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C^0 Approximation on the Spatially Homogeneous Boltzmann Equation for Maxwellian Molecules*

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Abstract

In this paper we study the viscosity analysis of the spatially homogeneous Boltzmann equation for Maxwellian molecules. We first show that the global existence in time of the mild solution of the viscosity equation $\partial_t f_\varepsilon = Q(f_\varepsilon, f_\varepsilon) + \varepsilon\Delta_v f_\varepsilon$. We then study the asymptotic behaviour of the mild solution as the coefficients $\varepsilon \to 0^+$, and an estimate on $\|f_\varepsilon - f\|_0$ is derived.

Keywords: Viscosity Boltzmann Equation, Mild Solution, Viscosity Approximation, Collision Kernel

1. Introduction

In this paper we shall investigate the asymptotic properties of the solution of the viscosity Boltzmann equation for Maxwellian molecules

$$\partial_t f_\varepsilon = Q(f_\varepsilon, f_\varepsilon) + \varepsilon\Delta_v f_\varepsilon \quad \text{in } [0,\infty)\times R^3 \tag{1}$$

as the viscosity coefficients $\varepsilon \to 0^+$. Here, $Q(f,f)$ is the Boltzmann collision operator for Maxwellian molecules defined by its quadratic form

$$Q(f,g) = 2\pi\int_{R^3}\int_0^\pi (f'g'_* - fg_*)\cdot b(\cos\theta)\sin\theta d\theta dv_*$$

where function b is nonnegative and continuous, and $b(\cos\theta)\sin\theta \in L^1[0,\pi]$. Here the shorthand $f' = f(t,v')$, $f'_* = f(t,v'_*)$ are used; v', v'_* are the post-collisional velocities corresponding to the pre-collisional velocities v, v_* respectively, which submit to the elastic collision law

$$\begin{cases} v' = v - (v - v_*, \omega)\omega \\ v'_* = v' + (v - v_*, \omega)\omega \end{cases} \tag{2}$$

where $(\cdot,\cdot)$ denotes the scalar product. $\omega \in S^2$ the 2-D unit sphere and $\omega = (v' - v)/|v' - v|$. θ is the angle between $v_* - v$ and ω, $\theta \in [0,\pi]$. On physically, Q satisfies the symmetrization and translation invariance.

For Maxwellian potential Q can be split into Q^+ and Q^-:

$$Q(f,f) = Q^+(f,f) - Q^-(f,f)$$

$$Q^+(f,f) = \int_{R^3\times S^2} f' f'_* b(\cos\theta)\, d\omega dv_*$$

$$Q^-(f,f) = fL(f),$$

$$L(f) = \int_{R^3\times S^2} f_* b(\cos\theta)\, d\omega dv_*$$

The problem of viscosity approximation of the spatially homogeneous Boltzmann equation, namely whether the solution of (1) converges to the solution of the equation

$$\partial_t f = Q(f,f) \quad \text{in } [0,\infty)\times R^3 \tag{3}$$

as $\varepsilon \to 0^+$, is very interested for mathematical theory of Boltzmann equation as well as practical applications. We know that the energy of the solution of (1) is increasing with the time t due to the diffusion effect. We cannot expect that the solution of (1) approaches to the Maxwellian equilibrium in large time. This observation has recently been shown by Li-Matsumura [1]. In early work of the authors an explicit estimate of $\|f_\varepsilon - f\|$ in L^1_k was derived which indicates also the dependence of time [2]. It must be stressed this result excludes the case of Maxwellian molecules. Actually, the produce of moments for cutoff potential is not valid for Maxwellian molecules. In this paper we shall study the viscosity approximation for Maxwellian molecules. Our goal is to

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study the existence and uniqueness of the global solution of the viscosity equation (1) in time, and to estimate $\| f_\varepsilon - f \|$ explicitly in C^0-norm. The new tool is the Gagliardo-Nirenberg inequality.

Let us mention some works about the spatially homogeneous Boltzmann equation with cutoff potential, see [3-11] for example. For the Maxwellian molecules Morgenstern first deduced the existence and uniqueness of the solution in L^1 space [12]. We also remark that the approximation with diffusion term in velocity variable was present in the work of DiPerna-Lions [13].

Now we complement the equation (1) and (3) with the same initial condition:

$$f|_{t=0} = f_\varepsilon|_{t=0} = \varphi(v), \quad v \in R^3. \tag{4}$$

In the sequel we always assume that

$$0 \le \varphi(v) \in L^1(R^3). \tag{5}$$

It must be emphasized that the nonnegative hypothesis of $\varphi(v)$ is not necessary in present paper.

In the following we denote the C^m norm by $|\cdot|_m$, and

$$|f|_m = \max_{0\le j\le m} [f]_j, \quad and \quad [f]_j = \max_{|\alpha|=j} \sup_{v\in R^3} |D^\alpha f|$$

Here α is the multi-index.

This paper is organized as follows. We introduce a mild solution to the Cauchy problem (1) and (4) in Section 2. We prove local existence of the mild solution by the contracted mapping principle. In Section 3, we propose the global existence of the mild solution. Our main tool is the interpolation inequalities. Finally, we study the $W^{2,p}$ estimate of f_ε in Section 4 and deduce the following asymptotic expression

$$|f_\varepsilon - f|_0 \le \varepsilon A e^{kt}.$$

2. The Local Existence

In this section we shall study the local existence of the solution of the Cauchy problem (1), (4).

Definition 1. Given $\varepsilon > 0$. We call f_ε is the mild solution to the Cauchy problem (1) and (4), if $f_\varepsilon \in C([0,\infty); W^{1,p}(R^3))(1 \le p < \infty)$ and satisfies

$$f_\varepsilon(t,v) = \begin{cases} \varphi(v), & t = 0 \\ \int_{R^3} G(t, v-\xi)\varphi(\xi)d\xi + \int_0^t \int_{R^3} G(t-s, v-\xi) & \\ Q(f_\varepsilon, f_\varepsilon)(s,\xi)dsd\xi & t > 0 \end{cases} \tag{6}$$

where

$$G_t = G(t,v) \equiv \frac{1}{(4\pi\varepsilon t)^{3/2}} \exp\left(\frac{-|v|^2}{4\varepsilon t}\right), \ t > 0$$

The following is the local existence theorem.

Theorem 1. Given $\varepsilon > 0$. Let $\varphi \in W^{1,p}(R^3)$, $1 \le p < \infty$ and satisfy (5). Then there exists $T > 0$ such that the Cauchy problem (1) and (4) has a unique mild solution $f_\varepsilon, \forall 0 < t \le T$.

In order to prove Theorem 1, let us recall a well-known result which is often called convolution property.

Proposition 2 ([10,14]). For any $1 \le p < \infty$, if $f \in L^1(R^3), g \in L^p(R^3)$ then there exist constant $C > 0$ dependent on b only, such that

$$\|Q(g,f)\|_{L^p} \le C\|g\|_{L^p} \|f\|_{L^1}$$

The proof of Theorem 1 Consider the following space

$$\mathbb{S} = \left\{ f_\varepsilon \in C\left([0,T]; W^{1,p}\left(R^3\right)\right)\right\}$$

T is determined. Defined the mapping $\mathbb{F}: f_\varepsilon \to \overline{f_\varepsilon}$ by

$$\overline{f_\varepsilon}(t,v) = \begin{cases} \varphi(v), \ t = 0 \\ G_t *_v \varphi(v) + \int_0^t G_{t-s} *_v Q(f_\varepsilon, f_\varepsilon)(v)ds, t > 0 \end{cases} \tag{7}$$

where $*_v$ denotes the convolution in variable v.

By $\varphi \in L^1(R^3)$ and the definition of the mild solution (6), we have

$$\|f_\varepsilon\|_{L^1} \le \|\varphi\|_{L^1} + \int_0^t \|Q(f_\varepsilon, f_\varepsilon)\|_{L^1} ds$$

Making use of the Prop. 2 and Gronwall's lemma, we obtain the estimate of $\|f_\varepsilon\|_{L^1}$.

In terms of (7), we denote $\overline{f_\varepsilon}(t,v)(t>0)$ by I_1 and I_2. Obviously,

$$\|I_1\|_{W^{1,p}} \le \|\varphi\|_{W^{1,p}}$$

By Prop. 2 and Young's inequality, noting that $\|G_{t-s}\|_{L^1} = 1$, one obtain

$$\|I_2\|_{L^p} \le \int_0^t \|G_{t-s}\|_{L^1} \|Q(f_\varepsilon, f_\varepsilon)\|_{L^p} ds \le C\|f_\varepsilon\|_{L^p} \|f_\varepsilon\|_{L^1} t$$

Here the nonnegative constant C depends on b only. In what following, we denote C for various nonnegative constants independent of ε unless special statements. On the other hand,

$$\begin{aligned} \|\nabla I_2\|_{L^p} &\le \int_0^t \|\nabla G_{t-s}\|_{L^1} \|Q(f_\varepsilon, f_\varepsilon)\|_{L^p} ds \\ &\le C\int_0^t \varepsilon^{-\frac{1}{2}} (t-s)^{-\frac{1}{2}} \|f_\varepsilon\|_{L^1} \|f_\varepsilon\|_{L^p} ds \\ &\le C\varepsilon^{-\frac{1}{2}} \|f_\varepsilon\|_{L^1} \|f_\varepsilon\|_{L^p} t^{\frac{1}{2}}. \end{aligned} \tag{8}$$

Therefore,

$$\|I_2\|_{W^{1,p}} \le C\left(t+\varepsilon^{-\frac{1}{2}}\sqrt{t}\right)\|f_\varepsilon\|_{W^{1,p}}$$

This follows the mapping $\mathbb{F}$ is closed. Let $f_1, f_2 \in \mathbb{S}$, by the bilinearity of Q

$$Q(f_1, f_1)-Q(f_2, f_2)=Q^+(f_1, f_1-f_2)+$$
$$Q^+(f_1-f_2, f_2)-f_1L(f_1-f_2)-(f_1-f_2)L(f_2)$$

Thus

$$\|Q(f_1, f_1)-Q(f_2, f_2)\|_{L^p} \le C\|f_1-f_2\|_{L^p}$$
$$\le C\left(t+\varepsilon^{-\frac{1}{2}}\sqrt{t}\right)\|Q(f_1, f_1)-Q(f_2, f_2)\|_{L^p}$$
$$\le C\left(T+\varepsilon^{-\frac{1}{2}}\sqrt{T}\right)\|f_1-f_2\|_{W^{1,p}} \quad (9)$$

So, one deduces that the mapping $\mathbb{F}$ is locally Lipschitz continuous. By choosing $T>0$ suitably, such that $0<t\le T$, the Cauchy problem (1) and (4) exists a unique mild solution.

3. The Global Existence of the Wild Solution

In order to prove the global existence of the mild solution above, it suffices to show that

$$\|f_\varepsilon\|_{W^{1,p}} < \infty,\ \forall t>0$$

First, let us recall the N dimensional Gagliardo-Nirenberg's inequality: let $1\le q,r<\infty$, j, m are integers and $0\le j<m$. Suppose that $a\in\left[\frac{j}{m},1\right]$, ($a<1$ if $m-j-N/r$ is a nonnegative integer). Then there exists a constant C dependent on q, r, j, m, a, N such that for any $u\in\mathfrak{D}\ (R^N)$,

$$\sum_{|\alpha|<j}\|D^\alpha u\|_{L^p} \le C\left(\sum_{|\alpha|<m}\|D^\alpha u\|_{L^r}\right)^a\|u\|_{L^q}^{1-a}$$

where

$$\frac{1}{p}=\frac{j}{m}+a\left(\frac{1}{r}-\frac{m}{N}\right)+\frac{1-a}{q}$$

Lemma 3. Given $\varepsilon>0$. Let $\varphi\in W^{1,p}(R^3)$, $1\le p<\infty$ and satisfy (5). Then $\forall T>0, t\in[0,T]$, the solution of Cauchy problem (1) and (4) satisfies

$$\|f_\varepsilon\|_{L^p} \le C\left(T,\|\varphi\|_{L^p}\right)$$

Proof From (1), we have

$$\frac{1}{p}\frac{d}{dt}\|f_\varepsilon\|_{L^p}^p=\int_{R^3}\operatorname{sgn} f_\varepsilon|f_\varepsilon|^{p-1}\partial_t f_\varepsilon dv$$
$$=\int_{R^3}\operatorname{sgn} f_\varepsilon|f_\varepsilon|^{p-1}\left(\varepsilon\Delta f_\varepsilon+Q(f_\varepsilon, f_\varepsilon)\right)dv$$
$$=-\int_{R^3}\varepsilon(p-1)|\nabla f_\varepsilon|^2|f_\varepsilon|^{p-2}dv+$$
$$\int_{R^3}\operatorname{sgn} f_\varepsilon|f_\varepsilon|^{p-1}Q(f_\varepsilon, f_\varepsilon)dv=I_1+I_2 \quad (10)$$

Obviously $I_1<0$. By Prop. 2 and Holder's inequality

$$\left|\int_{R^3}|f_\varepsilon|^{p-1}Q(f_\varepsilon, f_\varepsilon)dv\right|\le\|Q(f_\varepsilon, f_\varepsilon)\|_{L^p}\|f_\varepsilon\|_{L^p}^{p-1}$$
$$\le C\|f_\varepsilon\|_{L^1}\|f_\varepsilon\|_{L^p}^p\le C\|f_\varepsilon\|_{L^p}^p \quad (11)$$

By these estimates above and Gronwall's lemma, we derive

$$\|f_\varepsilon\|_{L^p}\le C\left(T,\|\varphi\|_{L^p}\right)$$

This finishes the proof of the lemma.

Lemma 4. Given $\varepsilon>0$. Let $\varphi\in W^{1,p}(R^3)$, $1\le p<\infty$ and satisfy (5). Then $\forall T>0, t\in[0,T]$, the solution of Cauchy problem (1) and (4) satisfies

$$\|\nabla f_\varepsilon\|_{L^p}\le C\left(T,\|\varphi\|_{W^{1,p}}\right).$$

Proof In terms of (6), for any $t>0$

$$f_\varepsilon(t,v)=G_t*\varphi+\int_0^t G_{t-s}*Q(f_\varepsilon, f_\varepsilon)ds.$$

Therefore,

$$\nabla f_\varepsilon=G_t*\nabla\varphi+\int_0^t\nabla G_{t-s}*Q(f_\varepsilon, f_\varepsilon)ds \quad (12)$$

By Young's inequality

$$\|\nabla f_\varepsilon\|_{L^p}\le\|G_t\|_{L^1}\|\nabla\varphi\|_{L^p}+\int_0^t\|\nabla G_{t-s}\|_{L^q}\|Q(f_\varepsilon, f_\varepsilon)\|_{L^r}ds \quad (13)$$

where

$$1+\frac{1}{p}=\frac{1}{q}+\frac{1}{r},\ 1\le p,q,r<\infty.$$

Next we estimate $\|\nabla G_{t-s}\|_{L^q}$ and $\|Q(f_\varepsilon, f_\varepsilon)\|_{Lr}$ respectively. Noting that

$$|z|e^{-|z|^2}\le\frac{\sqrt{2}}{2}e^{-\frac{1}{2}},\forall z\in R^3.$$

Thus

$$|\nabla G_{t-s}|=\left|\frac{|v|}{2\varepsilon(t-s)}G_{t-s}\right|$$
$$=\pi^{-\frac{3}{2}}\varepsilon^{-\frac{1}{2}}(t-s)^{-\frac{1}{2}}|z|e^{-|z|^2}$$
$$\le\frac{\sqrt{2}}{2}e^{-\frac{1}{2}}\pi^{-\frac{3}{2}}\varepsilon^{-\frac{1}{2}}(t-s)^{-\frac{1}{2}}\le C(t-s)^{-\frac{1}{2}} \quad (14)$$

Therefore

$$\|\nabla G_{t-s}\|_{L^q}\le\|\nabla G_{t-s}\|_{L^1}^{\frac{1}{q}}\|\nabla G_{t-s}\|_{L^\infty}^{1-\frac{1}{q}}\le C(t-s)^{-\frac{1}{2}} \quad (15)$$

By Gagliardo-Nirenberg's inequality,

$$\left\|Q\left(f_\varepsilon, f_\varepsilon\right)\right\|_{L^r} \le \left(\left\|Q\left(f_\varepsilon, f_\varepsilon\right)\right\|_{L^p} + \left\|\nabla Q\left(f_\varepsilon, f_\varepsilon\right)\right\|_{L^p}\right)^a \cdot \left\|Q\left(f_\varepsilon, f_\varepsilon\right)\right\|_{L^p}^{1-a} \quad (16)$$

where

$$\frac{1}{r} = a\left(\frac{1}{p} - \frac{1}{3}\right) + \frac{1-a}{p},\ 0 \le a \le 1$$

By the translation invariance of Q, it is easily to show that

$$\nabla Q\left(f_\varepsilon, f_\varepsilon\right) = Q\left(\nabla f_\varepsilon, f_\varepsilon\right) + Q\left(f_\varepsilon, \nabla f_\varepsilon\right)$$

So making us of Prop. 2 again

$$\left\|\nabla Q\left(f_\varepsilon, f_\varepsilon\right)\right\|_{L^p} \le C\left(\|f_\varepsilon\|_{L^1}\|\nabla f_\varepsilon\|_{L^p} + \|f_\varepsilon\|_{L^p}\|\nabla f_\varepsilon\|_{L^1}\right). \quad (17)$$

By (12), it gives

$$\|\nabla f_\varepsilon\|_{L^1} \le \|G_t\|_{L^1}\|\nabla\varphi\|_{L^1} + \int_0^t \|G_{t-s}\|_{L^1}\left\|\nabla Q\left(f_\varepsilon, f_\varepsilon\right)\right\|_{L^1} ds$$

$$\le \|\nabla\varphi\|_{L^1} + C\int_0^t \|\nabla f_\varepsilon\|_{L^1} ds \quad (18)$$

that is

$$\|\nabla f_\varepsilon\|_{L^1} \le C\left(T, \|\nabla\varphi\|_{L^1}\right) \quad (19)$$

Plugging (19) into (17), gives

$$\left\|\nabla Q\left(f_\varepsilon, f_\varepsilon\right)\right\|_{L^p} \le C\left(\|\nabla f_\varepsilon\|_{L^p} + \|f_\varepsilon\|_{L^p}\right). \quad (20)$$

By (20) and (16),

$$\left\|Q\left(f_\varepsilon, f_\varepsilon\right)\right\|_{L^r} \le C\left(\|f_\varepsilon\|_{L^p} + \|\nabla f_\varepsilon\|_{L^p}\right)^a \|f_\varepsilon\|_{L^p}^{1-a}$$

$$\le C\left(\|f_\varepsilon\|_{L^p}^a + \|\nabla f_\varepsilon\|_{L^p}^a\right)\|f_\varepsilon\|_{L^p}^{1-a} \quad (21)$$

Combining (15), (21) and (13), we deduce

$$\|\nabla f_\varepsilon\|_{L^p} \le \|\nabla\varphi\|_{L^p} + C\int_0^t (t-s)^{-\frac{1}{2}}\left(\|f_\varepsilon\|_{L^p}^a + \|\nabla f_\varepsilon\|_{L^p}^a\right)\|f_\varepsilon\|_{L^p}^{1-a} ds \quad (22)$$

According to the Lemma 3, one has

$$\|\nabla f_\varepsilon\|_{L^p} \le \|\nabla\varphi\|_{L^p} + C\int_0^t (t-s)^{-\frac{1}{2}} ds + C\int_0^t (t-s)^{-\frac{1}{2}}\|\nabla f_\varepsilon\|_{L^p}^a ds. \quad (23)$$

By Gronwall type inequality we obtain the desired result.

Next using the basic theory of parabolic equation and the *a priori* estimate above, we have the following theorem.

Theorem 5. Given $\varepsilon > 0$. Let $\varphi \in W^{1,p}\left(R^3\right)$, $1 \le p < \infty$ and satisfy (5). Then for any $0 < T < \infty$ the Cauchy problem (1) and (4) exists a unique mild solution f_ε such that

$$f_\varepsilon \in C\left(\left[0,T\right]; W^{1,p}\left(R^3\right)\right) \cap C^\infty\left(\left[0,T\right] \times R^3\right)$$

4. $W^{2,p}$ Estimate and C^0 Approximation

In this section we shall make $W^{2,p}$ estimate on f_ε and deduce the explicit estimate on the viscosity approximation.

Theorem 6. Given $\varepsilon > 0$, $2 < p < \infty$. Then for any $\varphi \in W^{2,p}\left(R^3\right)$ satisfying (5) the mild solution f_ε of the Cauchy problem (1) and (4) belongs to $C\left(\left[0,\infty\right); W^{2,p}\left(R^3\right)\right)$.

Proof By the equation of (1), one has

$$\frac{\partial}{\partial t}\left(\Delta f_\varepsilon\right) = \varepsilon\Delta\left(\Delta f_\varepsilon\right) + \Delta Q\left(f_\varepsilon, f_\varepsilon\right). \quad (24)$$

Thus

$$\frac{1}{p}\frac{d}{dt}\|\Delta f_\varepsilon\|_{L^p}^p = \int_{R^3} \operatorname{sgn}\left(\Delta f_\varepsilon\right)\left|\Delta f_\varepsilon\right|^{p-1}\partial_t\left(\Delta f_\varepsilon\right)dv$$

$$= \int_{R^3} \operatorname{sgn}\left(\Delta f_\varepsilon\right)\left|\Delta f_\varepsilon\right|^{p-1}\left(\varepsilon\Delta\left(\Delta f\right)_\varepsilon + \Delta Q\left(f_\varepsilon, f_\varepsilon\right)\right)dv$$

$$\equiv I_1 + I_2 \quad (25)$$

Next we estimate I_1 and I_2 respectively.

$$I_1 = \int_{R^3} \varepsilon \operatorname{sgn}\left(\Delta f_\varepsilon\right)\left|\Delta f_\varepsilon\right|^{p-1}\Delta\left(\Delta f_\varepsilon\right)dv$$

$$= -\int_{R^3} \varepsilon\left(p-1\right)\left|\Delta f_\varepsilon\right|^{p-2}\left|\nabla\left(\Delta f_\varepsilon\right)\right|^2 dv \quad (26)$$

and

$$I_2 = \int_{R^3} \operatorname{sgn}\left(\Delta f_\varepsilon\right)\left|\Delta f_\varepsilon\right|^{p-1}\Delta Q\left(f_\varepsilon, f_\varepsilon\right)dv$$

$$= -\int_{R^3}\left(p-1\right)\left|\Delta f_\varepsilon\right|^{p-2}\nabla\left(\Delta f_\varepsilon\right)\cdot\nabla Q\left(f_\varepsilon, f_\varepsilon\right)dv$$

$$= -\int_{R^3}\left(p-1\right)\left|\Delta f_\varepsilon\right|^{\frac{p-2}{2}}\nabla\left(\Delta f_\varepsilon\right)\cdot\nabla Q\left(f, f\right)\left|\Delta f\right|^{\frac{p-2}{2}} dv \quad (27)$$

By Young's inequality, for any $\zeta > 0$,

$$\left|I_2\right| \le \int_{R^3}\left(p-1\right)\zeta\left|\Delta f_\varepsilon\right|^{p-2}\left|\nabla\left(\Delta f_\varepsilon\right)\right|^2 dv + \int_{R^3}\frac{p-1}{4\zeta}\left|\nabla Q\left(f_\varepsilon, f_\varepsilon\right)\right|^2\left|\Delta f_\varepsilon\right|^{p-2} dv \quad (28)$$

Employing Young's inequality again, the second term of the above formulation can be estimated by

$$\frac{p-1}{2p\zeta}\int_{R^3}\left|\nabla Q\left(f_\varepsilon, f_\varepsilon\right)\right|^p dv + \frac{\left(p-1\right)\left(p-2\right)}{4p\zeta}\int_{R^3}\left|\Delta f_\varepsilon\right|^p dv \quad (29)$$

Taking $\zeta = \frac{\varepsilon}{4}$, and plugging (28), (29) and (26) into (25), it gives

$$\frac{1}{p}\frac{d}{dt}\left\|\Delta f_\varepsilon\right\|_{L^p}^p \le \frac{3(1-p)\varepsilon}{4}\int_{R^3}\left|\Delta f_\varepsilon\right|^{p-2}\left|\nabla\left(\Delta f_\varepsilon\right)\right|^2 dv$$
$$+\frac{2(p-1)}{p\varepsilon}\int_{R^3}\left|\nabla Q\left(f_\varepsilon, f_\varepsilon\right)\right|^p dv+\frac{(p-1)(p-2)}{p\varepsilon}\int_{R^3}\left|\Delta f_\varepsilon\right|^p dv \tag{30}$$

By (20) and Gronwall's lemma, and the Schauder theory, we conclude the desired result.

Now, we consider whether the mild solution of the Cauchy problem (1) and (4) converges to the solution of (3) and (4) in C^0-norm as $\varepsilon \to 0^+$. The following theorem is our main result.

Theorem 7. Let $\varphi \in C^2 \bigcap W^{2,p}(R^3)$ and satisfying (5), $2 < p < \infty$. For any $T > 0$ and $\varepsilon > 0$, set f_ε is the mild solution of the viscosity equation

$$\begin{cases}\partial_t f_\varepsilon = \varepsilon\Delta f_\varepsilon + Q\left(f_\varepsilon, f_\varepsilon\right) \ in \left[0,T\right]\times R^3 \\ f_\varepsilon|_{t=0} = \varphi \quad in \ R^3\end{cases} \tag{31}$$

and f is the solution of

$$\begin{cases}\partial_t f = Q\left(f, f\right) \quad in \ \left[0,T\right]\times R^3 \\ f|_{t=0} = \varphi \quad in \ R^3\end{cases} \tag{32}$$

Then,

$$\left|f_\varepsilon\left(t,\cdot\right) - f\left(t,\cdot\right)\right|_0 \le \varepsilon A e^{kt} \tag{33}$$

where A and k are constants independent of ε. Furthermore, for any $\varepsilon > 0$

$$\left|f_\varepsilon\left(t,\cdot\right) - f\right|_0 \le \varepsilon \tag{34}$$

if

$$0 \le t \le \min\left\{\frac{1}{k}\log\left(\frac{\varepsilon}{\varepsilon A}\right), T\right\}$$

Proof By the theorem above and the result of spatially homogenous Boltzmann equation, we know that $f_\varepsilon\left(t,v\right),\ f \in W^{2,p}\left(R^3\right)$.

Let $w = f_\varepsilon - f$, then

$$\frac{\partial}{\partial t}w = \varepsilon\Delta f_\varepsilon + Q\left(f_\varepsilon, f_\varepsilon\right) - Q\left(f, f\right)$$

Therefore,

$$\frac{\partial}{\partial t}|w| = \operatorname{sgn}\left(w\right)\varepsilon\Delta f_\varepsilon + \operatorname{sgn}\left(w\right)\cdot \left(Q\left(f_\varepsilon, f_\varepsilon\right) - Q\left(f, f\right)\right). \tag{35}$$

Noting that

$$\left|Q\left(f_\varepsilon, f_\varepsilon\right) - Q\left(f, f\right)\right|_0$$
$$= \left|Q\left(f_\varepsilon, f_\varepsilon - f\right) + Q\left(f_\varepsilon - f, f\right)\right|_0$$
$$\le C\left|f_\varepsilon - f\right|_0\left(\left\|f_\varepsilon\right\|_{L^1} + \left\|f\right\|_{L^1}\right) \tag{36}$$

By the estimate on $\left\|f_\varepsilon\right\|_{L^1}$ in Section 2, we know the estimate (36) is uniform in ε. Therefore

$$\left|Q\left(f_\varepsilon, f_\varepsilon\right) - Q\left(f, f\right)\right| \le C\left|w\right|_0 \tag{37}$$

Next, we estimate $\left|f_\varepsilon\right|_2$. Indeed, by (6) for any multi-indices α, $\left|\alpha\right| = 2$, we have

$$\left|D^\alpha f_\varepsilon\right|_0 \le \left|\int_{R^3} G\left(t,\xi\right) D^\alpha \varphi\left(v-\xi\right) d\xi\right|_0$$
$$+\left|\int_0^t\int_{R^3} G\left(t-s,\xi\right) D^\alpha Q\left(f_\varepsilon, f_\varepsilon\right)\left(s, v-\xi\right) ds d\xi\right|_0$$
$$\equiv I_1 + I_2 \tag{38}$$

noting that $\int_{R^3} G\left(t,v\right) dv = 1$, $I_1 \le \left|\varphi\right|_2$. Making use of Leibniz formula, for $\left|\alpha\right| = 2$

$$D^\alpha Q\left(f_\varepsilon, f_\varepsilon\right) = \sum_{|\mu|\le 2}\binom{\alpha}{\mu} Q\left(D^\mu f_\varepsilon, D^{\alpha-\mu} f_\varepsilon\right) \tag{39}$$

Thus

$$I_2 \le C\int_0^t\left(\left|f_\varepsilon\right|_2\left\|f_\varepsilon\right\|_{L^1} + \left\|f_\varepsilon\right\|_{W^{1,1}}^2\right) ds. \tag{40}$$

Noting that

$$\left\|\nabla f_\varepsilon\right\|_{L^1} \le \left\|\nabla\varphi\right\|_{L^1} + \int_0^t\left\|\nabla Q\left(f_\varepsilon, f_\varepsilon\right)\right\|_{L^1} ds$$

So,

$$\left|f_\varepsilon\right|_2 \le \left|\varphi\right|_2 + C\int_0^t\left|f_\varepsilon\right|_2 ds + CT \tag{41}$$

By Gronwall's inequality we can deduce the bound of $\left|f_\varepsilon\right|_2$ and the bound is independent of ε.

Together these estimate with (35) we have

$$\frac{\partial}{\partial t}\left|w\right|_0 \le 3\varepsilon\left|f_\varepsilon\right|_2 + \left|Q\left(f_\varepsilon, f_\varepsilon\right) - Q\left(f, f\right)\right|_0$$
$$\le C\varepsilon + C\left|w\right|_0 \tag{42}$$

Therefore,

$$\left|w\left(t\right)\right|_0 \le \varepsilon A e^{kt}$$

This finishes the proof of Theorem 7.

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