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On The Eneström-Kakeya Theorem

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Abstract

In this paper, we prove some generalizations of results concerning the Eneström-Kakeya theorem. The results obtained considerably improve the bounds by relaxing the hypothesis in some cases.

Keywords: Polynomial, Zeros, Eneström-Kakeya theorem

1. Introduction and Statement of Results

The following result due to Eneström and Kakeya [1] is well known in the theory of distribution of the zeros of polynomials.

Theorem A. *If* $P(z):=\sum_{j=0}^{n}a_j z^j$ *is a polynomial of degree n such that*

$$a_n \ge a_{n-1} \ge a_{n-2} \ge \cdots \ge a_1 \ge a_0 \ge o,$$

then $P(z)$ does not vanish in $|z|>1$

In the literature, [2-8], there exist extensions and generalizations of Eneström-Kakeya theorem. Joyal, Labelle and Rahman [9] extended this theorem to a polynomial whose coefficients are monotonic but not necessarily non negative by proving the following result.

Theorem B. *Let* $P(z):=\sum_{j=0}^{n}a_j z^j$ *be a polynomial of degree n such that*

$$a_n \ge a_{n-1} \ge a_{n-2} \ge \cdots \ge a_1 \ge a_0,$$

then all the zeros of $P(z)$ lie in

$$|z| \le \frac{1}{|a_n|}\left\{a_n - a_0 + |a_0|\right\}.$$

Dewan and Bidkham [10] generalized Theorem B and proved the following:

Theorem C. *Let* $P(z):=\sum_{j=0}^{n}a_j z^j$ *be a polynomial of degree n such that for some* $t>0$ *and* $0<\lambda\le n$,

$$a_n t^n \le a_{n-1}t^{n-1} \le \cdots \le a_\lambda t^\lambda \ge a_{\lambda-1}t^{\lambda-1} \ge \cdots \ge t a_1 \ge a_0,$$

then $P(z)$ has all the zeros in the circle

$$|z| \le \frac{t}{|a_n|}\left\{\left(\frac{2a_\lambda}{t^{n-\lambda}} - a_n\right) + \frac{1}{t^n}\left(|a_0| - a_0\right)\right\}.$$

By using Schwarz's Lemma, Aziz and Mohammad [11] generalized Eneström-Kakeya theorem in a different way and proved the following:

Theorem D. *Let* $P(z):=\sum_{j=0}^{n}a_j z^j$ *be a polynomial of degree n with real positive coefficients. If* $t_1 > t_2 \ge 0$ *can be found such that*

$$a_r t_1 t_2 + a_{r-1}(t_1 - t_2) - a_{r-2} \ge 0, \text{ for } r = 1,2,\cdots,n+1$$
$$(a_{-1} = a_{n+1} = 0),$$

then all the zeros of $P(z)$ lie in $|z| \le t_1$

Aziz and Zargar [12] also relaxed the hypothesis of Eneström-Kakeya theorem in a different way and proved the following result.

Theorem E. *Let* $P(z):=\sum_{j=0}^{n}a_j z^j$ *be a polynomial of degree n such that for some* $K \ge 1$,

$$Ka_n \ge a_{n-1} \ge a_{n-2} \ge \cdots \ge a_1 \ge a_0 > 0,$$

then all the zeros of $P(z)$ lie in

$$|z + K - 1| \le K.$$

While studying Theorem E, a natural question arises that what happens if we relax the hypothesis of Theorem D in a similar way and only assume that

$$a_r t_1 t_2 + a_{r-1}(t_1 - t_2) - a_{r-2} \ge 0, \text{ for } r = 2,3,\cdots,n$$

In this paper, we study such a case and prove a more general result from which many known results follow on a fairly uniform procedure. Infact we prove:

Theorem 1. *Let* $P(z):=\sum_{j=0}^{n}\alpha_j z^j$ *be a polynomial of degree n such that* $\alpha_j = a_j + ib_j$ *where* a_j *and* b_j, $j = 0,1\cdots,n$ *are real numbers and if* $t_1 > t_2 \ge 0$ *can be found such that for* $r = 2,3,\cdots,n$

$$a_r t_1 t_2 + a_{r-1}(t_1 - t_2) - a_{r-2} \geq 0,$$

$$b_r t_1 t_2 + b_{r-1}(t_1 - t_2) - b_{r-2} \geq 0,$$

and for some $K \geq 1$,

$$Ka_n(t_1 - t_2) - a_{n-1} \geq 0,$$

$$Kb_n(t_1 - t_2) - b_{n-1} \geq 0,$$

then all the zeros of $P(z)$ lie in $|z + (K-1)(t_1 - t_2)| \leq R,$

where

$$R = \frac{1}{|\alpha_n|}\left\{K(a_n + b_n)(t_1 - t_2) + (a_n + b_n)t_2 - (a_1 + b_1)\frac{t_2}{t_1^{n-1}} - (a_0 + b_0)\frac{1}{t_1^{n-1}} + (|a_1 t_1 t_2 + a_0(t_1 - t_2)| + |b_1 t_1 t_2 + b_0(t_1 - t_2)|)\frac{1}{t_1^n} + (|a_0| + |b_0|)\frac{t_2}{t_1^n}\right\}.$$

The following interesting result immediately follows from Theorem 1, if we assume that all the coefficients of the polynomial $P(z)$ are real.

Corollary 1. *Let* $P(z) := \sum_{j=0}^{n} a_j z^j$ *be a polynomial of degree n with real coefficients. If* $t_1 > t_2 \geq 0$ *can be found such that*

$$a_r t_1 t_2 + a_{r-1}(t_1 - t_2) - a_{r-2} \geq 0, \quad \text{for } r = 2,3,\cdots,n$$

and for some $K \geq 1$,

$$Ka_n(t_1 - t_2) - a_{n-1} \geq 0,$$

then all the zeros of $P(z)$ lie in $|z + (K-1)(t_1 - t_2)| \leq R^*,$

where

$$R^* = \frac{1}{|a_n|}\left\{Ka_n(t_1 - t_2) + a_n t_2 - a_1\frac{t_2}{t_1^{n-1}} - a_0\frac{1}{t_1^{n-1}} + |a_1 t_1 t_2 + a_0(t_1 - t_2)|\frac{1}{t_1^n} + |a_0|\frac{t_2}{t_1^n}\right\}.$$

Remark 1. If we assume that all the coefficients of $P(z)$ are real and positive, then for $K = 1$, Corollary 1 satisfies the statement of Theorem D and a simple calculation shows that in this case also all the zeros of $P(z)$ lie in $|z| \leq t_1$.

Next, if in the Theorem 1, we take $t_2 = 0$ and assume that coefficients to be real, we get the following:

Corollary 2. *Let* $P(z) := \sum_{j=0}^{n} a_j z^j$ *be a polynomial of degree n with real coefficients. If for some* $t > 0$ *and* $K \geq 1$,

$$Ka_n t^n \geq a_{n-1}t^{n-1} \geq a_{n-2}t^{n-2} \geq \cdots \geq a_1 t > a_0,$$

then all the zeros of $P(z)$ lie in

$$|z + (K-1)t| \leq \frac{t}{|a_n|}\left(Ka_n + \frac{|a_0|}{t^n} - \frac{a_0}{t^n}\right).$$

Remark 2. If we put $t = 1$ in Corollary 2, we get the result due to Aziz and Zargar [2] and for $t = 1$, $K = 1$, Corollary 2 reduces to Theorem B.

We next prove the following more general result which is of independent interest.

Theorem 2. *Let* $P(z) := \sum_{j=0}^{n} \alpha_j z^j$ *be a polynomial of degree n such that* $\alpha_j = a_j + ib_j$ *where* a_j *and* b_j, $j = 0,1,2,\cdots,n$ *are real numbers. If* $t_1 > t_2 \geq 0$ *can be found such that for* $r = 2,3,\cdots,n+1$

$$a_r t_1 t_2 + a_{r-1}(t_1 - t_2) - a_{r-2} \geq 0,$$

$$b_r t_1 t_2 + b_{r-1}(t_1 - t_2) - b_{r-2} \geq 0,$$

and for some real numbers u and v , $u \geq 1, v \geq 1$

$$ua_n(t_1 - t_2) - a_{n-1} \geq 0,$$

$$vb_n(t_1 - t_2) - b_{n-1} \geq 0,$$

then all the zeros of $P(z)$ lie in

$$\left|z + (t_1 - t_2)\left(\frac{ua_n + ivb_n}{\alpha_n} - 1\right)\right| \leq R_1,$$

where

$$R_1 = \frac{1}{|\alpha_n|}\left\{(ua_n + vb_n)(t_1 - t_2) + (a_n + b_n)t_2 - (a_1 + b_1)\frac{t_2}{t_1^{n-1}} - (a_0 + b_0)\frac{1}{t_1^{n-1}} + (|a_1 t_1 t_2 + a_0(t_1 - t_2)| + |b_1 t_1 t_2 + b_0(t_1 - t_2)|)\frac{1}{t_1^n} + (|a_0| + |b_0|)\frac{t_2}{t_1^n}\right\}.$$

If in Theorem 2, we take

$$u = \frac{a_{n-1}}{a_n(t_1 - t_2)} \quad \text{and} \quad v = \frac{b_{n-1}}{b_n(t_1 - t_2)},$$

so that $u \geq 1, v \geq 1$, we get the following:

Corollary 3. *Let* $P(z) := \sum_{j=0}^{n} \alpha_j z^j$ *be a polynomial of degree n such that* $\alpha_j = a_j + ib_j$ *where* a_j *and* b_j, $j = 0,1,2,\cdots,n$ *are real numbers. If* $t_1 > t_2 \geq 0$ *can be found such that*

$$a_r t_1 t_2 + a_{r-1}(t_1 - t_2) - a_{r-2} \geq 0, \text{ for } r = 2,3,\cdots,n$$

$$a_r t_1 t_2 + a_{r-1}(t_1 - t_2) - a_{r-2} \leq 0, \text{ for } r = n+1$$

$$b_r t_1 t_2 + b_{r-1}(t_1 - t_2) - b_{r-2} \geq 0, \text{ for } r = 2,3,\cdots,n$$

$$b_r t_1 t_2 + b_{r-1}(t_1 - t_2) - b_{r-2} \le 0, \text{ for } r = n+1,$$

then all the zeros of $P(z)$ lie in

$$\left| z + \frac{\alpha_{n-1}}{\alpha_n} - (t_1 - t_2) \right| \le R_1^*,$$

where

$$R_1^* = \frac{1}{|\alpha_n|}\Big\{ (a_n + b_n) t_2 + (a_{n-1} + b_{n-1}) - (a_1 + b_1)\frac{t_2}{t_1^{n-1}} - (a_0 + b_0)\frac{1}{t_1^{n-1}} + |a_1 t_1 t_2 + a_0 (t_1 - t_2)| \frac{1}{t_1^n} + |b_1 t_1 t_2 + b_0 (t_1 - t_2)| \frac{1}{t_1^n} + (|a_0| + |b_0|)\frac{t_2}{t_1^n} \Big\}.$$

In particular, if

$$a_r t_1 t_2 + a_{r-1}(t_1 - t_2) - a_{r-2} \ge 0, \text{ for } r = 1, 2, \cdots, n$$

$$a_r t_1 t_2 + a_{r-1}(t_1 - t_2) - a_{r-2} \le 0, \text{ for } r = n+1,$$

$$b_r t_1 t_2 + b_{r-1}(t_1 - t_2) - b_{r-2} \ge 0, \text{ for } r = 1, 2, \cdots, n$$

$$b_r t_1 t_2 + b_{r-1}(t_1 - t_2) - b_{r-2} \le 0, \text{ for } r = n+1,$$

then

$$a_1 t_1 t_2 + a_0 (t_1 - t_2) \ge 0,$$

$$b_1 t_1 t_2 + b_0 (t_1 - t_2) \ge 0$$

and we get in this case all the zeros of $P(z)$ lie in

$$\left| z + \frac{\alpha_{n-1}}{\alpha_n} - (t_1 - t_2) \right| \le \frac{1}{|\alpha_n|}\left\{ (a_n + b_n) t_2 + (a_{n-1} + b_{n-1}) \right\}.$$

Remark 3. A result of Shah and Liman [7, Theorem 1] is a special case of Corollary 3, if we assume that all the coefficients of $P(z)$ are real.

The following result also follows from Theorem 2, if we assume that $t_2 = 0$ and $t_1 = 1$.]

Corollary 4. *Let* $P(z) := \sum_{j=0}^{n} \alpha_j z^j$ *be a polynomial of degree n such that* $\alpha_j = a_j + i b_j$ *where* a_j *and* b_j, $j = 0, 1, 2, \cdots, n$ *are real numbers. If for some* $u \ge 1$ *and* $v \ge 1$,

$$u a_n \ge a_{n-1} \ge \cdots \ge a_0 \ge 0,$$

$$v b_n \ge b_{n-1} \ge \cdots \ge b_0 \ge 0,$$

then all the zeros of $P(z)$ lie in

$$\left| z + \frac{u a_n + i v b_n}{\alpha_n} - 1 \right| \le \frac{u a_n + v b_n}{|\alpha_n|}$$

Many other known results and generalizations similarly follows from Theorem 2 with suitable substitutions. We leave this to the readers.

2. Proofs of the Theorems

Proof of Theorem 1. Consider the polynomial

$$\begin{aligned} f(z) &= (t_2 + z)(t_1 - z)P(z) \\ &= -\alpha_n z^{n+2} + (\alpha_n (t_1 - t_2) - \alpha_{n-1}) z^{n+1} + (\alpha_n t_1 t_2 + \alpha_{n-1}(t_1 - t_2) - \alpha_{n-2}) z^n + \cdots \\ &\quad + (\alpha_2 t_1 t_2 + \alpha_1 (t_1 - t_2) - \alpha_0) z^2 + (\alpha_1 t_1 t_2 + \alpha_0 (t_1 - t_2)) z + \alpha_0 t_1 t_2 \qquad (1) \\ &= -\alpha_n z^{n+2} - (K-1)\alpha_n (t_1 - t_2) z^{n+1} + (K\alpha_n (t_1 - t_2) - \alpha_{n-1}) z^{n+1} + (\alpha_n t_1 t_2 + \alpha_{n-1}(t_1 - t_2) - \alpha_{n-2}) z^n + \cdots \\ &\quad + (\alpha_2 t_1 t_2 + \alpha_1 (t_1 - t_2) - \alpha_0) z^2 + (\alpha_1 t_1 t_2 + \alpha_0 (t_1 - t_2)) z + \alpha_0 t_1 t_2 \qquad (2) \\ &= -\alpha_n z^{n+2} - (K-1)\alpha_n (t_1 - t_2) z^{n+1} + (K a_n (t_1 - t_2) - a_{n-1}) z^{n+1} + (a_n t_1 t_2 + a_{n-1}(t_1 - t_2) - a_{n-2}) z^n + \ldots \\ &\quad + (a_2 t_1 t_2 + a_1 (t_1 - t_2) - a_0) z^2 + (a_1 t_1 t_2 + a_0 (t_1 - t_2)) z + a_0 t_1 t_2 + i\Big[(K b_n (t_1 - t_2) - b_{n-1}) z^{n+1} + \\ &\quad (b_n t_1 t_2 + b_{n-1}(t_1 - t_2) - b_{n-2}) z^n + \cdots + (b_2 t_1 t_2 + b_1 (t_1 - t_2) - b_0) z^2 + (b_1 t_1 t_2 + b_0 (t_1 - t_2)) z + b_0 t_1 t_2 \Big]. \end{aligned}$$

This gives

$$\begin{aligned} |f(z)| &\ge |\alpha_n| |z|^{n+1} |z + (K-1)(t_1 - t_2)| - |K a_n (t_1 - t_2) - a_{n-1}| |z|^{n+1} - |a_n t_1 t_2 + a_{n-1}(t_1 - t_2) - a_{n-2}| |z|^n - \cdots \\ &\quad - |a_2 t_1 t_2 + a_1 (t_1 - t_2) - a_0| |z|^2 - |a_1 t_1 t_2 + a_0 (t_1 - t_2)| |z| - |a_0 t_1 t_2| - \Big[|K b_n (t_1 - t_2) - b_{n-1}| |z|^{n+1} \\ &\quad + |b_n t_1 t_2 + b_{n-1}(t_1 - t_2) - b_{n-2}| |z|^n + \cdots + |b_2 t_1 t_2 + b_1 (t_1 - t_2) - b_0| |z|^2 + |b_1 t_1 t_2 + b_0 (t_1 - t_2)| |z| + |b_0 t_1 t_2| \Big]. \end{aligned}$$

$$= |z|^{n+1}\Big\{ |z+(K-1)(t_1-t_2)||\alpha_n| - \left(|Ka_n(t_1-t_2)-a_{n-1}| + |Kb_n(t_1-t_2)-b_{n-1}|\right) - \left(|a_nt_1t_2+a_{n-1}(t_1-t_2)-a_{n-2}|\right.$$

$$\left. + |b_nt_1t_2+b_{n-1}(t_1-t_2)-b_{n-2}|\right)\frac{1}{|z|} - \cdots - \left(|a_2t_1t_2+a_1(t_1-t_2)-a_0| + |b_2t_1t_2+b_1(t_1-t_2)-b_0|\right)\frac{1}{|z|^{n-1}}$$

$$-\left(|a_1t_1t_2+a_0(t_1-t_2)| + |b_1t_1t_2+b_0(t_1-t_2)|\right)\frac{1}{|z|^{n}} - \left(|a_0t_1t_2| + |b_0t_1t_2|\right)\frac{1}{|z|^{n+1}} \Big\}.$$

For $|z| > t_1$, we have by using hypothesis

$$|f(z)| \ge |z|^{n+1}\Big\{ |z+(K-1)(t_1-t_2)||\alpha_n| - \left(|Ka_n(t_1-t_2)-a_{n-1}| + |Kb_n(t_1-t_2)-b_{n-1}|\right)$$

$$-\left(|a_nt_1t_2+a_{n-1}(t_1-t_2)-a_{n-2}| + |b_nt_1t_2+b_{n-1}(t_1-t_2)-b_{n-2}|\right)\frac{1}{t_1} - \cdots$$

$$-\left(|a_2t_1t_2+a_1(t_1-t_2)-a_0| + |b_2t_1t_2+b_1(t_1-t_2)-b_0|\right)\frac{1}{t_1^{n-1}}$$

$$-\left(|a_1t_1t_2+a_0(t_1-t_2)| + |b_1t_1t_2+b_0(t_1-t_2)|\right)\frac{1}{t_1^{n}} - \left(|a_0t_1t_2| + |b_0t_1t_2|\right)\frac{1}{t_1^{n+1}} \Big\} > 0,$$

if

$$|z+(K-1)(t_1-t_2)||\alpha_n| > Ka_n(t_1-t_2) + Kb_n(t_1-t_2) + a_nt_2 + b_nt_2 - a_1\frac{t_2}{t_1^{n-1}} - b_1\frac{t_2}{t_1^{n-1}} - \frac{a_0}{t_1^{n-1}} - \frac{b_0}{t_1^{n-1}}$$

$$+\left(|a_1t_1t_2+a_0(t_1-t_2)| + |b_1t_1t_2+b_0(t_1-t_2)|\right)\frac{1}{t_1^{n}} + |a_0|\frac{t_2}{t_1^{n}} + |b_0|\frac{t_2}{t_1^{n}}.$$

Therefore, for $|z| \ge t_1$, $|f(z)| > 0$, if

$$|z+(K-1)(t_1-t_2)| > \frac{1}{|\alpha_n|}\Big\{K(a_n+b_n)(t_1-t_2) + (a_n+b_n)t_2 - (a_1+b_1)\frac{t_2}{t_1^{n-1}} - (a_0+b_0)\frac{1}{t_1^{n-1}}$$

$$+\left(|a_1t_1t_2+a_0(t_1-t_2)| + |b_1t_1t_2+b_0(t_1-t_2)|\right)\frac{1}{t_1^{n}} + \left(|a_0|+|b_0|\right)\frac{t_2}{t_1^{n}}\Big\}.$$

Hence all the zeros of $f(z)$ whose modulus is greater than t_1 lie in the circle

$$|z+(K-1)(t_1-t_2)| \le \frac{1}{|\alpha_n|}\Big\{K(a_n+b_n)(t_1-t_2) + (a_n+b_n)t_2 - (a_1+b_1)\frac{t_2}{t_1^{n-1}} - (a_0+b_0)\frac{1}{t_1^{n-1}}$$

$$+\left(|a_1t_1t_2+a_0(t_1-t_2)| + |b_1t_1t_2+b_0(t_1-t_2)|\right)\frac{1}{t_1^{n}} + \left(|a_0|+|b_0|\right)\frac{t_2}{t_1^{n}}\Big\}.$$

Since all the zeros whose modulus is less than t_1 already lie in this circle, we conclude that all the zeros of $f(z)$ and therefore $P(z)$ lies in

$$|z+(K-1)(t_1-t_2)| \le \frac{1}{|\alpha_n|}\Big\{K(a_n+b_n)(t_1-t_2) + (a_n+b_n)t_2 - (a_1+b_1)\frac{t_2}{t_1^{n-1}} - (a_0+b_0)\frac{1}{t_1^{n-1}}$$

$$+\left(|a_1t_1t_2+a_0(t_1-t_2)| + |b_1t_1t_2+b_0(t_1-t_2)|\right)\frac{1}{t_1^{n}} + \left(|a_0|+|b_0|\right)\frac{t_2}{t_1^{n}}\Big\}.$$

This completes the proof of the Theorem 1.

Proof of Theorem 2. Consider the polynomial

$$
\begin{aligned}
f(z) &= (t_2+z)(t_1-z)P(z)\\
&= -\alpha_n z^{n+2} + \left(\alpha_n(t_1-t_2)-\alpha_{n-1}\right)z^{n+1} + \left(\alpha_n t_1t_2 + \alpha_{n-1}(t_1-t_2)-\alpha_{n-2}\right)z^n + \dots\\
&\quad + \left(\alpha_2 t_1t_2 + \alpha_1(t_1-t_2)-\alpha_0\right)z^2 + \left(\alpha_1 t_1 t_2 + \alpha_0(t_1-t_2)\right)z + \alpha_0 t_1 t_2\\
&= -\alpha_n z^{n+2} + \left(a_n(t_1-t_2)-a_{n-1}\right)z^{n+1} + \left(a_n t_1t_2 + a_{n-1}(t_1-t_2)-a_{n-2}\right)z^n + \dots + \left(a_2t_1t_2 + a_1(t_1-t_2)-a_0\right)z^2\\
&\quad + \left(a_1t_1t_2 + a_0(t_1-t_2)\right)z + a_0t_1t_2 + i\Big[\left(b_n(t_1-t_2)-b_{n-1}\right)z^{n+1} + \left(b_nt_1t_2 + b_{n-1}(t_1-t_2)-b_{n-2}\right)z^n + \cdots\\
&\quad + \left(b_2t_1t_2 + b_1(t_1-t_2)-b_0\right)z^2 + \left(b_1t_1t_2 + b_0(t_1-t_2)\right)z + b_0t_1t_2\Big]\\
&= -\alpha_n z^{n+2} - (u-1)a_n(t_1-t_2)z^{n+1} + \left(ua_n(t_1-t_2)-a_{n-1}\right)z^{n+1} + \left(a_nt_1t_2 + a_{n-1}(t_1-t_2)-a_{n-2}\right)z^n + \cdots\\
&\quad + \left(a_2t_1t_2 + a_1(t_1-t_2)-a_0\right)z^2 + \left(a_1t_1t_2 + a_0(t_1-t_2)\right)z + a_0t_1t_2 + i\Big[-(v-1)b_n(t_1-t_2)z^{n+1} + \left(vb_n(t_1-t_2)-b_{n-1}\right)z^{n+1}\\
&\quad + \left(b_nt_1t_2 + b_{n-1}(t_1-t_2)-b_{n-2}\right)z^n + \dots + \left(b_2t_1t_2 + b_1(t_1-t_2)-b_0\right)z^2 + \left(b_1t_1t_2 + b_0(t_1-t_2)\right)z + b_0t_1t_2\Big]\\
&= -\alpha_n z^{n+2} + \left\{-(ua_n+ivb_n)+\alpha_n\right\}(t_1-t_2)z^{n+1} + \left(ua_n(t_1-t_2)-a_{n-1}\right)z^{n+1} + \left(a_nt_1t_2 + a_{n-1}(t_1-t_2)-a_{n-2}\right)z^n + \cdots\\
&\quad + (a_2t_1t_2 + a_1(t_1-t_2)-a_0)z^2 + \left(a_1t_1t_2 + a_0(t_1-t_2)\right)z + a_0t_1t_2 + i\Big[\left(vb_n(t_1-t_2)-b_{n-1}\right)z^{n+1} +\\
&\quad \left(b_nt_1t_2 + b_{n-1}(t_1-t_2)-b_{n-2}\right)z^n + \cdots + \left(b_2t_1t_2 + b_1(t_1-t_2)-b_0\right)z^2 + \left(b_1t_1t_2 + b_0(t_1-t_2)\right)z + b_0t_1t_2\Big].
\end{aligned}
$$

This gives

$$
\begin{aligned}
|f(z)| &\ge |\alpha_n||z|^{n+1}\left|z + (t_1-t_2)\left(\frac{ua_n+ivb_n}{\alpha_n}-1\right)\right| - \left|ua_n(t_1-t_2)-a_{n-1}\right||z|^{n+1} - \left|a_nt_1t_2 + a_{n-1}(t_1-t_2)-a_{n-2}\right||z|^n - \cdots\\
&\quad - \left|a_2t_1t_2 + a_1(t_1-t_2)-a_0\right||z|^2 - \left|a_1t_1t_2 + a_0(t_1-t_2)\right||z| - |a_0t_1t_2| - \left|vb_n(t_1-t_2)-b_{n-1}\right||z|^{n+1}\\
&\quad - \left|b_nt_1t_2 + b_{n-1}(t_1-t_2)-b_{n-2}\right||z|^n - \cdots - \left|b_2t_1t_2 + b_1(t_1-t_2)-b_0\right||z|^2 - \left|b_1t_1t_2 + b_0(t_1-t_2)\right||z| - |b_0t_1t_2|.\\
&= |z|^{n+1}\Bigg\{|\alpha_n|\left|z + (t_1-t_2)\left(\frac{ua_n+ivb_n}{\alpha_n}-1\right)\right| - \left(\left|ua_n(t_1-t_2)-a_{n-1}\right| + \left|vb_n(t_1-t_2)-b_{n-1}\right|\right)\\
&\quad - \left(\left|a_nt_1t_2 + a_{n-1}(t_1-t_2)-a_{n-2}\right| + \left|b_nt_1t_2 + b_{n-1}(t_1-t_2)-b_{n-2}\right|\right)\frac{1}{|z|} - \cdots - \left(\left|a_1t_1t_2 + a_0(t_1-t_2)\right| + \left|b_1t_1t_2 + b_0(t_1-t_2)\right|\right)\frac{1}{|z|^n}\\
&\quad - \left(|a_0|t_1t_2 + |b_0|t_1t_2\right)\frac{1}{|z|^{n+1}}\Bigg\}
\end{aligned}
$$

For $|z| > t_1$, we have

$$
\begin{aligned}
|f(z)| &\ge |z|^{n+1}\Bigg\{\left|z + (t_1-t_2)\left(\frac{ua_n+ivb_n}{\alpha_n}-1\right)\right||\alpha_n| - \left(\left|ua_n(t_1-t_2)-a_{n-1}\right| + \left|vb_n(t_1-t_2)-b_{n-1}\right|\right)\\
&\quad - \left(\left|a_nt_1t_2 + a_{n-1}(t_1-t_2)-a_{n-2}\right| + \left|b_nt_1t_2 + b_{n-1}(t_1-t_2)-b_{n-2}\right|\right)\frac{1}{t_1} - \cdots - \left(\left|a_1t_1t_2 + a_0(t_1-t_2)\right| + \left|b_1t_1t_2 + b_0(t_1-t_2)\right|\right)\frac{1}{t_1^n}\\
&\quad - \left(|a_0|t_1t_2 + |b_0|t_1t_2\right)\frac{1}{t_1^{n+1}}\Bigg\}.
\end{aligned}
$$

By using hypothesis, this gives

$$
|f(z)| \ge |z|^{n+1}\Bigg\{\left|z + (t_1-t_2)\left(\frac{ua_n+ivb_n}{\alpha_n}-1\right)\right||\alpha_n| - \Bigg[(ua_n+vb_n)(t_1-t_2) + (a_n+b_n)t_2 - (a_1+b_1)\frac{t_2}{t_1^{n-1}} - (a_0+b_0)\frac{1}{t_1^{n-1}}\Bigg]
$$

$$-\left(\left|a_1t_1t_2+a_0\left(t_1-t_2\right)\right|+\left|b_1t_1t_2+b_0\left(t_1-t_2\right)\right|\right)\frac{1}{t_1^n}-\left(\left|a_0\right|+\left|b_0\right|\right)\frac{t_2}{t_1^n}\Bigg\}>0,$$

if

$$\left|z+\left(t_1-t_2\right)\left(\frac{ua_n+ivb_n}{\alpha_n}-1\right)\right|>\frac{1}{\left|\alpha_n\right|}\Bigg\{\left(ua_n+vb_n\right)\left(t_1-t_2\right)+\left(a_n+b_n\right)t_2-\left(a_1+b_1\right)\frac{t_2}{t_1^{n-1}}-\left(a_0+b_0\right)\frac{1}{t_1^{n-1}}$$

$$+\left(\left|a_1t_1t_2+a_0\left(t_1-t_2\right)\right|+\left|b_1t_1t_2+b_0\left(t_1-t_2\right)\right|\right)\frac{1}{t_1^n}+\left(\left|a_0\right|+\left|b_0\right|\right)\frac{t_2}{t_1^n}\Bigg\}.$$

Hence all the zeros of $f(z)$ whose modulus is greater than t_1 lie in the circle

$$\left|z+\left(t_1-t_2\right)\left(\frac{ua_n+ivb_n}{\alpha_n}-1\right)\right|\le\frac{1}{\left|\alpha_n\right|}\Bigg\{\left(ua_n+vb_n\right)\left(t_1-t_2\right)+\left(a_n+b_n\right)t_2-\left(a_1+b_1\right)\frac{t_2}{t_1^{n-1}}-\left(a_0+b_0\right)\frac{1}{t_1^{n-1}}$$

$$+\left(\left|a_1t_1t_2+a_0\left(t_1-t_2\right)\right|+\left|b_1t_1t_2+b_0\left(t_1-t_2\right)\right|\right)\frac{1}{t_1^n}+\left(\left|a_0\right|+\left|b_0\right|\right)\frac{t_2}{t_1^n}\Bigg\}.$$

Since all the zeros whose modulus is less than t_1 already lie in this circle, we conclude that all the zeros of $f(z)$ and therefore $P(z)$ lies in

$$\left|z+\left(t_1-t_2\right)\left(\frac{ua_n+ivb_n}{\alpha_n}-1\right)\right|\le R_1,$$

where

$$R_1=\frac{1}{\left|\alpha_n\right|}\Bigg\{\left(ua_n+vb_n\right)\left(t_1-t_2\right)+\left(a_n+b_n\right)t_2-\left(a_1+b_1\right)\frac{t_2}{t_1^{n-1}}-\left(a_0+b_0\right)\frac{1}{t_1^{n-1}}+\left(\left|a_1t_1t_2+a_0\left(t_1-t_2\right)\right|+\left|b_1t_1t_2+b_0\left(t_1-t_2\right)\right|\right)\frac{1}{t_1^n}$$

$$+\left(\left|a_0\right|+\left|b_0\right|\right)\frac{t_2}{t_1^n}\Bigg\}.$$

This proves Theorem 2 completely.

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4. References

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