Scientific
Research

The Structure of Reflective Function of Higher Dimensional Differential System

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Received March 16, 2010; *revised April* 18, 2010; *accepted April* 29, 2010

Abstract

In this article, we discuss the structure of reflective function of the higher dimensional differential systems and apply the results to study the existence of periodic solutions of these systems.

1. Introduction

As we know, to study the property of the solutions of differential system

$$x' = X(t,x) \qquad (1)$$

is very important not only for the theory of ordinary differential equation but also for practical reasons. If $X(t+2\omega,x) = X(t,x)$ (ω is a positive constant), to study the solutions' behavior of (1), we could use, as introduced in [1], the Poincare mapping. But it is very difficult to find the Poincare mapping for many systems which cannot be integrated in quadratures. In the 1980's the Russian mathematician Mironenko [2] first established the theory of reflective functions (RF). Since then a quite new method to study (1) has been found.

In the present section, we introduce the concept of the reflective function, which will be used throughout the rest of this article.

Now consider the system (1) with a continuously differentiable right-hand side and with a general solution $\psi(t;t_0,x_0)$. For each such system, the reflective function (RF) of (1) is defined as $F(t,x(t)) = \psi(-t;t,x)$. Then for any solution $x(t)$ of (1), we have $F(t,x(t)) = x(-t)$. If system (1) is 2ω–periodic with respect to t, and $F(t,x)$ is its RF, then $F(-\omega,x) = \psi(\omega;-\omega,x)$ is the Poincare [1-2] mapping of (1) over the period $[-\omega,\omega]$. So, for any solution $x(t)$ of (1) defined on $[-\omega,\omega]$, it will be 2ω--periodic if and only if $x(-\omega)$ is a fixed point of the Poincare mapping $T(x) = F(-\omega,x)$. A function $F(t,x)$ is a reflective function of system (1) if and only if it is a solution of the partial differential

equation (called a basic relation, BR)

$$F_t' + F_x'X(t,x) + X(-t,x) = 0 \qquad (2)$$

with the initial condition $F(0,x) = x$. It implies that for non-integrable periodic systems we also can find out its Poincare mapping. If, for example,

$X(t,x) + X(-t,x) = 0$, then $T(x) = x$.

If $F(t,x)$ is the RF of (1), then it is also the RF of the system

$$x' = X(t,x) + F_x^{-1}R(t,x) - R(-t,F(t,x)),$$

where $R(t,x)$ is an arbitrary vector function such that the solutions of the above systems are uniquely determined by their initial conditions. Therefore, all these 2ω−periodic systems have a common Poincare mapping over the period $[-\omega,\omega]$, and the behavior of the periodic solutions of these systems are the same.

To find out the reflective function is very important for studying the qualitative behavior of solutions of differential systems. The literatures [5-8] have discussed the structure of the reflective function of some second order quadric systems and linear systems and obtained many good results.

Now, we consider the higher dimensional polynomial differential system

$$\begin{cases} x' = p_1 + p_2 y + p_3 z = P(t,x,y,z) \\ y' = q_1 + q_2 y + q_3 z + q_4 y^2 + q_5 yz + q_6 z^2 = Q(t,x,y,z) \\ z' = r_1 + r_2 y + r_3 z + r_4 y^2 + r_5 yz + r_6 z^2 = R(t,x,y,z) \end{cases} (3)$$

where

$$p_i = p_i(t,x), q_j = q_j(t,x), r_j = r_j(t,x)$$
$$(i = 1,2,3; j = 1,2,...,6)$$

are continuously differentiable functions in R^2, and $p_2^2 + p_3^2 \neq 0$ (in some deleted neighborhood of $t = 0$

and $|t|$ being small enough, $p_2^2 + p_3^2 \neq 0$ is different from zero）, and there exists a unique solution for the initial value problem of (3). And suppose that $F(t,x,y,z) = (F_1(t,x,y,z), F_2(t,x,y,z), F_3(t,x,y,z))^T$ is the RF of (3).

In this paper, we will discuss the structure of $F_i(t,x,y,z)$ $(i=2,3)$ when $F_1(t,x,y,z) = f(t,x)$. At the same time, we obtain the good results that $F_i(t,x,y,z) = f_{i1}(t,x) + f_{i2}(t,x)y + f_{i3}(t,x)z$ $(i=2,3)$. The obtained results are used for research of problems of existence of periodic solution of the system (3) and establish the sufficient conditions under which the first component of the solution of (3) is even function.

In the following, we will denote

$\bar{p}_i = p_i(-t,x); \bar{q}_j = q_j(-t,x); \bar{r}_j = r_j(t,x); F_i = F_i(t,x,y,z),$
$i = 1,2,3, j = 1,2,...,6$. The notation $p_i(t,x) \neq 0$ means that, in some deleted neighborhood of $t=0$ and $|t|$ being small enough, $p_i(t,x)$ is different from zero,

$$DA = \frac{\partial A}{\partial t} + \frac{\partial A}{\partial x}P(t,x,y,z) + \frac{\partial A}{\partial y}Q(t,x,y,z) + \frac{\partial A}{\partial z}R(t,x,y,z).$$

2. Main Results

Without loss of generality, we suppose that

$f(t,x) = x$. Otherwise, we take the transformation $\xi = f(t,x), \eta = y, \zeta = z$.

Now, let's consider the system (3)

Lemma 1. For the system (3), suppose that $F_1 = x$. Then

$$p_i(t,x) = 0, \ i = 1,2,3. \tag{4}$$

Proof. Using the relation (2), we get

$$P(t,x,y,z) + P(-t,x,F_2,F_3) = 0,$$

i.e.

$$p_1 + \bar{p}_1 + p_2 y + p_3 z + \bar{p}_2 F_2 + \bar{p}_3 F_3 = 0. \tag{5}$$

Putting $t = 0$, we get

$$p_1(0,x) + p_2(0,x)y + p_3(0,x)z \equiv 0, \ \forall x,y,z.$$

It implies that the relation (4) is valid.

In the following discussion, we always assume (4) holds without further mention.

Case 1. $p_3 \neq 0$.

From (5), we get

$$F_3 = \lambda_1 + \lambda_2 F_2, \tag{6}$$

where

$$\lambda_1 = -\frac{p_1 + \bar{p}_1}{\bar{p}_3} - \frac{p_2}{\bar{p}_3}y - \frac{p_3}{\bar{p}_3}z = \lambda_{11} + \lambda_{12}y + \lambda_{13}z,$$

$$\lambda_2 = -\frac{\bar{p}_2}{\bar{p}_3}.$$

Differentiating relation (6) respect to t implies

$$A_0 + A_1 F_2 + A_2 F_2^2 = 0, \tag{7}$$

where

$$A_0 = D\lambda_1 - \lambda_2 \bar{q}_1 + \bar{r}_1 + \lambda_1(\bar{r}_3 - \lambda_2 \bar{q}_3) + \lambda_1^2(\bar{r}_6 - \lambda_2 \bar{q}_6)$$
$$= a_{01} + a_{02}y + a_{03}z + a_{04}y^2 + a_{05}yz + a_{06}z^2;$$

$$A_1 = D\lambda_2 - \lambda_2 \bar{q}_2 + \bar{r}_2 + \lambda_2(\bar{r}_3 - \lambda_2 \bar{q}_3) + \lambda_1(\bar{r}_5 - \lambda_2 \bar{q}_5)$$
$$+ 2\lambda_1\lambda_2(\bar{r}_6 - \lambda_2\bar{q}_6) = a_{11} + a_{12}y + a_{13}z;$$

$$A_2 = A_2(t,x) = -\lambda_2 \bar{q}_4 + \bar{r}_4 + \lambda_2(\bar{r}_5 - \lambda_2 \bar{q}_5) + \lambda_2^2(\bar{r}_6 - \lambda_2 \bar{q}_6),$$

In which

$$a_{01} = \lambda_{11t}' + \lambda_{11x}'p_1 + \lambda_{12}q_1 + \lambda_{13}r_1 + \bar{r}_1 - \lambda_2\bar{q}_1 +$$
$$+ \lambda_{11}(\bar{r}_3 - \lambda_2\bar{q}_3) + (\bar{r}_6 - \lambda_2\bar{q}_6)\lambda_{11}^2;$$

$$a_{02} = \lambda_{12t}' + \lambda_{12x}'p_2 + \lambda_{12x}'p_1 + \lambda_{12}q_2 + \lambda_{13}r_2 +$$
$$+ \lambda_{12}(\bar{r}_3 - \lambda_2\bar{q}_3) + 2(\bar{r}_6 - \lambda_2\bar{q}_6)\lambda_{11}\lambda_{12};$$

$$a_{03} = \lambda_{13t}' + \lambda_{11x}'p_3 + \lambda_{13x}'p_3 + \lambda_{12}q_3 + \lambda_{13}r_3 +$$
$$+ \lambda_{13}(\bar{r}_3 - \lambda_2\bar{q}_3) + 2(\bar{r}_6 - \lambda_2\bar{q}_6)\lambda_{11}\lambda_{13};$$

$$a_{04} = \lambda_{12x}'p_2 + \lambda_{12}q_4 + \lambda_{13}r_4 + (\bar{r}_6 - \lambda_2\bar{q}_6)\lambda_{12}^2;$$

$$a_{05} = \lambda_{12x}'p_3 + \lambda_{13x}'p_2 + \lambda_{12}q_5 + \lambda_{13}r_5 + 2(\bar{r}_6 - \lambda_2\bar{q}_6)\lambda_{12}\lambda_{13};$$

$$a_{06} = \lambda_{13x}'p_3 + \lambda_{12}q_6 + \lambda_{13}r_6 + (\bar{r}_6 - \lambda_2\bar{q}_6)\lambda_{13}^2;$$

$$a_{11} = \lambda_{2t}' + \lambda_{2x}'p_1 + \bar{r}_2 - \lambda_2\bar{q}_2 + \lambda_2(\bar{r}_3 - \lambda_2\bar{q}_3) +$$
$$+ \lambda_{11}(\bar{r}_5 - \lambda_2\bar{q}_5) + 2(\bar{r}_6 - \lambda_2\bar{q}_6)\lambda_2\lambda_{11};$$

$$a_{12} = \lambda_{2x}'p_2 + \lambda_{12}(\bar{r}_5 - \lambda_2\bar{q}_5) + 2(\bar{r}_6 - \lambda_2\bar{q}_6)\lambda_2\lambda_{12};$$

$$a_{13} = \lambda_{2x}'p_3 + \lambda_{13}(\bar{r}_5 - \lambda_2\bar{q}_5) + 2(\bar{r}_6 - \lambda_2\bar{q}_6)\lambda_2\lambda_{13};$$

$$\lambda_{11} = -\frac{p_1 + \bar{p}_1}{\bar{p}_3}, \ \lambda_{12} = -\frac{p_2}{\bar{p}_3}, \ \lambda_{13} = -\frac{p_3}{\bar{p}_3}.$$

Lemma 2. Let $F_1 = x$, $A_2 \neq 0$ and $\lim_{t \to 0}\frac{a_{0i}}{A_2}$ $(i = 1,2,...,6)$ exist. Then $\lim_{t \to 0}\frac{a_{0j}}{A_2} = 0$ $(j = 1,3,6)$, $\lim_{t \to 0}\frac{a_{02} + a_{11}}{A_2} = 0$,

$$\lim_{t \to 0}\frac{a_{04} + a_{12}}{A_2} = -1, \lim_{t \to 0}\frac{a_{05} + a_{13}}{A_2} = 0.$$

Proof. Using the relation (7), we have

$$\lim_{t \to 0}\frac{a_{01} + a_{02}y + a_{03}z + a_{04}y^2 + a_{05}yz + a_{06}z^2}{A_2} +$$

$$\lim_{t \to 0}\frac{a_{11} + a_{12}y + a_{13}z}{A_2}F_2 + \lim_{t \to 0}F_2^2 = 0,$$

As $F_2(0,x,y,z) = y$, it follows that the results of Lemma 2 are true.

Theorem 1. Let the conditions of Lemma 1 and Lemma 2 satisfy and $\lim_{t \to 0}\frac{a_{12}}{A_2} + 2 > 0$. Then

$$F_i = f_{i1}(t,x) + f_{i2}(t,x)y + f_{i3}(t,x)z \ (i = 2,3).$$

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Proof. AS $A_2 \neq 0$. From (7), it follows

$$F_2^2 = -\frac{A_0 + A_1 F_2}{A_2}, \quad F_2^3 = \frac{A_1 A_0}{A_2^2} + \frac{A_1^2 - A_0 A_2}{A_2^2} F_2 \qquad (8)$$

Differentiating relation (7) respect to t implies

$$DA_0 + DA_1 F_2 + DA_2 F_2^2 - A_1 Q(-t, x, F_2, F_3)$$
$$-2A_2 Q(-t, x, F_2, F_3) F_2 = 0.$$

Substituting (6) into the above, we get

$$B_0 + B_1 F_2 + B_3 F_2^2 + B_4 F_2^3 = 0, \qquad (9)$$

where

$$B_0 = DA_0 - A_1 \mu_1, \quad B_1 = DA_1 - A_1 \mu_2 - 2A_2 \mu_1,$$
$$B_2 = DA_2 - A_1 \mu_3 - 2A_2 \mu_2, \quad B_4 = -2A_2 \mu_3,$$

in which

$$\mu_1 = \overline{q}_1 + \overline{q}_3 \lambda_1 + \overline{q}_6 \lambda_1^2, \mu_2 = \overline{q}_2 + \overline{q}_3 \lambda_2 + \overline{q}_5 \lambda_1 + 2\overline{q}_6 \lambda_2 \lambda_1,$$
$$\mu_3 = \overline{q}_4 + \overline{q}_5 \lambda_2 + \overline{q}_6 \lambda_2^2.$$

Substituting (8) into (9), we have

$$C_0 + C_1 F_2 = 0, \qquad (10)$$

where

$$C_0 = B_0 - B_2 \frac{A_0}{A_2} + B_3 \frac{A_1 A_0}{A_2^2},$$
$$C_1 = B_1 - B_2 \frac{A_1}{A_2} + B_3 \frac{A_1^2 - A_0 A_2}{A_2^2}.$$

1^*. If $C_1 \equiv 0$, from (10) follows $C_0 \equiv 0$. By simple computation, we obtain

$$D\frac{A_0}{A_2} = \mu_1 \frac{A_1}{A_2} - 2\mu_2 \frac{A_0}{A_2} + \mu_3 \frac{A_0 A_1}{A_2^2}, \qquad (11)$$

$$D\frac{A_1}{A_2} = 2\mu_1 - \mu_2 \frac{A_1}{A_2} + \mu_3 (\frac{A_1^2}{A_2^2} - 2\frac{A_0}{A_2}). \qquad (12)$$

Let $\Delta = (\frac{A_1}{A_2})^2 - 4\frac{A_0}{A_2}$. Using (11) (12) we get

$$D\Delta = 2(\mu_3 \frac{A_1}{A_2} - \mu_2)\Delta. \qquad (13)$$

Since

$$\Delta = (\frac{A_1}{A_2})^2 - 4\frac{A_0}{A_2} = \frac{1}{A_2^2}((a_{11} + a_{12}y + a_{13}z)^2 -$$

$$4A_2(a_{01} + a_{02}y + a_{03}z + a_{04}y^2 + a_{05}yz + a_{06}z^2))$$

$$= d_1 + d_2 y + d_3 z + d_4 y^2 + d_5 yz + d_6 z^2$$

$$= d_4(y + \frac{d_5}{2d_4}z + \frac{d_2}{2d_4})^2 + W,$$

where

$$W = \frac{1}{4d_4}[(4d_4 d_6 - d_5^2)z^2 + 2(2d_3 d_4 - d_2 d_5)z + 4d_1 d_4 - d_2^2],$$

$$d_1 = \frac{1}{A_2^2}(a_{11}^2 - 4A_2 a_{01}), d_2 = \frac{1}{A_2^2}(2a_{11}a_{12} - 4A_2 a_{02}),$$

$$d_3 = \frac{1}{A_2^2}(2a_{11}a_{13} - 4A_2 a_{03}), d_4 = \frac{1}{A_2^2}(a_{12}^2 - 4A_2 a_{04}),$$

$$d_5 = \frac{1}{A_2^2}(2a_{13}a_{12} - 4A_2 a_{05}), d_6 = \frac{1}{A_2^2}(a_{13}^2 - 4A_2 a_{06}).$$

By **Lemma 2** we get

$$\lim_{t \to 0}(4d_4 d_6 - d_5^2) = 0, \lim_{t \to 0}(3d_3 d_4 - d_2 d_5) = 0,$$
$$\lim_{t \to 0}(4d_4 d_1 - d_2^2) = 0,$$

thus, $\lim_{t \to 0} W(t, x, z) = 0$. In the identity (13) taking

$$y = \phi = -\frac{d_5}{2d_4}z - \frac{d_2}{2d_4}. \text{ We obtain}$$

$$W_t' + W_x' P(t, x, \phi) + W_z' Q(t, x, \phi, z) =$$

$$2W(\frac{A_1(t, x, \phi, z)}{A_2(t, x)} \mu_3(t, x) - \mu_2(t, x, \phi, z)).$$

By the uniqueness of solution of initial problem of linear partial differential equation, we get $W(t, x, z) \equiv 0$. Therefore

$$\Delta = d_4(y + \frac{d_5}{2d_4}z + \frac{d_2}{2d_4})^2.$$

Using the relation (7), we obtain

$$F_2 = -\frac{a_{11} + a_{12}y + a_{13}z}{2A_2} + \frac{1}{2}\sqrt{d_4}(y + \frac{d_5}{2d_4}z + \frac{d_2}{2d_4})$$

$$= f_{21}(t, x) + f_{22}(t, x)y + f_{23}(t, x)z.$$

By the relation (6), we get

$$F_3 = \lambda_1 + \lambda_2 F_2 = f_{31}(t, x) + f_{32}(t, x)y + f_{33}(t, x)z.$$

2^*. If $C_1 \neq 0$. From (10) follows $F_2 = -\frac{C_0}{C_1}$. By the express of A_i, B_j $(i = 0, 1, 2, j = 0, 1, 2, 3)$, we know that C_1 is a quadratic polynomial respect of y, z, C_0 is a cubic polynomial respect to y, z. Substituting $F_2 = -C_0/C_1$ into relation (7), we get $C_1(C_0 A_1 - C_1 A_0) = A_2 C_0^2$. It implies that C_1 is divided by C_0 or A_2, and $F_2 = \sum_{i+j=0}^{3} f_{2ij}(t, x)y^i z^j$, substituting it into (7) and equating the coefficients of like powers of y and z implies $f_{2ij}(t, x) = 0, i + j > 1$. Thus,

$$F_i = f_{i1}(t, x) + f_{i2}(t, x)y + f_{i3}(t, x)z, i = 2, 3.$$

Summarizing the above, the proof is completed.

Obviously, from the relation (7) implies

Theorem 2. Let

$$F_1 = x, \ p_i(0,x) = 0 \ (i=1,2,3), \ A_2 = 0, \ A_1 \neq 0.$$

Then

$$F_2 = -\frac{A_0}{A_1} = -\frac{a_{01} + a_{02}y + a_{03}z + a_{04}y^2 + a_{05}yz + a_{06}z^2}{a_{11} + a_{12}y + a_{13}z},$$

$$F_3 = \lambda_1 + \lambda_2 F_2.$$

Case 2. $p_3(t,x) \equiv 0, p_2 \neq 0$.

Applying identity (5) yields

$$F_2 = -\frac{p_1 + \overline{p}_1}{\overline{p}_2} - \frac{p_2}{\overline{p}_2}y = \zeta_1 + \zeta_2 y,$$

where $\zeta_1 = -\dfrac{p_1 + \overline{p}_1}{\overline{p}_2}, \zeta_2 = -\dfrac{p_2}{\overline{p}_2}$.

Differentiating this identity respect to t implies

$$M_0 + M_1 F_3 + M_2 F_3^2 = 0,$$

where

$$M_0 = D(\zeta_1 + \zeta_2 y) + \overline{q}_1 + \overline{q}_2 F_2 + \overline{q}_4 F_3^2 =$$
$$m_{01} + m_{02}y + m_{03}z + m_{04}y^2 + m_{05}yz + m_{06}z^2,$$

$$M_1 = \overline{q}_3 + \overline{q}_5 F_2 = m_{11} + m_{12}y, \ M_2 = \overline{q}_6,$$

$$m_{01} = \zeta'_{1t} + \zeta'_{1x}p_1 + \zeta_2 q_1 + \overline{q}_1 + \overline{q}_2 \zeta_1 + \overline{q}_4 \zeta_1^2;$$

$$m_{02} = \zeta'_{1x}p_2 + \zeta'_{2x}p_1 + \zeta'_{2t} + \zeta_2 q_2 + \overline{q}_2 \zeta_2 + 2\overline{q}_4 \zeta_1 \zeta_2;$$

$$m_{04} = \zeta'_{2x}p_2 + \zeta_2 q_4 + \overline{q}_4 \zeta_2^2;$$

$$m_{03} = \zeta_2 q_3, \ m_{05} = \zeta_2 q_5, \ m_{06} = \zeta_2 q_6,$$

$$m_{11} = \overline{q}_3 + \overline{q}_5 \zeta_1, \ m_{12} = \overline{q}_5 \zeta_2.$$

Similarly, we obtain the following conclusion:

Lemma 3. Let $p_3 = 0, p_2 \neq 0, q_6 \neq 0, F_1 = x$ and

$$\lim_{t \to 0}\frac{m_{0i}}{q_6} \ (i=1,2,...,5) \text{ and } \lim_{t \to 0}\frac{p_1}{\overline{p}_2} \text{ exist. Then}$$

$$\lim_{t \to 0}\frac{m_{0j}}{q_6} = 0 \ (j=1,2,4), \lim_{t \to 0}\frac{p_1 + \overline{p}_1}{\overline{p}_2} = 0, \ \lim_{t \to 0}\frac{p_2}{\overline{p}_2} = -1$$

$$\lim_{t \to 0}\frac{m_{03} + m_{11}}{q_6} = 0, \lim_{t \to 0}\frac{m_{05} + m_{12}}{q_6} = 0, \lim_{t \to 0}\frac{q_6}{\overline{q}_6} = -1.$$

Theorem 3. Let the conditions of Lemma 1 and Lemma 3 satisfy. Then

$$F_2 = \zeta_1(t,x) + \zeta_2(t,x)y,$$

$$F_3 = f_{31}(t,x) + f_{32}(t,x)y + f_{33}(t,x)z.$$

Theorem 4. Let

$$p_i(0,x) = 0 \ (i=1,2), q_3(t,x) \equiv 0, \ q_6(t,x) \equiv 0$$
$$m_{11}^2 + m_{12}^2 \neq 0, F_1 = x \qquad , \text{ Then}$$

$$F_2 = \zeta_1(t,x) + \zeta_2(t,x)y,$$

$$F_3 = -\frac{m_{01} + m_{02}y + m_{03}z + m_{04}y^2 + m_{05}yz}{m_{11} + m_{12}y}.$$

Theorem 5. For the system (3), if the following conditions satisfy

$$p_1 + \overline{p}_1 + \overline{p}_2 f_{21} + \overline{p}_3 f_{31} = 0, \ p_2 + \overline{p}_2 f_{22} + \overline{p}_3 f_{32} = 0,$$

$$p_3 + \overline{p}_2 f_{23} + \overline{p}_3 f_{33} = 0, \ f_{21}(0,x) = 0, f_{31}(0,x) = 0,$$

$$\begin{pmatrix} f'_{21t} \\ f'_{31t} \end{pmatrix} + \begin{pmatrix} f'_{21x} \\ f'_{31x} \end{pmatrix}p_1 + \begin{pmatrix} f_{22} \\ f_{23} \end{pmatrix}q_1 + \begin{pmatrix} f_{23} \\ f_{33} \end{pmatrix}r_1 + \begin{pmatrix} Q(-t,x,f_{21},f_{31}) \\ R(-t,x,f_{21},f_{31}) \end{pmatrix}$$
$$= 0,$$

$$\begin{pmatrix} f'_{22t} & f'_{23t} \\ f'_{32t} & f'_{33t} \end{pmatrix} + \begin{pmatrix} f'_{22x} & f'_{23x} \\ f'_{32x} & f'_{33x} \end{pmatrix}p_1 + \begin{pmatrix} f'_{21x}p_2 & f'_{21x}p_3 \\ f'_{31x}p_2 & f'_{32x}p_3 \end{pmatrix} +$$
$$\begin{pmatrix} f_{22} & f_{23} \\ f_{32} & f_{33} \end{pmatrix}\begin{pmatrix} q_2 & q_3 \\ r_2 & r_3 \end{pmatrix} + \begin{pmatrix} \overline{Q}'_y & \overline{Q}'_z \\ \overline{R}'_y & \overline{R}'_z \end{pmatrix}_{(t,x,0,0)} = 0,$$

$$\begin{pmatrix} f'_{22x}p_2 & f'_{23x}p_3 \\ f'_{32x}p_2 & f'_{33x}p_3 \end{pmatrix} + \begin{pmatrix} f_{22} & f_{23} \\ f_{32} & f_{33} \end{pmatrix}\begin{pmatrix} q_4 & q_6 \\ r_4 & r_6 \end{pmatrix} +$$
$$\begin{pmatrix} \overline{q}_4 & \overline{q}_6 \\ \overline{r}_4 & \overline{r}_6 \end{pmatrix}\begin{pmatrix} f_{22}^2 & f_{23}^2 \\ f_{32}^2 & f_{33}^2 \end{pmatrix} + \begin{pmatrix} f_{22}f_{23}\overline{q}_5 & f_{23}f_{33}\overline{q}_5 \\ f_{22}f_{23}\overline{r}_5 & f_{23}f_{33}\overline{r}_5 \end{pmatrix} = 0,$$

$$\begin{pmatrix} f'_{22x} & f'_{23x} \\ f'_{32x} & f'_{33x} \end{pmatrix}\begin{pmatrix} p_3 \\ p_2 \end{pmatrix} + \begin{pmatrix} f_{22} & f_{23} \\ f_{32} & f_{33} \end{pmatrix}\begin{pmatrix} q_5 \\ r_5 \end{pmatrix} +$$
$$\begin{pmatrix} 2\overline{q}_4 f_{22}f_{23} + \overline{q}_5(f_{22}f_{33} + f_{23}f_{32}) + 2\overline{q}_6 f_{32}f_{33} \\ 2\overline{r}_4 f_{22}f_{23} + \overline{r}_5(f_{22}f_{33} + f_{23}f_{32}) + 2\overline{r}_6 f_{32}f_{33} \end{pmatrix} = 0,$$

$$\begin{pmatrix} f_{22} & f_{23} \\ f_{32} & f_{33} \end{pmatrix}\begin{pmatrix} \overline{f}_{22} & \overline{f}_{23} \\ \overline{f}_{32} & \overline{f}_{33} \end{pmatrix} = \begin{pmatrix} f_{22}(0,x) & f_{23}(0,x) \\ f_{32}(0,x) & f_{33}(0,x) \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

Then $F = \begin{pmatrix} x \\ f_{21} + f_{22}y + f_{22}z \\ f_{31} + f_{32}y + f_{33}z \end{pmatrix}$ is the RF of system (3).

Besides this, if the system (3) is 2ω--periodic with respect to t, then its solution $(x(t), y(t), z(t))$ defined on the interval $[-\omega, \omega]$ with initial condition $(x(-\omega), y(-\omega), z(-\omega)) = \chi$ is 2ω--periodic if and only if

$$F(-\omega, \chi) = \chi.$$

Proof. By checkout of the BR it is proved that the function $F = (x, f_{21} + f_{22}y + f_{23}z, f_{31} + f_{32}y + f_{33}z)^T$ is the RF of system (3). At this moment, the Poincare mapping of periodic system (3) is $T(x,y,z) = F(-\omega, x, y, z)$. By the previous introduction the assertions of the present theorem is hold. The proof is finished.

Under the hypotheses of Theorem 5, the first component of solution of system (3) is even function.

Example: Differential system

$$\begin{cases} x' = (1 - e^{\sin t}(1 - x^3 \sin t))y - x^2 z \sin t, \\ y' = \frac{1}{2} y \cos t (x^6 \sin^2 t + x^3 - 1) - \frac{1}{2} z x^2 \cos t e^{-\sin t}(1 + \sin t + x^3 \sin^2 t) \\ \quad + y^2(3x^2 \sin t - x^5 \sin^2 t) + yz e^{-\sin t}(x^4 \sin^2 t - 2x \sin t), \\ z' = \frac{1}{2} yx^4 \cos t e^{\sin t}(1 - \sin t + x^3 \sin^2 t) + \frac{1}{2} z \cos t(1 - x^3 - x^6 \sin^2 t) \\ \quad - y^2(x^6 \sin^2 t - 4x^3 \sin t)e^{\sin t} - yz(3x^2 \sin t - x^5 \sin^2 t) \end{cases}$$

has RF

$$F(t, x, y, z) = \begin{pmatrix} x \\ e^{\sin t}(1 - x^3 \sin t)y + x^2 z \sin t \\ -x^4 y \sin t + e^{-\sin t}(1 + x^3 \sin t)z \end{pmatrix}.$$

Since this system is a 2π--periodic system, and $F(-\pi, x, y, z) \equiv (x, y, z)^T$, by Theorem 5, all the solutions of the considered system defined $[-\pi, \pi]$ are 2π--periodic.

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