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On the Exponential Decay of Solutions for Some Kirchhoff-Type Modelling Equations with Strong Dissipation

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Abstract

This paper deals with the initial boundary value problem for a class of nonlinear Kirchhoff-type equations with strong dissipative and source terms $u_{tt}-\varphi(\|\nabla u\|_2^2)\Delta u - a\Delta u_t = b|u|^{\beta-2}u, x\in\Omega, t>0$ in a bounded domain, where $a,b>0$ and $\beta>2$ are constants. We obtain the global existence of solutions by constructing a stable set in $H_0^1(\Omega)$ and show the energy exponential decay estimate by applying a lemma of V. Komornik.

Keywords: Kirchhoff-type Equation; Initial Boundary Value Problem; Stable Set; Exponential Decay Estimate

1. Introduction

Let Ω be a bounded domain in R^n with smooth boundary $\partial\Omega$. In this paper, we investigate the existence and the energy exponential decay estimate of global solutions for the initial boundary value problem of the following Kirchhoff-type equation with strong dissipative and source terms in a bounded domain

$$u_{tt}-\varphi\left(\|\nabla u\|_2^2\right)\Delta u - a\Delta u_t = b|u|^{\beta-2}u, \quad x\in\Omega, t>0, \quad (1.1)$$

$$u(x,0)=u_0(x), \quad u_t(x,0)=u_1(x), \quad x\in\Omega, \quad (1.2)$$

$$u(x,t)=0, \quad x\in\partial\Omega, \quad t\ge 0, \quad (1.3)$$

where $a,b>0$ and $\beta>2$ are constants, $\varphi(s)$ is a C^1-class function on $[0,+\infty)$ satisfying

$$\varphi(s)\ge m_0, \ s\varphi(s)\ge \int_0^s \varphi(\theta)d\theta, \ \forall s\in[0,+\infty) \quad (1.4)$$

with $m_0\ge 1$ constant.

When $n=1$, the equation (1.1) describes a small amplitude vibration of an elastic string ([1]). The original equation is

$$\rho h\frac{\partial^2 u}{\partial t^2}+\tau\frac{\partial u}{\partial t}=\left(P_0+\frac{Eh}{2L}\int_0^L\left|\frac{\partial u}{\partial x}\right|ds\right)\frac{\partial^2 u}{\partial x^2}+f$$

where $0\le x\le L$ and $t>0$, $u(x,t)$ is the lateral displacement at the space coordinate x and the time t, ρ is the mass density, h is the cross-section area, L is the length, P_0 is the initial axial tension, τ is the resistance modulus, E is the Young modulus and f is the external force.

Many authors have studied the existence and uniqueness of solutions of (1.1)-(1.3) by using various methods. When $a,b>0$, and $\varphi(s)=s^r, r\ge 1,$ K. Nishihara and Y. Yamada [2] have proved the existence and the polynomial decay of global solution under the assumptions that the initial data u_0 and u_1 are sufficiently small and $u_0\ne 0$. However, the method in [2] can not be applied directly to the case that the equations have the blow-up term $|u|^{\beta-2}u$. M. Aassila and A. Benaissa [3] extend the global existence part of [2] to the case where $\varphi(s)>0$ with $\varphi\left(\|\nabla u_0\|^2\right)\ne 0$ and the nonlinear dissipative term $|u_t|^{\alpha-2}u_t$. K. Ono and K. Nishihara [4] have proved the global existence and decay structure of solutions of (1.1)-(1.3) without small condition of data using Galerkin method. K. Ono [5] has obtained the global existence of solutions for the problem (1.1)-(1.3) with dissipative term u_t instead of Δu_t.

In the case $a=0$, for large β and $\varphi(s)\ge r>0$, P.

D. Ancona and S. Spagnolo [6] proved that if $u_0, u_1 \in C_0^\infty(R^n)$ are small, then problem (1.1)-(1.3) has a global solution. When $\varphi(s) \geq 0$, M. Ghisi and M. Gobbino [7] proved the existence and uniqueness of a global solution $u(x,t)$ of (1.1)-(1.3) for small initial data

$$(u_0, u_1) \in \left(H_0^1(\Omega) \cap H^2(\Omega)\right) \times H_0^1(\Omega)$$

with $m\left(\|\nabla u_0\|^2\right) \neq 0$

and the asymptotic behavior

$$\left(u(t), u_t(t), u_{tt}(t)\right) \to (u_\infty, 0, 0)$$

in $(u_0, u_1) \in \left(H_0^1(\Omega) \cap H^2(\Omega)\right) \times H_0^1(\Omega) \times L^2(\Omega)$

as $t \to +\infty$, where either $u_\infty = 0$ or $\varphi\left(\|\nabla u_\infty\|^2\right) = 0$.

The case $\varphi(s) \geq r > 0$ has been considered by M. Hosoya and Y. Yamada [8] under the following condition:

$$0 \leq \beta < \frac{2}{n-4}, n \geq 5; \quad 0 \leq \beta < +\infty, n \leq 4.$$

They proved that, if the initial data are small enough, the problem (1.1)-(1.3) has a global solution which decays exponentially as $t \to +\infty$.

In this paper, we prove the global existence for the problem (1.1)-(1.3) by applying the potential well theory introduced by D. H. Sattinger [9] and L. Payne and D. H. Sattinger [10]. Meanwhile, we obtain the exponential decay estimate of global solutions by using the different method from paper [8].

We adopt the usual notation and convention. Let H^m denote the Sobolev space with the norm

$$\|u\|_{H^m(\Omega)} = \left(\sum_{|\alpha| \leq m} \|D^\alpha u\|_{L^2(\Omega)}^2\right),$$

$H_0^m(\Omega)$ denotes the closure in H^m of $C_0^\infty(\Omega)$. For simplicity of notations, hereafter we denote by $\|\cdot\|_p$ the Lebesgue space $L^p(\Omega)$ norm, $\|\cdot\|$ denotes $L^2(\Omega)$ norm and we write equivalent norm $\|\nabla \cdot\|$ instead of $H_0^1(\Omega)$ norm $\|\cdot\|_{H_0^1(\Omega)}$. Moreover, M denotes various positive constants depending on the known constants and it may be different at each appearance.

2. Preliminary

In order to state and prove our main results, we first define the following functionals

$$K(u) = m_0 \|\nabla u\|^2 - b\|u\|_\beta^\beta, \quad J(u) = \frac{m_0}{2}\|\nabla u\|^2 - \frac{b}{\beta}\|u\|_\beta^\beta$$

for $u \in H_0^1(\Omega)$. Then we define the stable set S by

$$S = \left\{u \in H_0^1(\Omega), K(u) > 0, J(u) < d\right\} \cup \{0\},$$

where

$$d = \inf\left\{\sup_{\lambda > 0} J(\lambda u), u \in H_0^1(\Omega) / \{0\}\right\}.$$

We denote the total energy functional associated with (1.1)-(1.3) by

$$E(t) = \frac{1}{2}\|u_t\|_2^2 + \frac{1}{2}\int_0^{\|\nabla u\|^2} \varphi(s)\,ds - \frac{b}{\beta}\|u\|_\beta^\beta \tag{2.1}$$

for $u \in H_0^1(\Omega), t \geq 0$, and

$$E(0) = \frac{1}{2}\|u_1\|_2^2 + \frac{1}{2}\int_0^{\|\nabla u_0\|^2} \varphi(s)\,ds - \frac{b}{\beta}\|u_0\|_\beta^\beta$$

is the total energy of the initial data.

Lemma 2.1 Let q be a number with $2 \leq q < +\infty$, $n \leq 2$ and $2 \leq q \leq \frac{2n}{n-2}$, $n > 2$. Then there exists a constant C depending on Ω and q such that

$$\|u\|_q \leq C\|u\|_{H_0^1(\Omega)}, \ \forall u \in H_0^1(\Omega).$$

Lemma 2.2 [11] Let $y(t): R^+ \to R^+$ be a nonincreasing function and assume that there is a constant $A > 0$, such that

$$\int_s^{+\infty} y(t)\,dt \leq Ay(s), \ 0 \leq s < +\infty,$$

then $y(t) \leq y(0)e^{1-\frac{t}{\lambda}}, \forall t \geq 0.$

We state a local existence result, which is known as a standard one.

Theorem 2.1 Suppose that β satisfies

$$2 < \beta < +\infty, n \leq 2; \ 2 < \beta \leq \frac{2n}{n-2}, n > 2. \tag{2.2}$$

If $(u_0, u_1) \in H_0^1(\Omega) \cap L^2(\Omega)$, then there exists $T > 0$ such that the problem (1.1)-(1.3) has a unique local solution $u(t)$ in the class

$$u \in C\left([0,T); H_0^1(\Omega)\right), \quad u_t \in C\left([0,T); L^2(\Omega)\right). \tag{2.3}$$

Lemma 2.3 Let $u(t,x)$ be a solutions of problem (1.1)-(1.3). Then $E(t)$ is a nonincreasing function for $t > 0$ and

$$\frac{d}{dt}E(t) = -a\|\nabla u_t(t)\|. \tag{2.4}$$

Proof By multiplying equation (1.1) by u_t and integrating over Ω, we get

$$\frac{d}{dt}E(t) = -a\|\nabla u_t(t)\| \leq 0.$$

Therefore, $E(t)$ is a nonincreasing function on t.

Lemma 2.4 Let $u \in H_0^1(\Omega)$, if (2.2) holds, then $d > 0$.

Proof Since

$$J(\lambda u) = \frac{m_0\lambda^2}{2}\|\nabla u\|^2 - \frac{b\lambda^\beta}{\beta}\|u\|_\beta^\beta,$$

so, we get

$$\frac{d}{d\lambda}J(\lambda u)=m_0\lambda\|\nabla u\|^2-b\lambda^{\beta-1}\|u\|_\beta^\beta.$$

Let $\frac{d}{d\lambda}J(\lambda u)=0$, which implies that

$$\lambda_1=\left(\frac{b}{m_0}\right)^{-\frac{1}{\beta-2}}\left(\frac{\|u\|_\beta^\beta}{\|\nabla u\|^2}\right)^{-\frac{1}{\beta-2}}.$$

As $\lambda=\lambda_1$, an elementary calculation shows that

$$\frac{d^2}{d\lambda^2}J(\lambda u)<0.$$

Hence, we have from Lemma 2.1 that

$$\sup_{\lambda\ge 0}J(\lambda u)=J(\lambda_1 u)=\frac{\beta-2}{2\beta}\left(\frac{b^2}{m_0^\beta}\right)^{-\frac{1}{\beta-2}}\left(\frac{\|u\|_\beta}{\|\nabla u\|}\right)^{-\frac{2\beta}{\beta-2}}$$

$$\ge\frac{\beta-2}{2\beta}\left(\frac{b^2}{m_0^\beta}\right)^{-\frac{1}{\beta-2}}C^{-\frac{2\beta}{\beta-2}}>0.$$

we get from the definition of d that $d>0$.

In order to prove the existence of global solutions for the problem (1.1)-(1.3), we need the following Lemma.

Lemma 2.5 Supposed that (2.2) hold, If $u_0\in S, u_1\in L^2(\Omega)$ and $E(0)<d$, then $u\in S$, for each $t\in[0,T)$.

Proof Assume that there exists a number $t^*\in[0,T)$. such that $u(t)\in S$ on $[0,t^*)$ and $u(t^*)\notin S$. Then, in virtue of the continuity of $u(t)$, we see $u(t^*)\in\partial S$. From the definition of S and the continuity of $J(u(t))$ and $K(u(t))$ in t, we have either
$J(u(t^*))=d$ or $K(u(t^*))=0$.
It follows from (1.4) and (2.1) that

$$J(u(t^*))=\frac{m_0}{2}\|\nabla u(t^*)\|^2-\frac{b}{\beta}\|u(t^*)\|_\beta^\beta \le E(t^*)\le E(0)<d. \tag{2.5}$$

So, the case $J(u(t^*))=d$ is impossible.

Assume that $K(u(t^*))=0$ holds, then we get that

$$\frac{d}{d\lambda}J(\lambda u(t^*))=m_0\lambda\left(1-\lambda^{\beta-2}\right)\|\nabla u\|^2.$$

We obtain from $\frac{d}{d\lambda}J(\lambda u(t^*))=0$ that $\lambda=1$.

Since

$$\frac{d^2}{d\lambda^2}J(\lambda u(t^*))|_{\lambda=1}=-m_0(\beta-2)\|\nabla u(t^*)\|<0.$$

Consequently, we get from (2.5) that

$$\sup_{\lambda\ge 0}J(\lambda u(t^*))=J(\lambda u(t^*))|_{\lambda=1}=J(u(t^*))<d$$

which contradicts the definition of d. Therefore, the case $K(u(t))=0$ is impossible as well. Thus, we conclude that $u(t)\in S$ on $[0,T)$.

3. Main Results and Proof

Theorem 3.1 Suppose that (2.2) holds, and $u(t)$ is a local solution of problem (1.1)-(1.3) on $[0,T)$. If $u_0\in S, u_1\in L^2(\Omega)$ and $E(0)<d$, then $u(x,t)$ is a global solution of the problem (1.1)-(1.3).

Proof It suffices to show that $\|\nabla u(t)\|^2+\|u_t(t)\|^2$ is bounded independently of t.

Under the hypotheses in Theorem 3.1, we get from Lemma 2.5 that $u(t)\in S$ on $[0,T)$. So the following formula holds on $[0,T)$.

$$\begin{aligned}J(u(t))&=\frac{m_0}{2}\|\nabla u(t)\|^2-\frac{b}{\beta}\|u(t)\|_\beta^\beta\\&\ge\frac{m_0}{2}\|\nabla u(t)\|^2-\frac{b}{\beta}\|\nabla u(t)\|^2\\&=\frac{(\beta-2)m_0}{2\beta}\|\nabla u(t)\|^2,\end{aligned}\tag{3.1}$$

Therefore, we have from (3.1) that

$$\begin{aligned}&\frac{1}{2}\|u_t(t)\|^2+\frac{(\beta-2)m_0}{2\beta}\|\nabla u(t)\|^2\\&\le\frac{1}{2}\|u_t(t)\|^2+J(u(t))=E(t)\le E(0)<d.\end{aligned}\tag{3.2}$$

Hence, we get

$$\|u_t(t)\|^2+\|\nabla u(t)\|^2\le\max\left(2,\frac{2\beta}{(\beta-2)m_0}\right)d<+\infty.$$

The above inequality and the continuation principle lead to the existence of global solution, that is, $T=+\infty$. Therefore, the solution $u(t)$ is a global solution of the problem (1.1)-(1.3).

The following Theorem shows the exponential decay estimate of global solutions for problem (1.1)-(1.3).

Theorem 3.2 If the hypotheses in Theorem 3.1 are valid, then the global solutions of problem (1.1)-(1.3) has the following exponential decay property

$$E(t)\le E(0)e^{1-\frac{t}{M}},$$

where $M>0$ is a constant.

Proof Multiplying by u on both sides of the Equation (1.1) and integrating over $\Omega\times[0,T)$, we obtain that

$$0=\int_S^T\int_\Omega u\left[u_{tt}-\varphi\left(\|\nabla u\|_2^2\right)\Delta u-a\Delta u_t-b\,|\,u\,|^{\beta-2}\,u\right]dxdt, \tag{3.3}$$

where $0\le S<T<+\infty.$

Since

$$\int_S^T\int_\Omega uu_{tt}dxdt=\int_\Omega uu_tdx\Big|_S^T-\int_S^T\int_\Omega|u_t|^2dxdt. \tag{3.4}$$

So, substituting the Formula (3.4) into the right-hand side of (3.3), we get that

$$\begin{aligned}0=&\int_S^T\left(\|u_t\|^2+\varphi\left(\|\nabla u\|_2^2\right)\|\nabla u\|^2-\frac{2b}{\beta}\|u\|_\beta^\beta\right)dt\\&-\int_S^T\int_\Omega\left[2|u_t|^2-a\nabla u_t\nabla u\right]dxdt\\&+\int_\Omega uu_tdx\Big|_S^T+\left(\frac{2}{\beta}-1\right)b\int_S^T\|u\|_\beta^\beta dt.\end{aligned} \tag{3.5}$$

It follows from (3.2) that

$$\|\nabla u(t)\|^2\le\frac{2\beta}{(\beta-2)m_0}E(t)\le\frac{2\beta}{(\beta-2)m_0}E(0)<\frac{2\beta}{(\beta-2)m_0}d \tag{3.6}$$

By exploiting Lemma 2.1 and (3.6), we easily arrive at

$$\begin{aligned}b\|u\|_\beta^\beta&\le bC^\beta\|\nabla u(t)\|^\beta=bC^\beta\|\nabla u(t)\|^{\beta-2}\|\nabla u(t)\|^2\\&<bC^\beta\left(\frac{2\beta}{(\beta-2)m_0}d\right)^{\frac{\beta-2}{2}}\|\nabla u(t)\|^2,\end{aligned} \tag{3.7}$$

We obtain from (3.6) and (3.7) that

$$\begin{aligned}b\left(1-\frac{2}{\beta}\right)\|u\|_\beta^\beta&\le bC^\beta\left(\frac{2\beta}{(\beta-2)m_0}d\right)^{\frac{\beta-2}{2}}\frac{\beta-2}{\beta}\|\nabla u(t)\|^2\\&\le bC^\beta\left(\frac{2\beta}{(\beta-2)m_0}d\right)^{\frac{\beta-2}{2}}\frac{\beta-2}{\beta}\bullet\frac{2\beta}{(\beta-2)m_0}E(t)\\&=\frac{2bC^\beta}{m_0}\left(\frac{2\beta}{(\beta-2)m_0}d\right)^{\frac{\beta-2}{2}}E(t).\end{aligned} \tag{3.8}$$

We derive from (1.4) that

$$\int_0^{\|\nabla u\|^2}\varphi(s)ds\le\varphi\left(\|\nabla u\|^2\right)\|\nabla u\|^2, \tag{3.9}$$

It follows from (3.5), (3.8) and (3.9) that

$$\begin{aligned}&2\left[1-\frac{bC^\beta}{m_0}\left(\frac{2\beta}{(\beta-2)m_0}d\right)^{\frac{\beta-2}{2}}\right]\int_S^TE(t)dt\\&\le\int_S^T\int_\Omega\left[2\,|\,u_t\,|^2-a\nabla u_t\nabla u\right]dxdt-\int_\Omega uu_tdx\Big|_S^T.\end{aligned} \tag{3.10}$$

We have from Lemma 2.1 and (3.2) that

$$\begin{aligned}\left|\int_\Omega uu_tdx\Big|_S^T\right|&\le\left(\frac12\|u\|^2+\frac12\|u_t\|^2\right)\Big|_S^T\\&\le\left(\frac{\beta C^2}{(\beta-2)m_0}\cdot\frac{(\beta-2)m_0}{2\beta}\|\nabla u\|^2+\frac12\|u_t\|^2\right)\Big|_S^T\\&\le\max\left(\frac{\beta C^2}{(\beta-2)m_0},1\right)E(t)\Big|_S^T\le ME(S),\end{aligned} \tag{3.11}$$

Substituting the estimate (3.11) into (3.10), we conclude that

$$\begin{aligned}&2\left[1-\frac{2bC^\beta}{m_0}\left(\frac{2\beta}{(\beta-2)m_0}d\right)^{\frac{\beta-2}{2}}\right]\int_S^TE(t)dt\\&\le\int_S^T\int_\Omega\left[2|u_t|^2-a\nabla u_t\nabla u\right]dxdt+ME(S).\end{aligned} \tag{3.12}$$

We get from Lemma 2.1 and Lemma 2.3 that

$$\begin{aligned}&2\int_S^T\int_\Omega|u_t|^2dxdt=2\int_S^T\|u_t\|^2dt\le2C^2\int_S^T\|\nabla u_t\|^2dt\\&=-\frac{2C^2}{a}\left(E(T)-E(S)\right)\le\frac{2C^2}{a}E(S).\end{aligned} \tag{3.13}$$

From Young inequality, Lemma 2.1, Lemma 2.3 and (3.6), We receive that

$$\begin{aligned}&-a\int_S^T\int_\Omega\nabla u_t\nabla udxdt\le a\int_S^T\left(\varepsilon\|\nabla u\|^2+M(\varepsilon)\|\nabla u_t\|^2\right)dt\\&\le\frac{2a\beta\varepsilon}{(\beta-2)m_0}\int_S^TE(t)dt+M(\varepsilon)\left(E(S)-E(T)\right)\\&\le\frac{2a\beta\varepsilon}{(\beta-2)m_0}\int_S^TE(t)dt+M(\varepsilon)E(S).\end{aligned} \tag{3.14}$$

Choosing small enough ε such that

$$\frac{2a\beta\varepsilon}{(\beta-2)m_0}+\frac{bC^\beta}{m_0}\left(\frac{2\beta}{(\beta-2)m_0}d\right)^{\frac{\beta-2}{2}}<1,$$

then, substituting (3.13) and (3.14) into (3.12),

$$\int_S^TE(t)dt\le ME(S). \tag{3.15}$$

Let $T\to+\infty$, then we have from (3.15) that

$$\int_S^{+\infty}E(t)dt\le ME(S). \tag{3.16}$$

Thus, we receive from (3.16) and Lemma 3.1 that

$$E(t)\le E(0)e^{1-\frac{t}{M}},\ t\in[0,+\infty). \tag{3.17}$$

The proof of Theorem 3.2 is finished.

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