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Implied Bond and Derivative Prices Based on Non-Linear Stochastic Interest Rate Models

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Abstract

In this paper we expand the Box Method of Sorwar *et al.* (2007) to value both default free bonds and interest rate contingent claims based on one factor non-linear interest rate models. Further we propose a one-factor non-linear interest rate model that incorporates features suggested by recent research. An example shows the extended Box Method works well in practice.

Keywords: Stochastic, Interest Rates, Derivatives, Box Method

1. Introduction

Stochastic differential equations are the foundations on which modern option pricing methodology is based. However, non-linear stochastic differential equations for interest rate models have been proposed that captures the non-linear dynamics of the spot interest rates. There are two aspects to the modeling of interest rate term structure models and interest rate contingent claims. The first concerns the econometric aspects (see for example, [1]) and the second the numerical implementation of the resulting models. With regard to the numerical aspects of interest rate modeling, there exist three different approaches. The first is the lattice approach introduced by Cox-Ross-Rubinstein (1979) [2]. However, as Barone-Adesi, Dinenis and Sorwar (1997) [3] have demonstrated the lattice approach does not always lead to meaningful bond and hence contingent claim prices. The second approach is the Monte-Carlo simulation approach introduced by Boyle (1977) is mainly used to value path dependent European type contingent claims. To date no single accepted Monte-Carlo simulation scheme has been put forward for the valuation of American type contingent claims. The third approach is the partial differential equation (PDE) approach. With this approach, the partial first and second order derivatives are discretized to produce a system of equations which are then solved iteratively to obtain the bond and contingent claim prices. However, Sorwar *et al.* (2007) have shown that the usual finite difference approach used to discretize the PDE does not always lead to bond and contingent claim prices that correspond with analytical prices where these prices are available.

Sorwar *et al.* (2007) introduced the Box Method from engineering to improve on the standard finite difference approach. Sorwar *et al.* (2007) focused on the CKLS (1992) model. Sorwar *et al.* (2007) did not attempt to value bonds and contingent claims based on non-linear interest rate models. Ait-Sahalia (1996) [4] non-and Conley *et al.* (1997) [5] propose parametric linear one-factor which allows non-linear parameterisation. Our main objective in this paper is to expand the Box Method of Sorwar *et al.* (1997) to price bonds and contingent claims based on both linear and non-linear interest rate models.

The outline of the paper is as follows: Section 2 the general non-linear parametric model and the resulting partial differential equation for default free bonds and contingent claims is outlined. We then derive the Expanded Box Method (EBM) for the valuation of default free bonds and contingent claims. Using US estimates we compute implied bond and contingent claims prices in Section 3. Section 4 contains a summary and conclusion.

2. Expanded Box Method (EBM)

In this section we discuss the valuation of the bond and contingent claim prices based on the extended Ait-Sahalia (1996) [4] and Conley *et al.* (1997) [5] framework. Following Sorwar *et al.* (2007) we let:

$B(r_t, t, T^*)$: price of a discount bond at time t which

Table 1. lternative Parametric Specifications of the Spot Interest Rate Process $dr_t = \mu(r_t)dt + \sigma^2(r_t)dW_t$.

Drift function $\mu(r)$	Diffusion function $\sigma^2(r)$	Reference
$\alpha_0 + \alpha_1 r$	β_0	Vasicek (1977) [6]
$\alpha_0 + \alpha_1 r$	$\beta_1 r$	Cox-Ingersoll-Ross(1985) [7] Brown-Dybvig(1986) [8] Gibbons-Ramaswamy(1993) [9]
$\alpha_0 + \alpha_1 r$	$\beta_2 r^2$	Courtadon (1982) [10]
$\alpha_0 + \alpha_1 r$	$\beta_2 r^{\beta_3}$	Chen *et al.* (1992)
$\alpha_0 + \alpha_1 r + \alpha_2 r^2 + \frac{\alpha_3}{r}$	$\beta_0 + \beta_1 r + \beta_2 r^{\beta_3}$	Ait-Sahalia (1996) [4]
$\alpha_0 + \alpha_1 r + \alpha_2 r^{\alpha_3} + \frac{\alpha_4}{r^{\alpha_5}}$	$\beta_0 + \beta_1 r + \beta_2 r^{\beta_3}$	

matures at time T^* with the generated spot rate r_t .

$P(t,T^*,T)$: price of a contingent claim at time t which expires at time T based on a discount bond which matures at time T^* subject to suitable boundary conditions.

In a risk-neutral world, the drift rate is adjusted by the market price of risk λr [1] so that the short-term interest process becomes:

$$dr_t = \left(\alpha_0 + (\alpha_1 + \lambda)r + \alpha_2 r^{\alpha_3} + \alpha_4 r^{-\alpha_5}\right)dt + \sqrt{\beta_0 + \beta_1 r + \beta_2 r^{\beta_3}}\, dW_t \tag{1}$$

The resulting partial differential equation is:

$$\frac{1}{2}\left[\beta_0 + \beta_1 r + \beta_2 r^{\beta_3}\right]\frac{\partial^2 U}{\partial r^2} + \left[\alpha_0 + (\alpha_1 + \lambda)r + \alpha_2 r^{\alpha_3} + \alpha_4 r^{-\alpha_5}\right]\frac{\partial U}{\partial r} - rU + \frac{\partial U}{\partial t} = 0 \tag{2}$$

In equation (2) $U(r_t,t)$ may represent either $B(r_t,t,T^*)$ or $P(t,T^*,T)$ subject to the appropriate boundary conditions (see [10] for more details). Following Sorwar *et al.* (2007) we transform the above pricing equation such that either the bond or the contingent claims evolves from the options expiration date or the bonds maturity date to the present, *i.e.* we let $\tau = T - t$.

[1]Risk premium is treated differently by researchers. Vasicek (1977) [6] takes $\lambda(r) = \lambda$, Chan *et al.* (1992) [1] take $\lambda(r) = 0$, ox *et al.* (1985), we take $\lambda(r) = \lambda r$.

The above equation then becomes:

$$\frac{\partial^2 U}{\partial r^2} + 2\left[\frac{\alpha_0 + (\alpha_1 + \lambda)r + \alpha_2 r^{\alpha_3} + \alpha_4 r^{-\alpha_5}}{\beta_0 + \beta_1 r + \beta_2 r^{\beta_3}}\right]\frac{\partial U}{\partial r} - \frac{2r}{\beta_0 + \beta_1 r + \beta_2 r^{\beta_3}}U = \frac{2}{\beta_0 + \beta_1 r + \beta_2 r^{\beta_3}}\frac{\partial U}{\partial \tau} \tag{3}$$

We now choose a general function $R(r,\alpha,\beta)$ such that:

$$\frac{1}{R}\frac{\partial}{\partial r}\left[R\frac{\partial U}{\partial r}\right] = \frac{\partial^2 U}{\partial r^2} + 2\left[\frac{\alpha_0 + (\alpha_1 + \lambda)r + \alpha_2 r^{\alpha_3} + \alpha_4 r^{-\alpha_5}}{\beta_0 + \beta_1 r + \beta_2 r^{\beta_3}}\right]\frac{\partial U}{\partial r} \tag{4}$$

The above expression simplifies to yield:

$$\frac{1}{R}\frac{\partial R}{\partial r} = 2\left[\frac{\alpha_0 + (\alpha_1 + \lambda)r + \alpha_2 r^{\alpha_3} + \alpha_4 r^{-\alpha_5}}{\beta_0 + \beta_1 r + \beta_2 r^{\beta_3}}\right] \tag{5}$$

We now integrate from the general value r $(r_{n-1} < r < r_{n+1})$ to the lower limit of integration $r = 0$ to obtain:

$$R(r,\alpha,\beta) = exp\left\{\phi + 2\int_{r_{n-1}}^{r}\left[\frac{\alpha_0 + (\alpha_1 + \lambda)r + \alpha_2 r^{\alpha_3} + \alpha_4 r^{-\alpha_5}}{\beta_0 + \beta_1 r + \beta_2 r^{\beta_3}}\right]dr\right\}$$

where $\phi = ln\, R(0,\alpha,\beta)$. We further note that:

$$\frac{1}{R}\frac{\partial}{\partial r}\left[R\frac{\partial U}{\partial r}\right] = \frac{1}{Q}\frac{\partial}{\partial r}\left[Q\frac{\partial U}{\partial r}\right]$$

where:

$$Q(r,\alpha,\beta)=$$
$$exp\left\{2\int_0^r\left[\frac{\alpha_0+(\alpha_1+\lambda)r+\alpha_2 r^{\alpha_3}+\alpha_4 r^{-\alpha_5}}{\beta_0+\beta_1 r+\beta_2 r^{\beta_3}}\right]dr\right\}$$

So equation (3) becomes:

$$\frac{1}{Q}\frac{\partial}{\partial r}\left(Q\frac{\partial U}{\partial r}\right)-\frac{2r}{\beta_0+\beta_1 r+\beta_2 r^{\beta_3}}U=$$

We now transform the interest rate as:

$$\frac{2}{\beta_0+\beta_1 r+\beta_2 r^{\beta_3}}\frac{\partial U}{\partial \tau} \quad (6)$$

$$s=\frac{cr}{1+cr} \quad \text{where } c \text{ is a constant.} \quad (7)$$

This leads to the transformation of equation (6) as:

$$\frac{1}{Q(s)}\frac{\partial}{\partial s}\left(\Psi(s)\frac{\partial U}{\partial s}\right)-\frac{2s}{c^2(1-s)^3}\frac{U}{\beta_0+\frac{\beta_1 s}{c(1-s)}+\beta_2\left[\frac{s}{c(1-s)}\right]^{\beta_3}}=\frac{2}{c(1-s)^2}\frac{1}{\beta_0+\frac{\beta_1 s}{c(1-s)}+\beta_2\left[\frac{s}{c(1-s)}\right]^{\beta_3}}\frac{\partial U}{\partial \tau} \quad (8)$$

where:

$$\Psi(s)=c(1-s)^2 Q(s)$$

$$Q(s)=exp\left\{\int_0^s\frac{2}{c(1-s)^2}\left\{\frac{\alpha_0+\frac{(\alpha_1+\lambda)s}{c(1-s)}+\alpha_2\left[\frac{s}{c(1-s)}\right]^{\alpha_3}+\alpha_4\left[\frac{s}{c(1-s)}\right]^{-\alpha_5}}{\beta_0+\frac{\beta_1 s}{c(1-s)}+\beta_2\left[\frac{s}{c(1-s)}\right]^{\beta_3}}\right\}dr\right\}$$

Following the set-up of Sorwar *et al.* (2007) a grid of size $M\times N$ is constructed for values of $U_n^m=U(n\Delta r,m\Delta t)$ - the value of U at time increment t_m and interest rate increment s_n, for each method, where:

$$t_m=t_0+m\Delta t \quad m=0,1,\ldots,M$$
$$^2\ \Delta s_{n+1}=\Delta s_n+a \quad n=1,\ldots,N$$

where *a* is an arbitrary constant.

Using the Euler backward difference for the time derivative gives: $\frac{\partial U}{\partial \tau}=\frac{U-U_0}{\Delta t}$,

where U_0 and U refers to bond or contingent claims prices at time step *m-1* and *m* respectively.

Integrating equation (8) from the point $s_{n-\frac{1}{2}}=\frac{s_n+s_{n-1}}{2}$ to point $s_{n+\frac{1}{2}}=\frac{s_{n+1}+s_n}{2}$, we have:

$$-\Delta t\int_{s_{n-\frac{1}{2}}}^{s_{n+\frac{1}{2}}}\frac{\partial}{\partial s}\left(\Psi(s)\frac{\partial U}{\partial s}\right)ds+2\Delta t\int_{s_{n-\frac{1}{2}}}^{s_{n+\frac{1}{2}}}\frac{s}{c^2(1-s)^3}Q(s)f(s)Uds+2\int_{s_{n-\frac{1}{2}}}^{s_{n+\frac{1}{2}}}\frac{1}{c(1-s)^2}Q(s)f(s)Uds$$
$$=2\int_{s_{n-\frac{1}{2}}}^{s_{n+\frac{1}{2}}}\frac{1}{c(1-s)^2}Q(s)f(s)U_0ds \quad (9)$$

Discretizing each of the above integrals, and rearranging gives us the following matrix equation:

$$\alpha_n U_n^{m-1}=\chi_n U_{n-1}^m+\eta_n U_n^m+\beta_n U_{n+1}^m \quad (10)$$

where:

$$-\Delta t\int_{s_{n-\frac{1}{2}}}^{s_{n+\frac{1}{2}}}\frac{\partial}{\partial s}\left(\Psi(s)\frac{\partial U}{\partial s}\right)ds+$$

$$2\Delta t\int_{s_{n-\frac{1}{2}}}^{s_{n+\frac{1}{2}}}\frac{s}{c^2(1-s)^3}Q(s)f(s)Uds+$$

[2] Where *a* and Δs_0 are arbitrary constants. A derivation of this expression can be found in Settari and Aziz (1972) [11].

$$2\int_{s_{n-\frac{1}{2}}}^{s_{n+\frac{1}{2}}} \frac{1}{c(1-s)^2} Q(s) f(s) U ds$$

$$= 2\int_{s_{n-\frac{1}{2}}}^{s_{n+\frac{1}{2}}} \frac{1}{c(1-s)^2} Q(s) f(s) U_0 ds \quad (9)$$

Discretizing each of the above integrals, and rearranging gives us the following matrix equation:

$$\alpha_n U_n^{m-1} = \chi_n U_{n-1}^m + \eta_n U_n^m + \beta_n U_{n+1}^m \quad (10)$$

where:

$$\alpha_n = 2I_1$$

$$\chi_n = \frac{-\Delta t}{s_n - s_{n-1}} \frac{\Psi\left(s_{n-\frac{1}{2}}\right)}{Q(s_n)}$$

$$\beta_n = \frac{-\Delta t}{s_{n+1} - s_n} \frac{\Psi\left(s_{n+\frac{1}{2}}\right)}{Q(s_n)}$$

$$\eta_n = \frac{\Delta t}{s_n - s_{n-1}} \frac{\Psi\left(s_{n-\frac{1}{2}}\right)}{Q(s_n)} + \frac{\Delta t}{s_{n+1} - s_n} \frac{\Psi\left(s_{n+\frac{1}{2}}\right)}{Q(s_n)} + 2\Delta t I_0 + 2I_1$$

$$I_0 = \frac{s_n f(s_n)}{c^2 (1-s_n)^3} \left(s_{n+\frac{1}{2}} - s_{n-\frac{1}{2}}\right)$$

$$I_1 = \frac{f(s_n)}{c(1-s_n)^2} \left(s_{n+\frac{1}{2}} - s_{n-\frac{1}{2}}\right)$$

The matrix equation linking all bond prices or contingent claim prices between two successive time steps *m-1* and *m* is:

$$\begin{pmatrix} \alpha_0 U_1^{m-1} \\ \alpha_0 U_1^{m-1} \\ \vdots \\ \vdots \\ \vdots \\ \vdots \\ \alpha_N U_N^{m-1} \end{pmatrix} =$$

$$\begin{pmatrix} \eta_1 & \beta_1 & 0 & 0 & 0 & \cdots & 0 \\ \chi_2 & \eta_2 & \beta_2 & 0 & 0 & \cdots & 0 \\ 0 & \chi_3 & \eta_3 & \beta_3 & 0 & \cdots & 0 \\ \cdots 0 & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \chi_{N-3} & \chi_{N-3} & \beta_{N-3} & 0 \\ \vdots & \ddots & \ddots & 0 & \chi_{N-2} & \eta_{N-2} & \beta_{N-2} \\ 0 & \cdots & \cdots & 0 & 0 & \chi_{N-1} & \eta_{N-1} \end{pmatrix} \begin{pmatrix} \alpha_0 U_1^m \\ \alpha_0 U_1^m \\ \vdots \\ \vdots \\ \vdots \\ \vdots \\ \alpha_N U_{N-1}^m \end{pmatrix}$$

Sorwar *et al.* [12] used the following SOR iteration process to determine bond and contingent claims prices:

$$z_n^m = \frac{1}{\eta_n}\left(\alpha_n U_n^{m-1} - \chi_n U_{n-1}^{m} - \beta_n U_{n+1}^{m-1}\right) \quad (11)$$

In particular they evaluated bond using the following expression:

$$U_n^m = \omega z_n^m + (1-\omega) U_n^{m-1} \quad (12)$$

Contingent claims were calculated using:

$$U_n^m = max\left[Z, \omega z_n^m + (1-\omega) U_n^{m-1}\right] \quad (13)$$

where Z is the intrinsic value of the contingent claim and for n=1,......,N-1, and $\omega \in (1,2]$ [3].

3. Analysis of Results

In this section we apply the EBM using recent estimates of the non-linear model of Ait-Sahalia (1996) [4] on 7-day Eurodollar deposit spot rate over 1973-1995 to demonstrate the method. Ait-Sahalia (1996, Table 4) [4] obtained the following estimates:

$$\alpha_0 = -4.643\times10^{-3},\ \alpha_1 = 4.333\times10^{-2},\ \alpha_2 = -1.143\times10^{-1}, \alpha_3 = 2,$$

$$\alpha_4 = 1.304\times10^{-4},\ \alpha_5 = 1.$$

$$\beta_0 = 1.108\times10^{-4},\ \beta_1 = -1.883\times10^{-3},\ \beta_2 = 9.681\times10^{-3},\ \beta_3 = 2.073.$$

Table 2 reports the bond prices for maturities ranging from 6 months to 30 years and across interest rates of 2% to 16%. Table III reports both the value of call and put options across a wide range of interest rates. We consider both short and long dated call and put options. The short dated call and put options are based on a 5-year bond with an expiry date of 1 year and is during the last year before the bond matures. Similarly long dated options are based on 10-year bond with an expiry date of 5 years during the last 5 years of the bond. Finally both call and put option prices are calculated across a wide range of exercise prices. The exercise prices are chosen so as to highlight variation of prices for both in-the-money and out-of-the-money options. We assume λ, the market price of risk is zero.

Turning to **Table 2**, we find that at lower interest rate bond prices decay slowly as the term to maturity increases. For example, at 2% interest rate a 1 year maturity bond is valued at 98.1119, whilst a 30 year bond is valued at 74.8290. At high interest rates, the bond price decay is more rapid for example at 16% interest rate, a 1 year maturity bond is valued at 85.2915, whist a 30 year maturity bond is valued at 1.1770. Turning to **Table 3**, we observe the following features. Short expiry call op

[3] ω is determined by numerical experimentation. For all our calculations we took $\omega = 1.85$

Table 2. All options written on zero coupon bonds with a face value of $100.00.

	Interest Rate							
Maturity of Bond	**2%**	**4%**	**6%**	**8%**	**10%**	**12%**	**14%**	**16%**
0.5	99.0286	98.0370	96.9855	96.0885	95.1315	94.1844	93.2506	92.3403
1	98.1119	96.1434	94.0805	92.3406	90.5050	88.7059	86.9566	85.2915
5	92.2400	83.3035	74.3413	67.4685	60.8623	54.9010	49.7324	45.6212
10	87.0431	71.9535	56.7017	46.1717	37.2750	30.1193	24.7834	21.3038
15	83.1089	64.1538	44.6651	32.1800	22.9491	16.5267	12.3933	10.1317
20	79.9228	58.6473	36.4723	22.9644	14.3178	9.0809	6.2237	4.8889
25	77.2156	54.6338	30.8731	16.8832	9.0110	5.0032	3.1400	2.3870
30	74.8290	51.6021	27.0075	12.8582	5.7491	2.7679	1.5921	1.1770

Table 3. All options written on zero coupon bonds with a face value of $100.00.

r (%)	**Exercise-Price**	**5 year maturity 1 year expiry**	**5 year maturity 1 year expiry**	**Exercise-Price**	**10 year maturity 5 year expiry**	**10 year maturity 5 year expiry**
	(83.3035)	**call**	**put**	***(71.9535)***	**call**	**put**
4	70	16.0031	0.0000	60	21.9713	0.0007
	75	11.1959	0.0000	65	17.8062	0.0493
	80	6.3895	0.0050	70	13.6418	0.6489
	85	1.9369	1.6966	75	9.5270	3.1894
	90	0.1421	6.6966	80	5.7979	8.0466
	(67.4685)			***(46.1717)***		
8	55	16.6811	0.0000	35	22.5578	0.0000
	60	12.0641	0.0000	40	19.1843	0.0000
	65	7.4471	0.0000	45	15.8109	0.0058
	70	2.8302	2.5315	50	12.4375	3.8283
	75	0.0203	7.5315	55	9.0641	8.8283
12	***(54.9010)***			***(30.1193)***		
	45	14.9341	0.0000	20	19.1395	0.0000
	50	10.4996	0.0000	25	16.3942	0.0000
	55	6.0652	0.1561	30	13.6492	0.0183
	60	1.6310	5.1561	35	10.9042	4.8804
	65	0.0000	10.1561	40	8.1591	9.8804
16	***(45.6212)***			***(21.3038)***		
	35	15.7692	0.0000	10	16.7416	0.0000
	40	11.5046	0.0000	15	14.4606	0.0000
	45	7.2400	0.0005	20	12.1795	0.0001
	50	2.9755	4.3788	25	9.8985	3.6962
	55	0.0129	9.3782	30	7.6174	8.6962

tions decay faster than longer expiry call options; for example at r = 4%; the price of a call option decreases from 16.0031 to 11.1959 when the exercise price in creases from 70 to 75. For a similar 5 year call option the price decreases from 21.9713 to 17.8062, when the exercise price increases from 60 to 65. Furthermore, the call option prices decrease at a slower rate at high interests. This feature becomes more pronounced for longer expiry call options. With regard to put options we find, the prices are very close to zero, when the options are at-the-money or out-of-the-money. Finally, we find that the value of in-the-money put options is dominated by the intrinsic-value.

4. Conclusions

The introduction of non-linear stochastic interest rate models has led to the possibility of valuing interest contingent claims that reflects the characteristics of the yield curve more accurately. In this paper we have expanded the Box Method to value both bond and American type interest rate contingent claims based on single factor non-linear interest rate models. We have found that the

Expanded Box Method works well with the example considered.

5. References

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Appendix

$$\int_{s_{n-\frac{1}{2}}}^{s_{n+\frac{1}{2}}} \frac{\partial}{\partial s}\left(\Psi(s)\frac{\partial U}{\partial s}\right) ds \approx \Psi\left(s_{n+\frac{1}{2}}\right)\frac{\partial U}{\partial s}\bigg|_{s_{n+\frac{1}{2}}} - \Psi\left(s_{n-\frac{1}{2}}\right)\frac{\partial U}{\partial s}\bigg|_{s_{n-\frac{1}{2}}}$$

Further:

$$\frac{\partial U}{\partial s}\bigg|_{s_{n+\frac{1}{2}}} \approx \frac{U_{n+1}^m - U_n^m}{s_{n+1} - s_n}$$

$$\frac{\partial U}{\partial s}\bigg|_{s_{n-\frac{1}{2}}} \approx \frac{U_n^m - U_{n-1}^m}{s_n - s_{n-1}}$$

Substitution of the above approximation yields:

$$\int_{s_{n-\frac{1}{2}}}^{s_{n+\frac{1}{2}}} \frac{\partial}{\partial s}\left(\Psi(s)\frac{\partial U}{\partial s}\right) ds \approx \frac{\Psi\left(s_{n+\frac{1}{2}}\right)}{s_{n+1} - s_n} U_{n+1}^m - \left[\frac{\Psi\left(s_{n+\frac{1}{2}}\right)}{s_{n+1} - s_n} + \frac{\Psi\left(s_{n-\frac{1}{2}}\right)}{s_n - s_{n-1}}\right] U_n^m + \frac{\Psi\left(s_{n-\frac{1}{2}}\right)}{s_n - s_{n-1}} U_{n-1}^m$$

$$2\Delta t \int_{s_{n-\frac{1}{2}}}^{s_{n+\frac{1}{2}}} \frac{s}{c^2(1-s)^3} f(s) Q(s) U ds \approx$$

$$2\Delta t Q(s_n) U_n^m \int_{s_{n-\frac{1}{2}}}^{s_{n+\frac{1}{2}}} \frac{s}{c^2(1-s)^3} f(s) ds$$

We further take:

$$\int_{s_{n-\frac{1}{2}}}^{s_{n+\frac{1}{2}}} \frac{s}{c^2(1-s)^3} f(s) ds \approx \frac{s_n}{c^2(1-s_n)^3} f(s_n)\left(s_{n+\frac{1}{2}} - s_{n-\frac{1}{2}}\right)$$

Similar approximation yields:

$$2\int_{s_{n-\frac{1}{2}}}^{s_{n+\frac{1}{2}}} \frac{1}{c(1-s)^2} f(s) Q(s) U ds \approx$$

$$2Q(s_n) U_n^m \frac{1}{c(1-s_n)^2} f(s_n)\left(s_{n+\frac{1}{2}} - s_{n-\frac{1}{2}}\right)$$

$$2\int_{s_{n-\frac{1}{2}}}^{s_{n+\frac{1}{2}}} \frac{1}{c(1-s)^2} f(s) Q(s) U_0 ds \approx$$

$$2Q(s_n) U_n^{m-1} \frac{1}{c(1-s_n)^2} f(s_n)\left(s_{n+\frac{1}{2}} - s_{n-\frac{1}{2}}\right)$$

