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Hyperbolic Fibonacci and Lucas Functions, "Golden" Fibonacci Goniometry, Bodnar's Geometry, and Hilbert's Fourth Problem

——Part I. Hyperbolic Fibonacci and Lucas Functions and "Golden" Fibonacci Goniometry

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Abstract

This article refers to the "*Mathematics of Harmony*" by Alexey Stakhov in 2009, a new interdisciplinary direction of modern science. The main goal of the article is to describe two modern scientific discoveries—*New Geometric Theory of Phyllotaxis* (*Bodnar's Geometry*) and *Hilbert's Fourth Problem* based on the *Hyperbolic Fibonacci and Lucas Functions* and "Golden" Fibonacci λ*-Goniometry* ($\lambda > 0$ is a given positive real number). Although these discoveries refer to different areas of science (mathematics and theoretical botany), however they are based on one and the same scientific ideas—the "*golden mean*", which had been introduced by Euclid in his *Elements*, and its generalization—the "*metallic means*", which have been studied recently by Argentinian mathematician Vera Spinadel. The article is a confirmation of interdisciplinary character of the "*Mathematics of Harmony*", which originates from Euclid's Elements.

Keywords: Euclid's Fifth Postulate, Lobachevski's Geometry, Hyperbolic Geometry, Phyllotaxis, Bodnar's Geometry, Hilbert's Fourth Problem, The "Golden" and "Metallic" Means, Binet Formulas, Hyperbolic Fibonacci and Lucas Functions, Gazale Formulas, "Golden" Fibonacci λ-Goniometry

1. Introduction

In the second half of 20 century the interest in the "Golden Section" and Fibonacci numbers increased in modern mathematics very fastly [1-4]. In 2009 the International Publishing House "World Scientific" published the book "The Mathematics of Harmony. From Euclid to Contemporary Mathematics and Computer Science" by Alexey Stakhov [5]. The book is a reflection of very important tendency of modern science—a revival of the interest in the Pythagorean Doctrine on the Numerical Harmony of the Universe, "Golden Mean" and Platonic Solids.

Many original mathematical results were obtained in the framework of the mathematics of harmony [5]. Possibly, the *hyperbolic Fibonacci and Lucas functions* [6-8] and "*golden*" *Fibonacci goniometry* [9] are the most important of them.

The main goal of the present article is to describe in brief form a theory of the hyperbolic Fibonacci and Lucas functions, and "golden" Fibonacci goniometry and to show their effectiveness for the solution of Hilbert's Fourth Problem [10] and the creation of new geometric theory of phyllotaxis (Bodnar's geometry) [4].

The article consists of three parts:

Part I. Hyperbolic Fibonacci and Lucas Functions and "Golden" Fibonacci Goniometry

Part II. A New Geometric Theory of Phyllotaxis (Bodnar's Geometry)

Part III. An Original Solution of Hilbert's Fourth Problem

2. Hyperbolic Fibonacci and Lucas Functions

2.1. The Golden Mean, Fibonacci and Lucas Numbers and Binet Formulas

A problem of the *Golden Section* came to us from Euclid's *Elements*. We are talking about the problem of the

division of line segment in extreme and mean ratio (Theorem II.11 of Euclid's *Elements*). In modern science this problem is named the *Golden Section* [1-4]. A solution to this problem is reduced to the simplest algebraic equation:

$$x^2 - x - 1 = 0 \tag{1.1}$$

A positive root of the Equation (1.1)

$$\Phi = \frac{1+\sqrt{5}}{2} \tag{1.2}$$

is the famous irrational number called *golden number, golden mean, golden proportion, divine proportion* and so on.

The algebraic Equation (1.1) and the *golden mean* (1.2) are connected closely with two remarkable numerical sequences—*Fibonacci numbers* $F(n)$ and *Lucas numbers* $L(n)$ given by the recurrence relations:

$$F(n+2) = F(n+1) + F(n);\ F(0) = 0,\ F(1) = 1 \tag{1.3}$$

$$L(n+2) = L(n+1) + L(n);\ L(0) = 2,\ L(1) = 1 \tag{1.4}$$

where $n = 0, \pm 1, \pm 2, \pm 3, \cdots$.

The arbitrary three adjacent Fibonacci numbers $F(n-1)$, $F(n), F(n+1)$ ($n = 0, \pm 1, \pm 2, \pm 3, \cdots$) are connected between themselves with the following mathematical identity:

$$F^2(n) - F(n-1)F(n+1) = (-1)^{n+1}. \tag{1.5}$$

The formula (1.5) is called *Cassini formula* after the French astronomer Giovanni Domenico Cassini (1625-1712) who deduced this formula for the first time.

In 19th century the French mathematician Jacques Philippe Marie Binet (1786-1856) deduced two remarkable formulas, which connect Fibonacci and Lucas numbers with the golden mean:

$$F(n) = \frac{\Phi^n - (-1)^n \Phi^{-n}}{\sqrt{5}};\quad L(n) = \Phi^n + (-1)^n \Phi^{-n} \tag{1.6}$$

Note that these formulas were discovered by de Moivre (1667-1754) and Nikolai Bernoulli (1687-1759) a one century before Binet. However, in modern mathematical literature these formulas are called *Binet formulas.*

2.2. Hyperbolic Fibonacci and Lucas Functions and a New Comprehension of the "Golden Mean" Role in Modern Science

Unfortunately, mathematicians of 19th and 20th century could not evaluate the true value of *Binet formulas*, although these formulas contained a hint on the important mathematical discovery—*hyperbolic Fibonacci and Lucas functions*.

In 1984 Alexey Stakhov published the book *Codes of the Golden Proportion* [3]. In this book *Binet formulas* (1.6) were represented in the form, which was not used earlier in mathematical literature:

$$F(n) = \begin{cases} \dfrac{\Phi^n + \Phi^{-n}}{\sqrt{5}}, & n = 2k+1 \\ \dfrac{\Phi^n - \Phi^{-n}}{\sqrt{5}}, & n = 2k \end{cases};$$
$$L(n) = \begin{cases} \Phi^n + \Phi^{-n}, & n = 2k \\ \Phi^n - \Phi^{-n}, & n = 2k+1 \end{cases} \tag{1.7}$$

A similarity of *Binet formulas*, presented in the form (1.7), in comparison with the hyperbolic functions

$$sh(x) = \frac{e^x - e^{-x}}{2},\ ch(x) = \frac{e^x + e^{-x}}{2}, \tag{1.8}$$

is so striking that the formulas (1.7) can be considered as a prototype of a new class of hyperbolic functions based on the golden mean, that is, Alexey Stakhov already in 1984 [3] predicted the appearance of a new class of hyperbolic functions—*hyperbolic Fibonacci and Lucas functions*.

According to the recommendation of the famous Ukrainian mathematician academician Yury Mitropolsky, the article on the *hyperbolic Fibonacci and Lucas functions* was published by the Ukrainian mathematicians Alexey Stakhov and Ivan Tkachenko in the *Reports of the National Academy of Sciences of Ukraine* in 1993 [6]. More lately, Alexey Stakhov and Boris Rosin developed this idea and introduced in [7,8] the so-called *symmetric hyperbolic Fibonacci and Lucas functions*.

Symmetric hyperbolic Fibonacci sine and cosine

$$sFs(x) = \frac{\Phi^x - \Phi^{-x}}{\sqrt{5}};\ cFs(x) = \frac{\Phi^x + \Phi^{-x}}{\sqrt{5}} \tag{1.9}$$

Symmetric hyperbolic Lucas sine and cosine

$$sLs(x) = \Phi^x - \Phi^{-x};\ cLs(x) = \Phi^x + \Phi^{-x} \tag{1.10}$$

Fibonacci and Lucas numbers are determined identically with the symmetric hyperbolic Fibonacci and Lucas functions as follows:

$$F(n) = \begin{cases} sFs(n), & n = 2k \\ cFs(n), & n = 2k+1 \end{cases};\ L(n) = \begin{cases} sLs(n), & n = 2k+1 \\ cLs(n), & n = 2k \end{cases}$$

The symmetric hyperbolic Fibonacci and Lucas functions (1.9) and (1.10) are connected among themselves by the following simple correlations:

$$sF(x) = \frac{sLs(x)}{\sqrt{5}};\ cFs(x) = \frac{cLs(x)}{\sqrt{5}}.$$

Note that the hyperbolic Fibonacci functions (1.9) and

(1.10) own the following unique mathematical properties:

$$\begin{aligned}\left[sFs(x)\right]^2 - cFs(x+1)cFs(x-1) &= -1\\ \left[cFs(x)\right]^2 - sFs(x+1)sFs(x-1) &= 1\end{aligned} \tag{1.11}$$

Independently on Stakhov, Tkachenko and Rosin, the Ukrainian researcher Oleg Bodnar came to the same ideas. He has introduced in [4] the so-called "golden" hyperbolic functions, which are different from hyperbolic Fibonacci and Lucas functions only constant coefficients. By using the "golden" hyperbolic functions, Bodnar created a new geometric theory of phyllotaxis in [4], where he showed that his "geometry of phyllotaxis" is a new variant of Non-Euclidean geometry based on the "golden" hyperbolic functions.

Thus, the works of Bodnar, Stakhov, Tkachenko and Rosin [4,6-8] can be considered as a contemporary breakthrough of "hyperbolic ideas" into theoretical natural sciences. First of all, in the works [6-8] a new class of hyperbolic functions based on Binet formulas (1.10) and (1.11) was developed. On the other hand, in Bodnar's book [4] it was shown that these functions are of direct relationship to botanical phenomenon of phyllotaxis (pine cones, cacti, pineapples, sunflowers, baskets of flowers, etc.), that is, the hyperbolic Fibonacci and Lucas functions lie in the base of important natural phenomenon called *phyllotaxis*.

However, the most important result of this study is comprehension of a new role of the golden mean in the structures of Nature. Obviously, the golden mean and the related to it Fibonacci and Lucas numbers are expressing "hidden harmony" of Nature, the essence of which is expressed in its hyperbolic character. Thus, the discovery of the golden mean or Fibonacci numbers in some natural phenomenon is a very clear signal that the geometric character of this phenomenon is hyperbolic.

3. Fibonacci and Lucas λ-Numbers and Metallic Means

A general theory of the hyperbolic Fibonacci and Lucas λ-functions are stated in [9]. That is why, we restrict ourselves to brief statement of mathematical results obtained in [9].

Let's give a positive real number $\lambda > 0$ and consider the following recurrence relation:

$$\begin{aligned}&F_\lambda(n+2) = \lambda F_\lambda(n+1) + F_\lambda(n);\\ &F_\lambda(0) = 0,\ F_\lambda(1) = 1.\end{aligned} \tag{1.12}$$

For the case $\lambda = 1$ the recurrence formula (1.12) is reduced to the recurrence relation (1.3) given the classical Fibonacci numbers. Based on this analogy, we will name the numerical sequences generated by more general recurrence relation (1.12) the *Fibonacci λ-numbers.*

Now let's represent the recurrence relation (1.12) in the form:

$$\frac{F_\lambda(n+2)}{F_\lambda(n+1)} = \lambda + \frac{1}{\dfrac{F_\lambda(n+1)}{F_\lambda(n)}} \tag{1.13}$$

For the case $n \to \pm\infty$ the expression (1.13) is reduced to the quadratic equation

$$x^2 - \lambda x - 1 = 0 \tag{1.14}$$

with the roots

$$x_1 = \frac{\lambda + \sqrt{4+\lambda^2}}{2} \quad \text{and} \quad x_2 = \frac{\lambda - \sqrt{4+\lambda^2}}{2} \tag{1.15}$$

For the proof of the Equation (1.14) let us consider auxiliary *point transformation*

$$\overline{s} = f(s) = \lambda + \frac{1}{s} \tag{1.16}$$

where *s* can take arbitrary real values different from zero, here at $s \to 0+0$ $\overline{s} \to +\infty$, and at $s \to 0-0$ $\overline{s} \to -\infty$.

In particular, if we take $s = \left(F_\lambda(n+1)\right)/\left(F_\lambda(n)\right)$, than by comparing (1.16) and (1.13), we get

$$\overline{s} = f(s) = \left(F_\lambda(n+2)\right)/\left(F_\lambda(n+1)\right).$$

Geometrically the fixed point of the transformation (1.16) can be obtained at the intersection of the curve (1.16) with the bisector $\overline{s} = s$, and algebraically they can be obtained as roots of the equation:

$$\lambda + \frac{1}{s} = s \tag{1.17}$$

The transformations (1.16) have exactly two fixed points $s^* = x_1$, $s^{**} = x_2$ of the kind (1.16), and hence they are the roots of the square Equation (1.14).

This consideration allows determining two characteristic fixed points—*attractive* and *repelling*. The *attractive point,* denoted by ξ, is a *limiting point* for the iterations $s_k = f^k(s)$, where *k* is a number of iterations. At $k \to +\infty$, all initial points *s* belong to any neighborhood $U(\xi)$ of the point ξ. The *repelling point* is a *limiting point* for the iterations $s_{-k} = f^{-k}(s)$, where *k* is a number of iterations. At $k \to +\infty$, all initial points *s* belong to any neighborhood $U(\xi)$ of the point ξ.

Note that in literature the *attractive* and *repelling* isolated fixed points are called *zero-dimensional attractor* and *zero-dimensional repeller*, respectively.

Let us denote by $f_s'(\xi)$ the first derivative on *s* of the function $f(s)$ at the fixed point ξ. It is proved in [11-13], that a sufficient condition for the fixed point ξ of the transformation $\overline{s} = f(s)$ to be *attractive* or *repelling* are the following inequalities for the derivative

$f_s'(\xi)$, respectively:

$$\left|f_s'(\xi)\right|<1 \quad \text{or} \quad \left|f_s'(\xi)\right|>1 \tag{1.18}$$

This sufficient indication is called *Kenig's theorem* [11-13].

Note, that in [11-13] the more old terms "*stable* and *unstable fixed points*" are used instead of the terms *attractor* and *repeller.*

Figure 1 demonstrates geometric picture for obtaining the fixed points in the transformation (1.16) for the case $\lambda = 2$.

Direct calculation at $\lambda = 2$ gives the following results. The transformation (1.16) has two fixed points:

1) the point $s^* = 2.41421$ is *attractive*, because $\left|f_s'(s^*)\right| = 0.171573 < 1$;

2) the point $s^{**} = -0.41421$ is *repelling*, because $\left|f_s'(s^{**})\right| = 5.82843 > 1$.

In general case for any fixed λ ($0 < \lambda < +\infty$), as it is stated above, the transformation (1.16) has two fixed points $s^* = x_1$ and $s^{**} = x_2$ of the kind (1.15); at that, because $\left|f_s'(s)\right| = 1/s^2$, then after simple calculations we get that for the fixed point $s^* = x_1 = \left(\lambda + \sqrt{4+\lambda^2}\right)/2$ we have $\left|f_s'(s^*)\right| < 1$, that is, the fixed point s^* is *attractive,* and hence

$$\lim_{k\to+\infty} f^k(s) = s^* \tag{1.19}$$

For the fixed point $s^{**} = x_2 = \left(\lambda - \sqrt{4+\lambda^2}\right)/2$ we have: $\left|f_s'(s^{**})\right| > 1$, that is, the fixed point s^{**} is *repelling* and hence

$$\lim_{k\to+\infty} f^{-k}(s) = s^{**} \tag{1.20}$$

Note that for the first case the initial point *s* should be chosen for any neighborhood $U(s^*)$ of *attractive* fixed point *s**; in the second case the initial point *s* should be chosen for any neighborhood $U(s^{**})$ of *repelling* fixed point s^{**}.

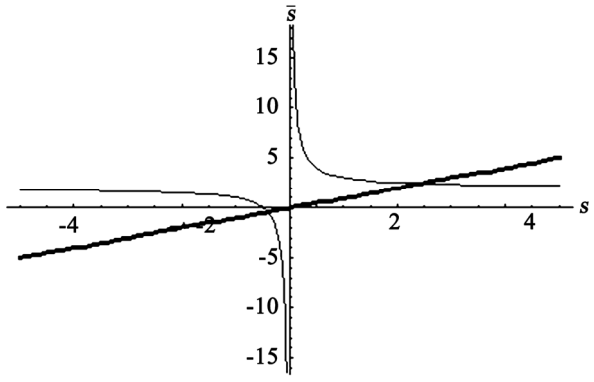

Figure 1. A geometric picture for obtaining the fixed points in the transformation (1.16) for the case $\lambda = 2$.

The graphs of the functions

$$h^*(\lambda) = \left|f_s'(s^*)\right| = \frac{4}{\left(\lambda + \sqrt{4+\lambda^2}\right)^2}$$

and

$$h^{**}(\lambda) = \left|f_s'(s^{**})\right| = \frac{4}{\left(\lambda - \sqrt{4+\lambda^2}\right)^2}$$

for all $\lambda > 0$ are represented in **Figures 2** and **3**. It follows from these figures that $h^*(\lambda) = \left|f_s'(s^*)\right| < 1$ and $h^{**}(\lambda) = \left|f_s'(s^{**})\right| > 1$.

Then, if we take the ratio $s = \left(F_\lambda(2)\right)/\left(F_\lambda(1)\right)$ as the initial value, by virtue of (1.13), we get the following iterations:

$$f^1(s) = \frac{F_\lambda(3)}{F_\lambda(2)},\ f^2(s) = \frac{F_\lambda(4)}{F_\lambda(3)},\ \cdots,\ f^k(s) = \frac{F_\lambda(k+2)}{F_\lambda(k+1)}, \tag{1.21}$$

Assume that $n = k+1$, then, taking into consideration (1.21), we get from (1.19):

$$\lim_{n\to+\infty} \frac{F_\lambda(n+1)}{F_\lambda(n)} = s^* = x_1 = \frac{\lambda + \sqrt{4+\lambda^2}}{2}. \tag{1.22}$$

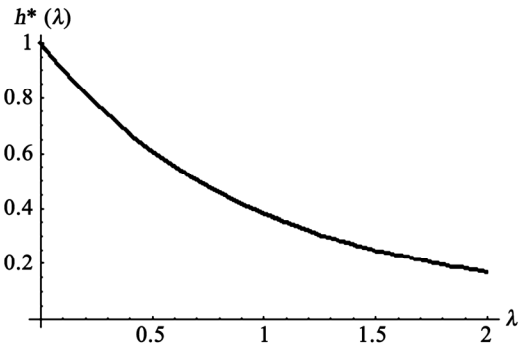

Figure 2. Graph of the function $h^*(\lambda) = \left|f_s'(s^*)\right|$.

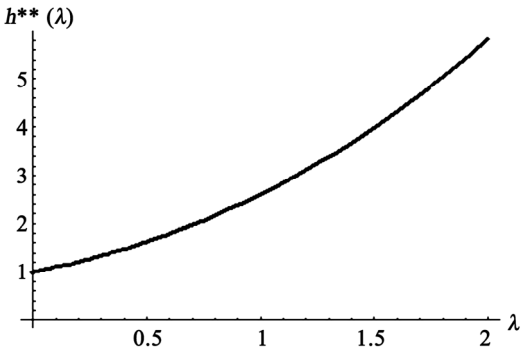

Figure 3. Graph of the function $h^{**}(\lambda) = \left|f_s'(s^{**})\right|$.

By analogy, if we take into consideration (1.21), we get:

$$\lim_{n\to+\infty}\frac{F_\lambda(-n)}{F_\lambda(-n-1)}=s^{\bullet\bullet}=x_2=\frac{\lambda-\sqrt{4+\lambda^2}}{2}. \quad (1.23)$$

Let us denote a positive root x_1 by Φ_λ and consider a new class of mathematical constants given by the following formula:

$$\Phi_\lambda=\frac{\lambda+\sqrt{4+\lambda^2}}{2}. \quad (1.24)$$

Note that for the case $\lambda=1$ the formula (1.24) takes the form (1.2) given the classical golden mean.

The Argentinean mathematician Vera W. Spinadel [14] named the mathematical constants generated by (1.24) *metallic means*. If we take λ =1, 2, 3, 4 in (1.24), then we get the following mathematical constants having according to Vera Spinadel special names:

$$\Phi_1=\frac{1+\sqrt{5}}{2}\ (\text{the Golden Mean},\ \lambda=1);$$
$$\Phi_2=1+\sqrt{2}\ (\text{the Silver Mean},\ \lambda=2);$$
$$\Phi_3=\frac{3+\sqrt{13}}{2}\ (\text{the Bronze Mean},\ \lambda=3);$$
$$\Phi_4=2+\sqrt{5}\ (\text{the Cooper Mean},\ \lambda=4).$$

Other metallic means ($\lambda\geq 5$) do not have special names:

$$\Phi_5=\frac{5+\sqrt{29}}{2};\ \Phi_6=3+2\sqrt{10};$$
$$\Phi_7=\frac{7+2\sqrt{14}}{2};\ \Phi_8=4+\sqrt{17}.$$

It is easy to prove that the root x_2 can be represented by the metallic mean (1.24) as follows:

$$x_2=-\frac{1}{\Phi_\lambda}=\frac{\lambda-\sqrt{4+\lambda^2}}{2}. \quad (1.25)$$

By using the algebraic Equation (1.14), it is easy to prove the following remarkable algebraic properties of the metallic means (1.24):

$$\Phi_\lambda=\sqrt{1+\lambda\sqrt{1+\lambda\sqrt{1+\lambda\sqrt{\cdots}}}};\ \Phi_\lambda=\lambda+\cfrac{1}{\lambda+\cfrac{1}{\lambda+\cfrac{1}{\lambda+\cdots}}} \quad (1.26)$$

They are a generalization of the following mathematical properties of the golden mean ($\lambda=1$):

$$\Phi=\sqrt{1+\sqrt{1+\sqrt{1+\sqrt{\cdots}}}};\ \Phi=1+\cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{1+\cdots}}}$$

4. Gazale Formulas for the Fibonacci and Lucas λ-Numbers

Based on the metallic means (1.24), Midchat Gazale in [15] has deduced remarkable formula, which gives *Fibonacci λ-numbers* (1.12) in analytical form:

$$F_\lambda(n)=\frac{\Phi_\lambda^n-(-1)^n\Phi_\lambda^{-n}}{\sqrt{4+\lambda^2}} \quad (1.27)$$

where n = 0, ±1, ±2, ±3, ⋯

Alexey Stakhov in [9] has deduced the similar analytical formula for the *Lucas λ-numbers*:

$$L_\lambda(n)=\Phi_\lambda^n+(-1)^n\Phi_\lambda^{-n}, \quad (1.28)$$

where n = 0, ±1, ±2, ±3, ⋯

The formulas (1.27) and (1.28) are named in [9] *Gazale formulas* after Midchat Gazale, who first has deduced the formula (1.27) in the book [15]. Note that for the case $\lambda=1$ Gazale formulas (1.27) and (1.28) are reduced to the Binet formulas (1.6).

As is shown in [9], the Lucas λ-numbers (1.28) can be given recursively in the form

$$L_\lambda(n)=\lambda L_\lambda(n-1)+L_\lambda(n-2);\ L_\lambda(0)=2,\ L_\lambda(1)=1. \quad (1.29)$$

Note that for the case $\lambda=1$ the Lucas λ-numbers, given by the recurrence relation (1.29), are reduced to the classical Lucas numbers.

Now let us represent the *Gazale formulas* (1.27) and (1.28) for the negative values of n as follows:

$$F_\lambda(-n)=\frac{\Phi_\lambda^{-n}-(-1)^{-n}\Phi_\lambda^n}{\sqrt{4+\lambda^2}} \quad (1.30)$$

$$L_\lambda(-n)=\Phi_\lambda^{-n}+(-1)^{-n}\Phi_\lambda^n \quad (1.31)$$

Comparing the formulas (1.27) and (1.30) for the even (n = 2k) and odd (n = 2k + 1) values of n, we can conclude that

$$F_\lambda(2k)=-F_\lambda(-2k)\quad\text{and}\quad F_\lambda(2k+1)=F_\lambda(-2k-1). \quad (1.32)$$

This means that for the given positive real number $\lambda>0$ the sequence of the Fibonacci λ-numbers (1.12) in the infinite range n=0, ±1, ±2, ±3, ⋯ is a symmetrical sequence relatively to the Fibonacci λ-number $F_\lambda(0)=0$ except that the Fibonacci λ-numbers $F_\lambda(2k)$ and $F_\lambda(-2k)$ are opposite by sign.

In **Table 1** we can see the Fibonacci λ-numbers $F_\lambda(n)$ for the cases $\lambda=1, 2, 3, 4$.

Note that for the case $\lambda=2$ the *Gazale formula* (1.27) generates a numerical sequence known as *Pell numbers.*

Table 1. The Fibonacci λ-sequences $F_\lambda(n)$ for the cases $\lambda = 1, 2, 3, 4$.

λ \ n	-5	-4	-3	-2	-1	0	1	2	3	4	5
1	5	-3	2	-1	1	0	1	1	2	3	5
2	29	-12	5	-2	1	0	1	2	5	12	29
3	109	-33	10	-3	1	0	1	3	10	33	109
4	305	-72	17	-4	1	0	1	4	17	72	305

Table 2. The Lucas λ-sequences $L_\lambda(n)$ for the cases $\lambda = 1, 2, 3, 4$.

λ \ n	-5	-4	-3	-2	-1	0	1	2	3	4	5
1	-11	7	-4	3	-1	2	1	3	4	7	11
2	-82	34	-14	6	-2	2	2	6	14	34	82
3	-393	119	-36	11	-3	2	3	11	36	119	393
4	-1364	322	-76	18	-4	2	4	18	76	322	1364

Comparing the formulas (1.28) and (1.31) for the even ($n = 2k$) and odd ($n = 2k + 1$) values of n, we can conclude that

$$L_\lambda(2k) = L_\lambda(-2k) \quad \text{and} \quad L_\lambda(2k+1) = -L_\lambda(-2k-1) \tag{1.33}$$

This means that for the given positive real number $\lambda > 0$ the sequence of the Lucas λ-numbers in the range n=0, ±1, ±2, ±3, ⋯ is a symmetrical sequence relative to the Lucas λ-number $L_\lambda(0) = 2$ except that the Lucas numbers $L_\lambda(2k+1)$ and $L_\lambda(-2k-1)$ are opposite by sign.

In **Table 2** we can see the Lucas λ-numbers $L_\lambda(n)$ for the cases $\lambda = 1, 2, 3, 4$.

Note that for the case $\lambda = 2$ the *Gazale formula* (1.28) generates the numerical sequence known as *Pell-Lucas numbers.*

It is easy to deduce the following identity for the Fibonacci λ-numbers similar to the *Cassini formula* (1.5):

$$F_\lambda^2(n) - F_\lambda(n-1)F_\lambda(n+1) = (-1)^{n+1} \tag{1.34}$$

5. "Golden" Fibonacci λ-Goniometry

5.1. A Definition of the Hyperbolic Fibonacci and Lucas λ-Functions

First of all, let us explain the term of *goniometry* used in this article. As is known, a goniometry is a part of geometry, which sets relations between trigonometric functions. In this article we use instead of trigonometric functions the so-called *symmetric hyperbolic Fibonacci and Lucas λ-functions* introduced in [9]. Let us consider these functions.

Hyperbolic Fibonacci λ-sine and λ-cosine

$$sF_\lambda(x) = \frac{\Phi_\lambda^x - \Phi_\lambda^{-x}}{\sqrt{4+\lambda^2}} = \frac{1}{\sqrt{4+\lambda^2}}\left[\left(\frac{\lambda+\sqrt{4+\lambda^2}}{2}\right)^x - \left(\frac{\lambda+\sqrt{4+\lambda^2}}{2}\right)^{-x}\right] \tag{1.35}$$

$$cF_\lambda(x) = \frac{\Phi_\lambda^x + \Phi_\lambda^{-x}}{\sqrt{4+\lambda^2}} = \frac{1}{\sqrt{4+\lambda^2}}\left[\left(\frac{\lambda+\sqrt{4+\lambda^2}}{2}\right)^x + \left(\frac{\lambda+\sqrt{4+\lambda^2}}{2}\right)^{-x}\right] \tag{1.36}$$

Hyperbolic Lucas λ-sine and λ-cosine

$$sL_\lambda(x) = \Phi_\lambda^x - \Phi_\lambda^{-x} = \left(\frac{\lambda+\sqrt{4+\lambda^2}}{2}\right)^x - \left(\frac{\lambda+\sqrt{4+\lambda^2}}{2}\right)^{-x} \tag{1.37}$$

$$cL_\lambda(x) = \Phi_\lambda^x + \Phi_\lambda^{-x} = \left(\frac{\lambda+\sqrt{4+\lambda^2}}{2}\right)^x + \left(\frac{\lambda+\sqrt{4+\lambda^2}}{2}\right)^{-x} \tag{1.38}$$

where x is continuous variable and $\lambda > 0$ is a given positive real number.

The Fibonacci and Lucas λ-numbers are determined identically by the hyperbolic Fibonacci and Lucas λ-functions as follows:

$$\begin{aligned} F_\lambda(n) &= \begin{cases} sF_\lambda(n), & n=2k \\ cF_\lambda(n), & n=2k+1 \end{cases}; \\ L_\lambda(n) &= \begin{cases} cL_\lambda(n), & n=2k \\ sL_\lambda(n), & n=2k+1 \end{cases}. \end{aligned} \tag{1.39}$$

It is easy to see that the functions (1.35)-(1.38) are connected by very simple correlations:

$$sF_\lambda(x)=\frac{sL_\lambda(x)}{\sqrt{4+\lambda^2}};\ cF_\lambda(x)=\frac{cL_\lambda(x)}{\sqrt{4+\lambda^2}}. \tag{1.40}$$

This means that the hyperbolic Lucas λ-functions (1.37) and (1.38) coincide with the hyperbolic Fibonacci λ-functions (1.35) and (1.36) to within of the constant coefficient $1/\sqrt{4+\lambda^2}$.

Note that for the case $\lambda=1$ the hyperbolic Fibonacci and Lucas λ-functions (1.35)-(1.38) are reduced to the symmetric hyperbolic Fibonacci and Lucas functions (1.9) and (1.10).

5.2. Graphs of the Hyperbolic Fibonacci and Lucas λ-Functions

The graphs of the hyperbolic Fibonacci and Lucas λ-functions are similar to the graphs of the symmetric hyperbolic Fibonacci and Lucas functions [7] (see **Figure 4**).

It is necessity to note that in the point $x = 0$, the hyperbolic Fibonacci λ-cosine $cF_\lambda(x)$ (36) takes the value $cF_\lambda(0)=2/\sqrt{4+\lambda^2}$, and the hyperbolic Lucas cosine $cL_\lambda(x)$ (38) takes the value $cL_\lambda(0)=2$. It is also important to note that the Fibonacci λ-numbers $F_\lambda(n)$ with the even values of $n = 0, \pm2, \pm4, \pm6, \cdots$ are "inscribed" into the graph of the hyperbolic Fibonacci λ-sine $sF_\lambda(x)$ in the discrete points $x = 0, \pm2, \pm4, \pm6, \cdots$ and the Fibonacci λ-numbers $F_\lambda(n)$ with the odd values of $n = \pm1, \pm3, \pm5, \cdots$ are "inscribed" into the hyperbolic Fibonacci λ-cosine $cF_\lambda(x)$ in the discrete points $x = \pm1, \pm3, \pm5 \cdots$.

On the other hand, the Lucas λ-numbers $L_\lambda(n)$ with the even values of n are "inscribed" into the graph of the hyperbolic Lucas λ-cosine $cL_\lambda(x)$ in the discrete points $x = 0, \pm2, \pm4, \pm6 \cdots$, and the Lucas λ-numbers $L_\lambda(n)$ with the odd values of n are "inscribed" into the graph of the hyperbolic Lucas λ-sine $sL_\lambda(x)$ in the discrete points $x = \pm1, \pm3, \pm5 \cdots$.

By analogy with the symmetric hyperbolic Fibonacci and Lucas functions [7], we can introduce other kinds of the hyperbolic Fibonacci and Lucas λ-functions, in particular, hyperbolic Fibonacci and Lucas λ-tangents and λ-cotangents, λ-secants and λ-cosecants and so on.

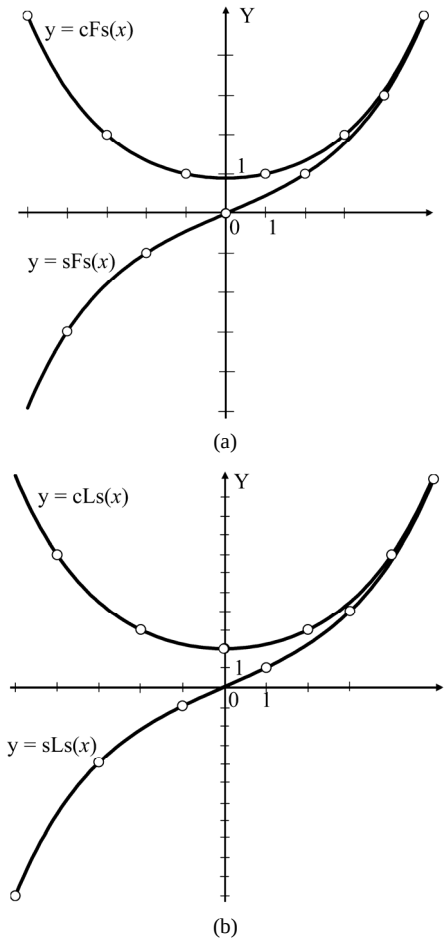

Figure 4. A graph of the symmetric hyperbolic Fibonacci functions (a) and Lucas functions (b).

5.3. Partial Cases of the Hyperbolic Fibonacci and Lucas λ-Functions

The formulas (1.35)-(1.38) set an infinite number of the different hyperbolic λ-functions because every real number $\lambda > 0$ generates its own variant of the hyperbolic Fibonacci and Lucas λ-functions of the kind (1.35)-(1.38).

Let us consider the partial cases of the hyperbolic Fibonacci and Lucas λ-functions (1.35)-(1.38) for the different values of λ.

For the case $\lambda=1$ the "*golden mean*" (1.2) is a base of the hyperbolic Fibonacci and Lucas 1-functions, which are reduced to the symmetric hyperbolic Fibonacci and Lucas functions (1.9) and (1.10). Therefore in further we will name the functions (1.9) and (1.10) the *"golden" hyperbolic Fibonacci and Lucas functions.*

For the case $\lambda=2$ the "*silver mean*" $\Phi_2=1+\sqrt{2}$ is a base of new class of hyperbolic functions. We will name them the "*silver" hyperbolic Fibonacci and Lucas functions*:

$$sF_2(x)=\frac{\Phi_2^x-\Phi_2^{-x}}{\sqrt{8}}=\frac{1}{2\sqrt{2}}\left[\left(1+\sqrt{2}\right)^x-\left(1+\sqrt{2}\right)^{-x}\right] \tag{1.41}$$

$$cF_2(x)=\frac{\Phi_2^x+\Phi_2^{-x}}{\sqrt{8}}=\frac{1}{2\sqrt{2}}\left[\left(1+\sqrt{2}\right)^x+\left(1+\sqrt{2}\right)^{-x}\right] \tag{1.42}$$

$$sL_2(x)=\Phi_2^x-\Phi_2^{-x}=\left(1+\sqrt{2}\right)^x-\left(1+\sqrt{2}\right)^{-x} \tag{1.43}$$

$$cL_2(x)=\Phi_2^x+\Phi_2^{-x}=\left(1+\sqrt{2}\right)^x+\left(1+\sqrt{2}\right)^{-x} \tag{1.44}$$

For the case $\lambda=3$ the "*bronze mean*" $\Phi_3=\left(3+\sqrt{13}\right)/2$ is a base of new class of hyperbolic functions. We will name them the "*bronze*" *hyperbolic Fibonacci and Lucas functions*:

$$sF_3(x)=\frac{\Phi_3^x-\Phi_3^{-x}}{\sqrt{13}}=\frac{1}{\sqrt{13}}\left[\left(\frac{3+\sqrt{13}}{2}\right)^x-\left(\frac{3+\sqrt{13}}{2}\right)^{-x}\right] \tag{1.45}$$

$$cF_3(x)=\frac{\Phi_3^x+\Phi_3^{-x}}{\sqrt{13}}=\frac{1}{\sqrt{13}}\left[\left(\frac{3+\sqrt{13}}{2}\right)^x+\left(\frac{3+\sqrt{13}}{2}\right)^{-x}\right] \tag{1.46}$$

$$sL_3(x)=\Phi_3^x-\Phi_3^{-x}=\left(\frac{3+\sqrt{13}}{2}\right)^x-\left(\frac{3+\sqrt{13}}{2}\right)^{-x} \tag{1.47}$$

$$cL_3(x)=\Phi_3^x+\Phi_3^{-x}=\left(\frac{3+\sqrt{13}}{2}\right)^x+\left(\frac{3+\sqrt{13}}{2}\right)^{-x} \tag{1.48}$$

For the case $\lambda=4$ the "*cooper mean*" $\Phi_4=2+\sqrt{5}$ is a base of new class of hyperbolic functions. We will name them the "*cooper*" *hyperbolic Fibonacci and Lucas functions*:

$$sF_4(x)=\frac{\Phi_4^x-\Phi_4^{-x}}{2\sqrt{5}}=\frac{1}{2\sqrt{5}}\left[\left(2+\sqrt{5}\right)^x-\left(2+\sqrt{5}\right)^{-x}\right] \tag{1.49}$$

$$cF_4(x)=\frac{\Phi_4^x+\Phi_4^{-x}}{2\sqrt{5}}=\frac{1}{2\sqrt{5}}\left[\left(2+\sqrt{5}\right)^x+\left(2+\sqrt{5}\right)^{-x}\right] \tag{1.50}$$

$$sL_4(x)=\Phi_4^x-\Phi_4^{-x}=\left(2+\sqrt{5}\right)^x-\left(2+\sqrt{5}\right)^{-x} \tag{1.51}$$

$$cL_4(x)=\Phi_4^x+\Phi_4^{-x}=\left(2+\sqrt{5}\right)^x+\left(2+\sqrt{5}\right)^{-x} \tag{1.52}$$

Note that a list of these functions can be continued ad infinitum. Note that, because $\lambda>0$ is a positive real number, the number of the hyperbolic Fibonacci and Lucas λ-functions is equal to the number of positive real numbers.

5.4. Comparison of the Classical Hyperbolic Functions with the Hyperbolic Lucas λ-Functions

Let us compare the hyperbolic Lucas λ-functions (1.37) and (1.38) with the classical hyperbolic functions (1.8). It is easy to prove [9] that for the case

$$\Phi_\lambda=\frac{\lambda+\sqrt{4+\lambda^2}}{2}=e \tag{1.53}$$

the hyperbolic Lucas λ-functions (1.37) and (1.38) coincide with the classical hyperbolic functions (8) to within of the constant coefficient $1/2$, that is,

$$sh(x)=\frac{sL_\lambda(x)}{2} \quad \text{and} \quad ch(x)=\frac{cL_\lambda(x)}{2}. \tag{1.54}$$

By using (1.53) after simple transformations we can calculate the value of λ_e, for which the expressions (1.54) are valid:

$$\lambda_e=e-\frac{1}{e}=2sh(1)\approx 2.35040238. \tag{1.55}$$

Thus, according to (1.54) the classical hyperbolic functions (1.8) are a partial case of the hyperbolic Lucas λ-functions for the case (1.55).

5.5. Some Identities for the "Golden" Fibonacci λ-Goniometry

The hyperbolic Fibonacci and Lucas λ-functions possess the *recursive* properties similar to the Fibonacci and Lucas λ-numbers given by the recurrence relations (1.12) and (1.29). On the other hand, they possess all *hyperbolic* properties similar to the properties of the classical hyperbolic functions (1.8).

First of all, we compare the "golden mean" (1.2) with the "metallic mean" (1.24) and the basic formulas generated by them (see **Table 3**).

A beauty of the formulas presented in **Table 3** is charming. This gives a right to suppose that Dirac's "Principle of Mathematical Beauty" is applicable fully to the metallic means (1.24) and hyperbolic Fibonacci and Lucas λ-functions (1.35)-(1.38). And this, in its turn, gives a hope that these mathematical results can be used as effective models of many phenomena in theoretical natural sciences.

Table 4 gives the basic formulas for the hyperbolic Fibonacci λ-functions $sF_\lambda(x)$ and $cF_\lambda(x)$ in comparison with corresponding formulas for the classical hyperbolic functions $sh(x)$ and $ch(x)$.

Remark. For the hyperbolic Lucas λ-functions $sL_\lambda(x)$ and $cL_\lambda(x)$ the corresponding formulas can be

Table 3. Comparative table for the Golden Mean and Metallic Means.

The Golden Mean ($\lambda = 1$)	The Metallic Means ($\lambda > 0$)
$\Phi = \frac{1+\sqrt{5}}{2}$	$\Phi_\lambda = \frac{\lambda+\sqrt{4+\lambda^2}}{2}$
$\Phi = \sqrt{1+\sqrt{1+\sqrt{1+\sqrt{\cdots}}}}$	$\Phi_\lambda = \sqrt{1+\lambda\sqrt{1+\lambda\sqrt{1+\lambda\sqrt{\cdots}}}}$
$\Phi = 1+\cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{1+\cdots}}}$	$\Phi_\lambda = \lambda+\cfrac{1}{\lambda+\cfrac{1}{\lambda+\cfrac{1}{\lambda+\cdots}}}$
$\Phi^n = \Phi^{n-1}+\Phi^{n-2} = \Phi\times\Phi^{n-1}$	$\Phi_\lambda^n = \lambda\Phi_\lambda^{n-1}+\Phi_\lambda^{n-2} = \Phi_\lambda\times\Phi_\lambda^{n-1}$
$F(n) = \frac{\Phi^n-(-1)^n\Phi^{-n}}{\sqrt{5}}$	$F_\lambda(n) = \frac{\Phi_\lambda^n-(-1)^n\Phi_\lambda^{-n}}{\sqrt{4+\lambda^2}}$
$L(n) = \Phi^n+(-1)^n\Phi^{-n}$	$L_\lambda(n) = \Phi_\lambda^n+(-1)^n\Phi_\lambda^{-n}$
$sFs(x) = \frac{\Phi^x-\Phi^{-x}}{\sqrt{5}}$ $cFs(x) = \frac{\Phi^x+\Phi^{-x}}{\sqrt{5}}$	$sF_\lambda(x) = \frac{\Phi_\lambda^x-\Phi_\lambda^{-x}}{\sqrt{4+\lambda^2}}$ $cF_\lambda(x) = \frac{\Phi_\lambda^x+\Phi_\lambda^{-x}}{\sqrt{4+\lambda^2}}$
$sLs(x) = \Phi^x-\Phi^{-x}$ $cLs(x) = \Phi^x+\Phi^{-x}$	$sL_\lambda(x) = \Phi_\lambda^x-\Phi_\lambda^{-x}$ $cL_\lambda(x) = \Phi_\lambda^x+\Phi_\lambda^{-x}$

Table 4. Stakhov's "golden" Fibonacci λ-goniometry.

Formulas for the classical hyperbolic functions	Formulas for the hyperbolic Fibonacci λ-functions
$sh(x) = \frac{e^x-e^{-x}}{2}; ch(x) = \frac{e^x+e^{-x}}{2}$	$sF_\lambda(x) = \frac{\Phi_\lambda^x-\Phi_\lambda^{-x}}{\sqrt{4+\lambda^2}}; cF_\lambda(x) = \frac{\Phi_\lambda^x+\Phi_\lambda^{-x}}{\sqrt{4+\lambda^2}}$
$sh(x+2) = 2sh(1)ch(x+1)+sh(x)$ $ch(x+2) = 2sh(1)sh(x+1)+ch(x)$	$sF_\lambda(x+2) = \lambda cF_\lambda(x+1)+sF_\lambda(x)$ $cF_\lambda(x+2) = \lambda sF_\lambda(x+1)+cF_\lambda(x)$
$sh^2(x)-ch(x+1)ch(x-1) = -ch^2(1)$ $ch^2(x)-sh(x+1)sh(x-1) = ch^2(1)$	$\left[sF_\lambda(x)\right]^2-cF_\lambda(x+1)cF_\lambda(x-1) = -1$ $\left[cF_\lambda(x)\right]^2-sF_\lambda(x+1)sF_\lambda(x-1) = 1$
$ch^2(x)-sh^2(x) = 1$	$\left[cF_\lambda(x)\right]^2-\left[sF_\lambda(x)\right]^2 = \frac{4}{4+\lambda^2}$
$sh(x+y) = sh(x)ch(y)+ch(x)sh(y)$ $sh(x-y) = sh(x)ch(y)-ch(x)sh(y)$	$\frac{2}{\sqrt{4+\lambda^2}}sF_\lambda(x+y) = sF_\lambda(x)cF_\lambda(x)+cF_\lambda(x)sF_\lambda(x)$ $\frac{2}{\sqrt{4+\lambda^2}}sF_\lambda(x-y) = sF_\lambda(x)cF_\lambda(x)-cF_\lambda(x)sF_\lambda(x)$
$ch(x+y) = ch(x)ch(y)+sh(x)sh(y)$ $ch(x-y) = ch(x)ch(y)-sh(x)sh(y)$	$\frac{2}{\sqrt{4+\lambda^2}}cF_\lambda(x+y) = cF_\lambda(x)cF_\lambda(x)+sF_\lambda(x)sF_\lambda(x)$ $\frac{2}{\sqrt{4+\lambda^2}}cF_\lambda(x-y) = cF_\lambda(x)cF_\lambda(x)-sF_\lambda(x)sF_\lambda(x)$
$ch(2x) = 2sh(x)ch(x)$	$\frac{1}{\sqrt{4+\lambda^2}}cF_\lambda(2x) = sF_\lambda(x)cF_\lambda(x)$
$\left[ch(x)\pm sh(x)\right]^n = ch(nx)\pm sh(nx)$	$\left[cF_\lambda(x)\pm sF_\lambda(x)\right]^n = \left(\frac{2}{\sqrt{4+\lambda^2}}\right)^{n-1}\left[cF_\lambda(nx)\pm sF_\lambda(nx)\right]$

got by multiplication of the hyperbolic Fibonacci λ-functions $sF_\lambda(x)$ and $cF_\lambda(x)$ by constant factor $\sqrt{4+\lambda^2}$ according to the correlations (1.41).

Table 4 for the hyperbolic Fibonacci λ-functions $sF_\lambda(x)$ and $cF_\lambda(x)$, with regard to the above remark for the hyperbolic Lucas λ-functions $sL_\lambda(x)$ and $cL_\lambda(x)$, makes up a base of *Stakhov's "golden" Fibonacci goniometry* [9]. This table is very convincing con-

firmation of the fact that we are talking about a new class of hyperbolic functions, which keep all well-known properties of the classical hyperbolic functions $sh(x)$ and $ch(x)$, but, in addition, they posses additional ("recursive") properties, which unite them with remarkable numerical sequences—*Fibonacci* and *Lucas* λ*-numbers* $F_\lambda(n)$ and $L_\lambda(n)$.

Thus, the main results of the works [6-9] is an introduction of new class of hyperbolic functions—*hyperbolic Fibonacci and Lucas functions* based on the *golden mean* [6-8]—and a proof of the existence of infinite number of similar hyperbolic functions - *hyperbolic Fibonacci and Lucas* λ*-functions* ($\lambda > 0$ is given real number) based on the *metallic means* [9]. These new hyperbolic functions are similar to the classical hyperbolic functions (1.8) and save all their useful mathematical properties (*hyperbolic properties*). Besides, they are a generalization of the classical Fibonacci and Lucas numbers and Fibonacci and Lucas λ-numbers, which coincide with *hyperbolic Fibonacci and Lucas functions* and *hyperbolic Fibonacci and Lucas* λ*-functions* for discrete values of continues variable $x = 0, \pm 1, \pm 2, \pm 3, \cdots$, and save all their useful mathematical properties (*recursive properties*).

At present, Oleg Bodnar in [4], Alexey Stakhov and Samuil Aranson in [10] have obtained many interesting applications of the hyperbolic Fibonacci and Lucas functions and "golden" Fibonacci goniometry in mathematics, theoretical physics and theoretical botany. We are talking on the following results:

1) Fibonacci-Lorentz transformations and "golden" interpretation of the Special Theory of Relativity (STR). This result has led in [10] to the original comological interpretation of the Universe evolution starting from Big Bang. This approach has a direct relation to the *hyperbolic Fibonacci functions,* because the *Fibonacci-Lorentz transformations* are based on the "golden" matrices [16]:

$$Q_0(x) = \begin{pmatrix} cFs(x+1) & sFs(x) \\ sFs(x) & cFs(x-1) \end{pmatrix}; \quad Q_1(x) = \begin{pmatrix} sFs(x+1) & cFs(x) \\ cFs(x) & sFs(x-1) \end{pmatrix}. \tag{1.56}$$

Note that the matrices (1.56) are functions of the continuous variable *x* and their elements are hyperbolic Fibonacci functions (1.9). However, the most unexpected property of the matrices (1.56) follows from the properties (1.11). By using the properties (1.11), it is proved in [16] that the determinants of the matrices (1.56) do not depend on the continuous variable *x* and are equal, respectively:

$$\det Q_0(x) = 1, \det Q_1(x) = -1. \tag{1.57}$$

Exactly these properties of the "golden" matrices (1.56) determine unusual properties of the *Fibonacci-Lorentz transformations* and unusual interpretation of STR.

2) A new geometric theory of phyllotaxis (Bodnar's geometry). As Bodnar's geometry is stated in Russian scientific literature [4], we have decided to describe this original geometric theory of phyllotaxis as example of effective use of the hyperbolic Fibonacci and Lucas functions in Part II of the article.

3) An original solution of *Hilbert's Fourth Problem*. This solution is stated in the article [10] without proof. In Part III of this article for the first time we give a full proof of this original solution.

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