Advances in Linear Algebra & Matrix Theory, 2019, 9, 73-81
http://www.scirp.org/journal/alamt
ISSN Online: 2165-3348
ISSN Print: 2165-3348

Maximum Principles for Normal Matrices

Achiya Dax

Hydrological Service, Jerusalem, Israel
Email: dax20@water.gov.il

How to cite this paper: Dax, A. (2019) Maximum Principles for Normal Matrices. *Advances in Linear Algebra & Matrix Theory*, **9**, 73-81.

https://doi.org/10.4236/alamt.2019.93005

Received: July 19, 2019
Accepted: September 27, 2019
Published: September 30, 2019

Abstract

Ky Fan maximum principle is a well-known observation about traces of certain hermitian matrices. In this note, we derive a powerful extension of this claim. The extension is achieved in three ways. First, traces are replaced with norms of diagonal matrices, and any unitarily invariant norm can be used. Second, hermitian matrices are replaced by normal matrices, so the rule applies to a larger class of matrices. Third, diagonal entries can be replaced with eigenvalues and singular values. It is shown that the new maximum principle is closely related to the problem of approximating one matrix by another matrix of a lower rank.

Keywords

Ky Fan Maximum Principle, Normal Matrices, Extended Maximum Principle, Unitarily Invariant Norms

1. Introduction and Main Results

Ky Fan maximum principle is a useful observation that characterizes an important property of hermitian matrices. It is interesting, therefore, to see whether it is possible to extend this rule to other types of matrices. In this note, we answer this question for normal matrices. The reader is referred to references [1–15] for detailed discussions of normal matrices and their properties. Let $N = (n_{ij}) \in \mathbb{C}^{n\times n}$ be a normal matrix with eigenvalues ν_j, $j = 1, \ldots, n$, that satisfy

$$|\nu_1| \geq |\nu_2| \geq \cdots \geq |\nu_n|. \tag{1.1}$$

Then N has a spectral decomposition of the form

$$N = VSV^*, \tag{1.2}$$

where $S \in \mathbb{C}^{n\times n}$ is a diagonal matrix

$$S = \mathrm{diag}\{\nu_1, \nu_2, \ldots, \nu_n\}, \tag{1.3}$$

and $V \in \mathbb{C}^{n\times n}$ is a unitary matrix whose columns are eigenvectors of N. The matrix V^* denotes the conjugate transpose of V, and the term unitary matrix means that $V^*V = VV^* = I$. Let $\boldsymbol{v}_j$, $j = 1, \ldots, n$, denote the jth column of V. Then (1.2) implies the equalities

$$N\boldsymbol{v}_j = \nu_j \boldsymbol{v}_j \quad \text{for} \quad j = 1, \ldots, n. \tag{1.4}$$

That is, $\boldsymbol{v}_j$ is an eigenvector of N that corresponds to ν_j. Let σ_j, $j = 1, \ldots, n$, denote the singular values of N arranged in decreasing order. Then the fact that N is a normal matrix implies the equalities

$$\sigma_j = |\nu_j| \quad \text{for} \quad j = 1, \ldots, n. \tag{1.5}$$

Let $1 \le k \le n$ be a given positive integer. Then the matrices $S_k \in \mathbb{C}^{k\times k}$ and $V_k \in \mathbb{C}^{n\times k}$ are defined by the rules

$$S_k = \text{diag}\{\nu_1, \ldots, \nu_k\}, \tag{1.6}$$

and

$$V_k = [\boldsymbol{v}_1, \ldots, \boldsymbol{v}_k]. \tag{1.7}$$

In other words, S_k is a principal submatrix of S, and V_k is composed from the first k columns of V. Observe that (1.2) implies the equalities $V^*NV = S$ and

$$V_k^* N V_k = S_k. \tag{1.8}$$

Note also that V_k belongs to the set

$$\mathbb{Q}_k = \{Q \,|\, Q \in \mathbb{C}^{n\times k} \quad \text{and} \quad Q^*Q = I\}, \tag{1.9}$$

which contains all the $n \times k$ matrices that have orthonormal columns.

The maximum problems that we solve consider Rayleigh quotient matrices of the form Q^*NQ where Q belongs to $\mathbb{Q}_k$. The first assertion ia about $\delta(Q^*NQ)$, the $k \times k$ diagonal matrix that shares the same diagonal entries as Q^*NQ.

Theorem 1. *Let N be a normal matrix as above, and let $\|\cdot\|$ be a unitarily invariant norm on $\mathbb{C}^{k\times k}$. Then*

$$\|S_k\| = \max_{Q\in\mathbb{Q}_k} \|\delta(Q^*NQ)\|, \tag{1.10}$$

and the maximal value is obtained when $Q = V_k$.

Recall that a matrix norm $\|\cdot\|$ on $\mathbb{C}^{n\times n}$ is called **unitarily invariant** if for any matrix $A \in \mathbb{C}^{n\times n}$ and any unitary matrix $U \in \mathbb{C}^{n\times n}$ we have the equalities $\|A\| = \|UA\| = \|AU\|$. The family of unitarily invariant norms includes several useful norms, such as Frobenius norm, the Schatten p-norms, Ky Fan k-norms, the trace norm, and the spectral norm. The relation between Theorem 1 and Ky Fan maximum principle can be seen by considering the trace norm $\|\cdot\|_{tr}$. Given a matrix $A = (a_{ij}) \in \mathbb{C}^{n\times n}$ its trace norm is defined as

$$\|A\|_{tr} = \sum_{j=1}^{n} \sigma_j(A), \tag{1.11}$$

where $\sigma_j(A)$, $j = 1, \ldots, n$, denote the singular values of A sorted in decreasing order. The conversion of Theorem 1 to handle the trace norm is simplified by applying the absolute-trace function

$$abstr(A) = \sum_{j=1}^{n} |a_{jj}|, \tag{1.12}$$

which turns (1.10) into the form

$$\sigma_1 + \cdots + \sigma_k = \max_{Q \in \mathbb{Q}_k} abstr(Q^* N Q). \tag{1.13}$$

The last result is the **absolute trace theorem** which was recently proved in [4]. Assume for a moment that N is a positive semidefinite hermitian matrix. In this case $\nu_j = \sigma_j$ for $j = 1, \ldots, n$, and the matrices $Q^* N Q$ are positive semidefinite. Hence the diagonal entries of these matrices are nonnegative and (1.13) is reduced to **Ky Fan maximum principle** [6]

$$\nu_1 + \cdots + \nu_k = \max_{Q \in \mathbb{Q}_k} \text{trace}(Q^* N Q). \tag{1.14}$$

It is also easy to verify that the positive semidefinite requrement is not essential for Ky Fan maximum principle, so (1.14) holds whenever N is hermitian. However, if N is not positive semidefinite then the two problems may have different solutions.

Summarizing the above observations we see that both Theorem 1 and the absolute trace theorem can be viewed as extensions of Ky Fan maximum principle. The absolute trace theorem allows the use of normal matrices instead of hermitian matrices, while Theorem 1 achieves further extension by replacing the trace norm with any unitarily invariant norm. Below we will show that further extensions are gained by replacing the diagonal entries with eigenvalues and singular values.

Let $\lambda_j(Q^* N Q)$, $j = 1, \ldots, k$, denote the eigenvalues of $Q^* N Q$. Then $\lambda(Q^* N Q)$ is defined to be the $k \times k$ diagonal matrix whose (j, j) diagonal entry equals $\lambda_j(Q^* N Q)$. That is,

$$\lambda(Q^* N Q) = \text{diag}\{\lambda_1(Q^* N Q), \ldots, \lambda_k(Q^* N Q)\}. \tag{1.15}$$

Theorem 2. *Let $N \in \mathbb{C}^{n \times n}$ be a normal matrix as above, and let $\|\cdot\|$ be a unitarily invariant norm on $\mathbb{C}^{k \times k}$. Then*

$$\|S_k\| = \max_{Q \in \mathbb{Q}_k} \|\lambda(Q^* N Q)\|, \tag{1.16}$$

and the maximal value is attained for V_k.

As before, it is interesting to consider the trace norm. In this case (1.16) takes the form

$$\sum_{j=1}^{k} |\nu_j| = \max_{Q \in \mathbb{Q}_k} \sum_{j=1}^{k} |\lambda_j(Q^* N Q)|, \tag{1.17}$$

and, when N is a positive semidefinite hermitian matrix, (1.17) is reduced to

$$\sum_{j=1}^{k} \nu_j = \max_{Q \in \mathbb{Q}_k} \sum_{j=1}^{k} \lambda_j(Q^* N Q), \tag{1.18}$$

Furthermore, since

$$\sum_{j=1}^{k} \lambda_j(Q^* N Q) = \text{trace}(Q^* N Q), \tag{1.19}$$

we see that (1.18) is essentially Ky Fan maximum principle (1.14).

The third maximum principle considers the singular values of Q^*NQ. Let $\sigma_j(Q^*NQ)$, $j = 1,\ldots,k$, denote the singular values of Q^*NQ arranged in decreasing order, and let $\sigma(Q^*NQ)$ denote the $k \times k$ diagonal matrix whose (j,j) diagonal entry equals $\sigma_j(Q^*NQ)$, $j = 1,\ldots,k$. That is

$$\sigma(Q^*NQ) = \text{diag}\{\sigma_1(Q^*NQ),\ldots,\sigma_k(Q^*NQ)\}. \tag{1.20}$$

Then, here we consider the maximization of $\|\sigma(Q^*NQ\|$. However, since the equality

$$\|Q^*NQ\| = \|\sigma(Q^*NQ\| \tag{1.21}$$

holds for any unitarily invariant norm, it is possible to replace $\|\sigma(Q^*NQ)\|$ with $\|Q^*NQ\|$.

Theorem 3. *Let N be a normal matrix as above, and let $\|\cdot\|$ be a unitarily invariant norm on $\mathbb{C}^{k\times k}$. Then*

$$\|S_k\| = \max_{Q\in\mathbb{Q}_k} \|Q^*NQ\|, \tag{1.22}$$

and the maximum value is attained for V_k.

As in the former cases it is easy to verify that when using the trace norm on a positive semidefinite hermitian matrix the last assertion is reduced to Ky Fan maximum principle. It should be noted, however, that in the general case, when using an arbitrary unitarily invariant norm and N is an arbitrary normal matrix, the objective functions of the three maximum problems can be quite different. Yet, as we have seen, the three problems share the same solution matrix, V_k, and the same optimal value, $\|S_k\|$.

The rest of the paper continues as follows: The next section introduces the necessary theoretical basis, while Section 3 provides the proofs of Theorems 1-3. Finally, in Section 4 we expose interesting relations between the new maximum principles and a minimum norm problem that arises when searching a rank-k matrix that is closest to N.

2. Theoretical Background and Tools

We shall start by introducing some useful notations. Let $A = (a_{ij}) \in \mathbb{C}^{n\times n}$ be a given arbitrary matrix with eigenvalues $\lambda_j(A)$, $j = 1,\ldots,n$, and singular values $\sigma_j(A)$, $j = 1,\ldots,n$. Then there is no loss of generality in assuming that

$$|\lambda_1(A))| \geq |\lambda_2(A)| \geq \cdots \geq |\lambda_n(A)|, \tag{2.1}$$

$$\sigma_1(A) \geq \sigma_2(A) \geq \cdots \geq \sigma_n(A), \tag{2.2}$$

and

$$|a_{11}| \geq |a_{22}| \geq \cdots \geq |a_{nn}|. \tag{2.3}$$

These inequalities enable us to define the related diagonal matrices

$$\lambda(A) = \text{diag}\{\lambda_1(A),\ldots,\lambda_n(A)\}, \tag{2.4}$$

$$\sigma(A) = \text{diag}\{\sigma_1(A),\ldots,\sigma_n(A)\}, \tag{2.5}$$

and

$$\delta(A) = \text{diag}\{a_{11},\ldots,a_{nn}\}. \tag{2.6}$$

Thus, for example, $\lambda(A)$ is an $n \times n$ diagonal matrix whose (j, j) entry equals $\lambda_j(A)$. Similarly, the matrix

$$S_k(A) = \operatorname{diag}\{\sigma_1(A), \dots, \sigma_k(A)\} \tag{2.7}$$

is a $k \times k$ principal submatrix of $\sigma(A)$.

Next, we will say a few words about majorization and dominance. Let $B \in \mathbb{C}^{n\times n}$ and $C \in \mathbb{C}^{n\times n}$ be a pair of matrices with singular values

$$\sigma_1(B) \geq \cdots \geq \sigma_n(B) \geq 0, \quad \text{and} \quad \sigma_1(C) \geq \cdots \geq \sigma_n(C) \geq 0,$$

respectively, that satisfy

$$\sum_{j=1}^{k} \sigma_j(B) \leq \sum_{j=1}^{k} \sigma_j(C) \quad \text{for} \quad k = 1, \dots, n. \tag{2.8}$$

In this case we say that the singular values of B are majorized by those of C. The importance of this relation comes from Ky Fan dominance theorem [6], which says that (2.8) ensures the inequality

$$\|B\| \leq \|C\| \tag{2.9}$$

for any unitarily invariant norm.

Another useful property stems from the interlacing theorems of Cauchy and Poincaré. The original statements of these theorems are about eigenvalues of hermitian matrices, e.g., [8] [11] [15]. Yet when these theorems are adapted to singular values we have the following results. Let $P_k(A)$ denote the $k \times k$ principal submatrix of A which is obtained by deleting from A the last $n - k$ columns and the last $n - k$ rows. Let

$$\sigma_1(P_k(A)) \geq \cdots \geq \sigma_k(P_k(A)) \geq 0 \tag{2.10}$$

denote the singular values of $P_k(A)$. Then

$$\sigma_j(P_k(A)) \leq \sigma_j(A) \quad \text{for} \quad j = 1, \dots, k. \tag{2.11}$$

Moreover, let $\mathbb{Q}_k$ be defined as in (1.9). Then the inequalities

$$\sigma_j(Q^*AQ) \leq \sigma_j(A), \quad j = 1, \dots, k, \tag{2.12}$$

hold for all $Q \in \mathbb{Q}_k$.

Combining the interlacing relations with Ky Fan dominance theorem yields two powerful tools. First note that (2.12) means that the singular values of Q^*AQ are majorized by those of $S_k(A)$, which yields the inequality

$$\|Q^*AQ\| \leq \|S_k(A)\|. \tag{2.13}$$

The second tool is based on the inequality

$$\sum_{j=1}^{n} |a_{jj}| \leq \sum_{j=1}^{n} \sigma_j(A), \tag{2.14}$$

whose proof can be found, for example, in [3, p. 1237], [9, p. 154], and [15, pp. 261–263]. Applying this inequality on $P_k(A)$ shows that

$$\sum_{j=1}^{k} |a_{jj}| \leq \sum_{j=1}^{k} \sigma_j(P_k(A)), \tag{2.15}$$

while (2.11) gives

$$\sum_{j=1}^{k} |a_{jj}| \leq \sum_{j=1}^{k} \sigma_j(A) \quad \text{for} \quad k = 1, \dots, n. \tag{2.16}$$

The last inequalities mean that the singular values of $\delta(A)$ are majorized by those of A, which shows that

$$\|\delta(A)\| \leq \|A\|. \tag{2.17}$$

Further results on majorization relations between $\delta(A), \sigma(A)$ and $\lambda(A)$, can be found in [1], [4], [9, p. 176], [10, pp. 313–318] and [15, p. 262].

3. Proofs

In this section we provide the proofs of Theorems 1-3.

The proof of Theorem 1. Let Q be some matrix from $\mathbb{Q}_k$. Then (2.17) implies

$$\|\delta(Q^* N Q)\| \leq \|Q^* N Q\|, \tag{3.1}$$

while from (2.13) we conclude that

$$\|Q^* N Q\| \leq \|S_k\|, \tag{3.2}$$

and

$$\|\delta(Q^* N Q)\| \leq \|S_k\|. \tag{3.3}$$

Finally, from (1.8) we obtain that

$$\|\delta(V_k^* N V_k)\| = \|\delta(S_k)\| = \|S_k\|. \tag{3.4}$$

□

The proof of Theorem 2. Let $\hat{Q}$ be some matrix from $\mathbb{Q}_k$. Then Schur's triangularization theorem ensures the existence of a $k \times k$ unitary matrix, $\tilde{Q}$, such that

$$\tilde{Q}^* (\hat{Q}^* N \hat{Q}) \tilde{Q} = T, \tag{3.5}$$

where $T \in \mathbb{C}^{k \times k}$ is an upper-triangular matrix. Therefore, since the diagonal entries of T are eigenvalues of $\hat{Q}^* N \hat{Q}$,

$$\lambda(\hat{Q}^* N \hat{Q}) = \delta(T). \tag{3.6}$$

Note also that the matrix

$$Q = \hat{Q}\tilde{Q} \tag{3.7}$$

belong to $\mathbb{Q}_k$, and

$$\lambda(\hat{Q}^* N \hat{Q}) = \delta(Q^* N Q). \tag{3.8}$$

Hence from (3.3) we conclude that

$$\|\lambda(\hat{Q}^* N \hat{Q})\| = \|\delta(Q^* N Q)\| \leq \|S_k\|, \tag{3.9}$$

while a further use of (1.8) shows that the above upper bound is achieved for V_k.

□

The proof of Theorem 3. From (2.13) we obtain that

$$\|Q^* N Q\| \leq \|S_k\|, \tag{3.10}$$

and (1.8) implies that the upper bound is gained when $Q = V_k$.

□

4. Relations with Low-Rank Approximations

In this section we reveal interesting relations between the new maximum principles and the low-rank approximation problem

$$\begin{array}{ll} \text{minimize} & F(B) = \|N - B\| \\ \text{subject to} & B \in \mathbb{C}^{n\times n} \quad \text{and} \quad \operatorname{rank}(B) \le k. \end{array} \tag{4.1}$$

It is well-known that the solution of (4.1) is obtained from the singular value decomposition (SVD) of N, e.g. [3, p. 1243] or [9, p. 215]. In our case the SVD is easily concluded from the spectral decomposition (1.1)-(1.8), and the matrix

$$T_k = V_k S_k V_k^* \tag{4.2}$$

is a rank-k truncated SVD of N that solves (4.1).

A second consequence of the spectral decomposition (1.1)-(1.8) is that the problem

$$\min_{Q\in\mathbb{Q}_k} \|N - Q(Q^*NQ)Q^*\| \tag{4.3}$$

is essentially equivalent to (4.1). This observation stems from the following facts. First note that for any $Q \in \mathbb{Q}_k$ the rank of the matrix $Q(Q^*NQ)Q^*$ can't exceed k. Hence the optimal value of (4.3) exceeds that of (4.1). Yet for $Q = V_k$ problem (4.3) attains this value. This shows that both problems share the same optimal value, $\|N - V_k S_k V_k^*\|$, that V_k solves (4.3), and that a solution for (4.3) provides a solution for (4.1).

The relation between (4.3) and the maximum principle (1.22) is exposed by using the Frobenius matrix norm $\|\cdot\|_F$. Recall that for any matrix $A = (a_{ij}) \in \mathbb{C}^{n\times n}$

$$\|A\|_F = \Big(\sum_{i=1}^n \sum_{j=1}^n |a_{ij}|^2\Big)^{1/2} = \Big(\sum_{j=1}^n (\sigma_j(A))^2\Big)^{1/2}. \tag{4.4}$$

When using this norm problem (4.1) is reduced to Eckart-Young low-rank approximation problem, e.g., [3, p. 1243] or [9, p. 217]. Observe that for any $Q \in \mathbb{Q}_k$ we have the equality

$$\|N - Q(Q^*NQ)Q^*\|_F^2 = \|N\|_F^2 - \|Q^*NQ\|_F^2\,, \tag{4.5}$$

which shows that the minimum norm problem

$$\min_{Q\in\mathbb{Q}_k} \|N - Q(Q^*NQ)Q^*\|_F^2 \tag{4.6}$$

is equivalent to the maximum problem

$$\max_{Q\in\mathbb{Q}_k} \|Q^*NQ\|_F^2. \tag{4.7}$$

The optimal values of (4.6) and (4.7) are $\sum_{j=k+1}^{n} |\nu_j|^2$ and $\sum_{j=1}^{k} |\nu_j|^2$, respectively, and the sum of these values equals $\|N\|_F^2 = \sum_{j=1}^{n} |\nu_j|^2$.

Another interesting equality characterizes the trace norm. In this case, the optimal values of (1.22) and (4.1) are $\sum_{j=1}^{k} |\nu_j|$ and $\sum_{j=k+1}^{n} |\nu_j|$, respectively, and the sum of these values equals $\|N\|_{tr} = \sum_{j=1}^{n} |\nu_j|$.

A similar situation occurs when using the Schatten p-norm, $\|\cdot\|_p$. Recall that $1 < p < \infty$ and

$$\|N\|_p = \Big(\sum_{j=1}^{n} |\nu_j|^p\Big)^{1/p}. \tag{4.8}$$

In practice it is convenient to use $\|\cdot\|_p^p$ instead of $\|\cdot\|_p$. Hence the resulting optimal values are $\sum_{j=1}^{k} |\nu_j|^p$ and $\sum_{j=k+1}^{n} |\nu_j|^p$, and the sum of these values equals $\|N\|_p^p$.

5. Concluding Remarks

The paper derives three maximum principles that apply to any normal matrix and any unitarily invariant norm. When using the trace norm on positive semidefinite hermitian matrices these principles coincide with Ky Fan maximum principle. Another interesting relation characterizes the Frobenius norm. In this case, the new maximum principle is closely related to Eckart-Young low-rank approximation problem.

The results for normal matrices pave the way for further extensions of the maximum principle. The difficulty here is that non-normal matrices don't have the spectral decomposition. Hence the maximum problems need some amendments and the solution is concluded from the SVD. See [3] for extended version of Theorem 3. This suggests that theorems 1 and 2 can be treated in a similar way. Another related question is whether a similar extension is possible for Ky Fan minimum principle [7]. However, these issues are left to future research.

Conflicts of Interest

The author declares no conflicts of interest regarding the publication of this paper.

References

[1] Bhatia, R. (1997) Matrix Analysis. Springer, New York. https://doi.org/10.1007/978-1-4612-0653-8

[2] Carlson, D. (1983) Minimax and Interlacing Theorems for Matrices. *Linear Algebra and Its Applications*, **54**, 153-172. https://doi.org/10.1016/0024-3795(83)90211-2

[3] Dax, A. (2010) On Extremum Properties of Orthogonal Quotient Matrices. *Linear Algebra and Its Applications*, **432**, 1234-1257. https://doi.org/10.1016/j.laa.2009.10.034

[4] Dax, A. (2019) Bounding Inequalities for Eigenvalues of Principal Submatrices. *Advances in Linear Algebra & Matrix Theory*, **9**, 21-34. https://doi.org/10.4236/alamt.2019.92002

[5] Fan, K. (1949) On a Theorem of Weyl Concerning Eigenvalues of Linear Transformations I. *Proceedings of the National Academy*

of Sciences of the United States of America, **35**, 652-655. https://doi.org/10.1073/pnas.35.11.652

[6] Fan, K. (1951) Maximum Properties and Inequalities for the Eigenvalues of Completely Continuous Operators. *Proceedings of the National Academy of Sciences of the United States of America*, **37**, 760-766. https://doi.org/10.1073/pnas.37.11.760

[7] Fan, K. (1953) A Minimum Property of the Eigenvalues of a Hermitian Transformation. *The American Mathematical Monthly*, **60**, 48-50. https://doi.org/10.2307/2306486

[8] Horn, R.A. and Johnson, C.R. (1985) Matrix Analysis. Cambridge University Press, Cambridge.

[9] Horn, R.A. and Johnson, C.R. (1991) Topics in Matrix Analysis. Cambridge University Press, Cambridge. https://doi.org/10.1017/CBO9780511840371

[10] Marshall, A.W., Olkin, I. and Arnold, B.C. (2011) Inequalities: Theory of Majorization and Its Applications. Springer Series in Statistics, 2nd Edition, Springer, New York. https://doi.org/10.1007/978-0-387-68276-1

[11] Parlett, B.N. (1980) The Symmetric Eigenvalue Problem. Prentice-Hall, Englewood Cliffs, NJ.

[12] Queiró, J.F. (1987) On the Interlacing Property for Singular Values and Eigenvalues. *Linear Algebra and Its Applications*, **97**, 23-28. https://doi.org/10.1016/0024-3795(87)90136-4

[13] Queiró, J.F. and Duarte, A.L. (2009) Imbedding Conditions for Normal Matrices. Linear Algebra and Its Applications, **430**, 1806-1811. https://doi.org/10.1016/j.laa.2008.08.015

[14] Thompson, R.C. (1972) Principal Submatrices IX: Interlacing Inequalities for Singular Values of Submatrices. *Linear Algebra and Its Applications*, **5**, 1-12. https://doi.org/10.1016/0024-3795(72)90013-4

[15] Zhang, F. (1999) Matrix Theory: Basic Results and Techniques. Springer-Verlag, New York.