{aligned}&|V(T_1)| + |V[T_2]| + \cdots + |V(T_{i-1})| \\ &= (2^m - 1) + 2^{m+1} + 2^{m+2} + \cdots + 2^{m+(i-1)} \\ &= 2^m(1 + 2 + \cdots + 2^{i-1}) - 1 = 2^{m+i} - 2^i - 1.\end{aligned}$$

Therefore $y_i - 1 = 2^{m+i} - 2^i$ and hence by Lemma 3, the edge e_i is of dilation 2.

Case 2 (v is not a child of u):

Let the label of v be x. Let $e_i = (x, y_i)$, $1 \leq i \leq p$.

By definition of pre-order traversal,

$$y_i = |V(T_1)| + |V(T_2)| + \cdots + |V(T_{i-1})| + 1 + x.$$

By definition of m-sequential k-ary tree,

$$|V(T_1)| + |V(T_2)| + \cdots + |V(T_{i-1})| = 3 + 4 + \cdots + 2^{i-1} = 2^i - 1.$$

Therefore $y_i - x = 2^i$ and hence by Lemma 1, the edge e_i is of dilation 2.

4. Conclusions

In this paper, we have obtained an embedding algorithm to embed an m-sequential k-ary tree into its optimal hypercube with dilation at most 2. We have also proved its correctness. This study is important as the embedding of binary trees into hypercubes received special attention in the computational structures of algorithm such as searching and storing.

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