Applied Mathematics, **2010, 1, 101-103**
doi:10.4236/am.2010.12013 Published Online July 2010 (http://www. SciRP.org/journal/am)

Bondage Number of 1-Planar Graph*

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Received May 8, 2010; *revised June* 1, 2010; *accepted June* 11, 2010

Abstract

The bondage number $b(G)$ of a nonempty graph *G* is the cardinality of a smallest set of edges whose removal from *G* results in a graph a domination number greater than the domination number of *G*. In this paper, we prove that $b(G) \leq 12$ for a 1-planar graph *G*.

Keywords: Domination Number, Bondage number, 1-Planar Graph, Combinatorial Problem

1. Introduction

Throughout this paper, we consider connected graphs without loops or multiple edges. A 1-planar graph is a graph which can be drawn on the plane so that every edge crosses at most one other edge. For a graph *G*, $V(G)$ and $E(G)$ are used to denote the vertex set and edge set of *G*, respectively. The degree of a vertex *u* in *G* is denoted by $d(u)$. For a vertex subset $S \subseteq V(G)$, define $N(S) = \{x \in V(G) \setminus S \mid$ there is a $y \in S$ such that $xy \in E(G)\}$. When $S = \{v\}$, we write $N(v) = N(S)$ for short. The minimum degree of vertices in *G* is denoted by $\delta(G)$ and the maximum degree by $\Delta(G)$. The distance between two vertices *u* and *v* in *G* is denoted by $d(u,v)$. For a subset $X \subseteq V(G)$, $G[X]$ denotes the subgraph of *G* induced by *X*.

A subset *D* of $V(G)$ is called a dominating set, if $D \cup N(D) = V(G)$. The domination number of *G*, denoted by $\gamma(G)$, is the minimum cardinality of a dominating set. The bondage number $b(G)$ of a nonempty graph *G* is the cardinality of a smallest set of edges whose removal from *G* results in a graph with domination number greater than $\gamma(G)$.

The bondage number was first introduced by Bauer *et al*. [1] in 1983. The following two main outstanding conjectures on bondage number were formulated by Teschner [2].

Conjecture 1.1 If *G* is a planar graph, then $b(G) \leq \Delta(G) + 1$.

Conjecture 1.2 For any graph *G*, $b(G) \leq \frac{3}{2}\Delta(G)$.

In 2000, Kang and Yuan [3] proved that $b(G) \leq \min\{8, \Delta(G) + 2\}$ for any planar graph. That is, conjecture 1.1 is showed for planar graph with $\Delta(G) \geq 7$. Up to now, conjecture 1.1 is still open for planar graph *G* with $\Delta(G) \leq 6$. Conjecture 1.2 is still open. In this paper, we prove that $b(G) \leq 12$ for a 1-planar graph *G*.

2. Preliminary Results

First of all, we recall some useful results that we will need

Lemma 2.1 [4] If *G* is a graph, then for every pair of adjacent vertices *u* and *v* in *G*, then $b(G) \leq d(u) + d(v) - 1 - |N(u) \cap N(v)|$.

Lemma 2.2 [5,6] If *u* and *v* are two vertices of *G* with $d(u,v) \leq 2$, then $b(G) \leq d(u) + d(v) - 1$.

Lemma 2.3 [7] Let *G* be a 1-planar graph, then $\delta(G) \leq 7$.

Lemma 2.4 [7] Let *G* be a 1-planar graph on *n* vertices and *m* edges, then $m \leq 4n - 8$.

Lemma 2.5 Let *G* be a bipartite 1-planar graph on *n* vertices and *m* edges, then $m \leq 3n - 6$.

Proof. Without loss of generality, let *G* be a maximal bipartite 1-planar graph on *n* vertices and *X*, *Y* is a bipartition of graph *G*. Form a 1-planar graph G' from *G* as follows: add some edges to join vertices in *X*, and add some edges to join the vertices in *Y*, such that G' is a maximal 1-planar graph with $G \subseteq G'$. By the maximal-

*This work was supported by NSFC (10901097) and the Nature Science Foundation of Shandong Province (Y2008A20), also was supported by the Scientific Research and Development Project of Shandong Provincial Education Department (TJY0706) and the Science and Technology Foundation of University of Jinan (XKY0705).

ity of G', the subgraph $G'[X]$ and $G'[Y]$ must be connected. Then we have

$$|E(G'[X])| \geq |X| - 1, \quad |E(G'[Y])| \geq |Y| - 1 .$$

By lemma 2.4, $|E(G')| \leq 4n - 8$. So

$$|E(G)| = |E(G')| - |E(G'[X])| - |E(G'[Y])|$$

$$\leq 4n - 8 - |X| + 1 - |Y| + 1 = 3n - 6 .$$

This completes the prove of lemma 2.5.

3. Bondage Number of 1-Planar Graph

Theorem 3.1 If G is a 1-planar graph, then $b(G) \leq 12$.

Proof. Suppose to the contrary that G is a 1-planar graph with $b(G) \geq 13$. Then we have

Claim 1. For two distinct vertices x, y of G, if $\max\{d(x), d(y)\} \leq 7$, and $\min\{d(x), d(y)\} \leq 6$, then it must be the case that $d(x, y) \geq 3$.

Otherwise, $d(x, y) \leq 2$. But then, by lemma 2.2, $b(G) \leq d(x) + d(y) - 1 \leq 12$, a contradiction.

Claim 2. If there is some $x \in V(G)$ such that $d(x) \leq 5$ then $d(y) \geq 9$ for all $y \in N(x)$.

Otherwise, $b(G) \leq d(x) + d(y) - 1 \leq 5 + 8 - 1 = 12$, a contradiction.

Now, we define

$$V_1 = \{x \in V(G) \mid d(x) \leq 5\},$$

$$V_2 = \{x \in V(G) \mid d(x) = 6\},$$

$$V_3 = \{x \in V(G) \mid d(x) = 7\}.$$

Let $A \subseteq V_3$ be the maximum and such that A is independent of G. By Claim 1 and the maximality of A, we have also

Claim 3. $N(V_2) \bigcap N(A) = \phi$ and $V_3 \subseteq A \bigcup N(A)$,

Let $V_2 \bigcup A = \{x_1, x_2 \cdots x_k\}$, and $H = G - V_1$. Define

$$H_0 = H ,$$

$$H_i = H_{i-1} + F_i, \quad 1 \leq i \leq k.$$

where $F_i \subseteq E_{x_i} = \{xy \mid x, y \in N(x_i), x \neq y, xy \notin E(H_{i-1})\}$ such that $H_{i-1} + F_i = H_i$ is still a 1-planar graph and such that $H_i[N(x_i)]$ is 2-connected. It is easy to see that $H_k[N(x_i)]$ is still 2-connected for $1 \leq i \leq k$.

Claim 4. If $V_2 \neq \phi$, then for each vertex $v \in N(V_2)$, v is of degree at least 9 in H_k .

In fact, let $x \in V_2$ and $v \in N(x)$. If $N(v) \bigcap N(x) = \varphi$ in G, then by lemma 2.2, $d(v) + d(x) - 1 \geq 13$, and $d(v) \geq 14 - d(x) = 8$, by the 2-connectivity of $H_k[N(x)]$, v is of degree at least 10 in H_k . If $N(v) \bigcap N(x) \neq \phi$, then by lemma 2.1, $d(v) + d(x) - 2 \geq 13$. Then $d(v) \geq 9$.

Analogously, we have

Claim 5. If $A \neq \phi$, then for each vertex $v \in N(A)$, v is of degree at least 9 in H_k .

Now, $G^* = H_k - V_2$ is a 1-planar graph, satisfying

(a) The minimum degree of G^* is 7,

(b) $A = \{v \in V(G^*) \mid d_{G^*}(v) = 7\}$,

(c) A is independent of G^*,

(d) For every vertex $v \in N_{G^*}(A) = N(A), d_{G^*}(v) \geq 9$.

Let $\partial(A) = \{xy \in E(G^*) \mid x \in A, y \in N(A)\}$. Then $(A, N(A); \partial(A))$ is a bipartite 1-planar graph with $7|A|$ edges. By lemma 2.5,

$$7|A| \leq 3|A| + 3|N(A)| - 6 .$$

Hence

$$|N(A)| \geq \frac{4}{3}|A| + 2 .$$

Then we have

$$|E(G^*)| = \frac{1}{2} \sum_{v \in V(G^*)} d_{G^*}(v)$$

$$\geq \frac{1}{2}(7|A| + 9|N(A)| + 8(|V(G^*)| - |A| - |N(A)|))$$

$$= 4|V(G^*)| + \frac{1}{2}|N(A)| - \frac{1}{2}|A|$$

$$\geq 4|V(G^*)| + \frac{1}{6}|A| + 1$$

$$> 4|V(G^*)| - 8 .$$

A contradiction.

This completes the proof of the theorem.

Theorem 3.2 If G is a 1-planar graph and there is no degree seven vertex, then $b(G) \leq 11$.

Proof. Suppose to the contrary that $b(G) \geq 12$.

Let $X = \{v \in V(G) \mid d(v) \leq 6\}$, and suppose that $X = \{x_1, x_2 \cdots x_k\}$

By lemma 2.2, for any two distinct vertices $x, y \in X$, $d(x, y) \geq 3$.

Define

$$H_0 = G ,$$

$$H_i = H_{i-1} + F_i, \quad 1 \leq i \leq k .$$

where $F_i \subseteq E_{x_i} = \{xy \mid x, y \in N(x_i), x \neq y, xy \notin E(H_{i-1})\}$ such that $H_{i-1} + F_i = H_i$ is still a 1-planar graph and such that $H_i[N(x_i)]$ is 2-connected when $d(x_i) \geq 3$.

Now, by lemma 2.1 and 2.2, for any $x \in X, y \in N(y)$, if $d(x) \leq 2$ then $d(y) \geq 11$ and so y is of degree at least 11 in H_k; If $d(x) \geq 3$ and $|N(x) \cap N(y)| \leq 1$, then $d(y) \geq 8$ and so y is of degree at least 9 in H_k; If $d(x) \geq 3$ and $|N(x) \cap N(y)| \geq 2$, then $d(y) \geq 9$ and So y is of degree at least 9 in H_k.

By the construction of H_k, we know that H_k is a 1-planar graph. But $H_k - X$ is a 1-planar graph with a minimum degree of at least 8. It contradicts with lemma 2.3, and the proof is completed.

4. References

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